to assume a specific value. These are simple boundary conditions. In other cases, a relationship is established among several degrees of freedom. These are called *multiple point constraints* (MPC). They can be linear or nonlinear. Linear multiple point constraints were encountered in Chapter 2, for instance, in Section 2.10 on cyclic symmetry. Examples of nonlinear equations are given in the following sections and include rigid body motion, incompressible behavior and others. In Section 2.6, it was shown that a linear multiple point constraint can be taken care of right away at the creation time of the stiffness matrix by expressing the dependent degree of freedom as a function of the independent degrees of freedom. A nonlinear multiple point constraint can be treated in the same way after linearization.

The linearization follows exactly the scheme sketched in Section 3.1. Let

$$\boldsymbol{U} := \left\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\right\} \tag{3.55}$$

be the degrees of freedom involved in the nonlinear multiple point constraint $f(\boldsymbol{U}) = F$. Then, a linearization at $\boldsymbol{U} = \boldsymbol{U}_0$ yields

$$f(\boldsymbol{U}_0) + \nabla f_U(\boldsymbol{U}_0) \cdot (\boldsymbol{U} - \boldsymbol{U}_0) = F \tag{3.56}$$

or

$$\nabla f_U(\boldsymbol{U}_0) \cdot \Delta \boldsymbol{U} = F - f(\boldsymbol{U}_0) \tag{3.57}$$

where

$$\Delta \boldsymbol{U} := \boldsymbol{U} - \boldsymbol{U}_0. \tag{3.58}$$

This equation is updated as soon as a new solution $\boldsymbol{U}_0$ is obtained. Notice that not only can the coefficients of a linearized multiple point constraint change from iteration to iteration, but also the degrees of freedom involved. This can lead to a change of the dependent degrees of freedom as the calculation proceeds.

Accordingly, a stream chart of a nonlinear solution procedure that includes nonlinear multiple point constraints looks like the one shown in Figure 3.8. The box "update MPC" not only stands for the update of the multiple point constraints but also for the update of any solution dependent boundary conditions such as contact areas or radiation heat flux rates.

3.5 Rigid Body Motion

A first example of nonlinear multiple point constraints constitutes rigid body motion. Here, nonlinearity arises because of large rotations. In what follows, rectangular coordinates are assumed and the spatial frame coincides with the material frame.

3.5.1 Large rotations

Consider a vector $\boldsymbol{\theta} = \theta \boldsymbol{n}$ along an axis AB (Figure 3.9), and a vector $\boldsymbol{r}_0$. Now, the vector $\boldsymbol{r}_0$ is rotated about the axis AB until the new vector $\boldsymbol{r}$ includes an angle $\theta = \|\boldsymbol{\theta}\|$ with $\boldsymbol{r}_0$. We would like to find an expression for $\boldsymbol{r}$ as a function of $\boldsymbol{r}_0$ and $\boldsymbol{\theta}$.

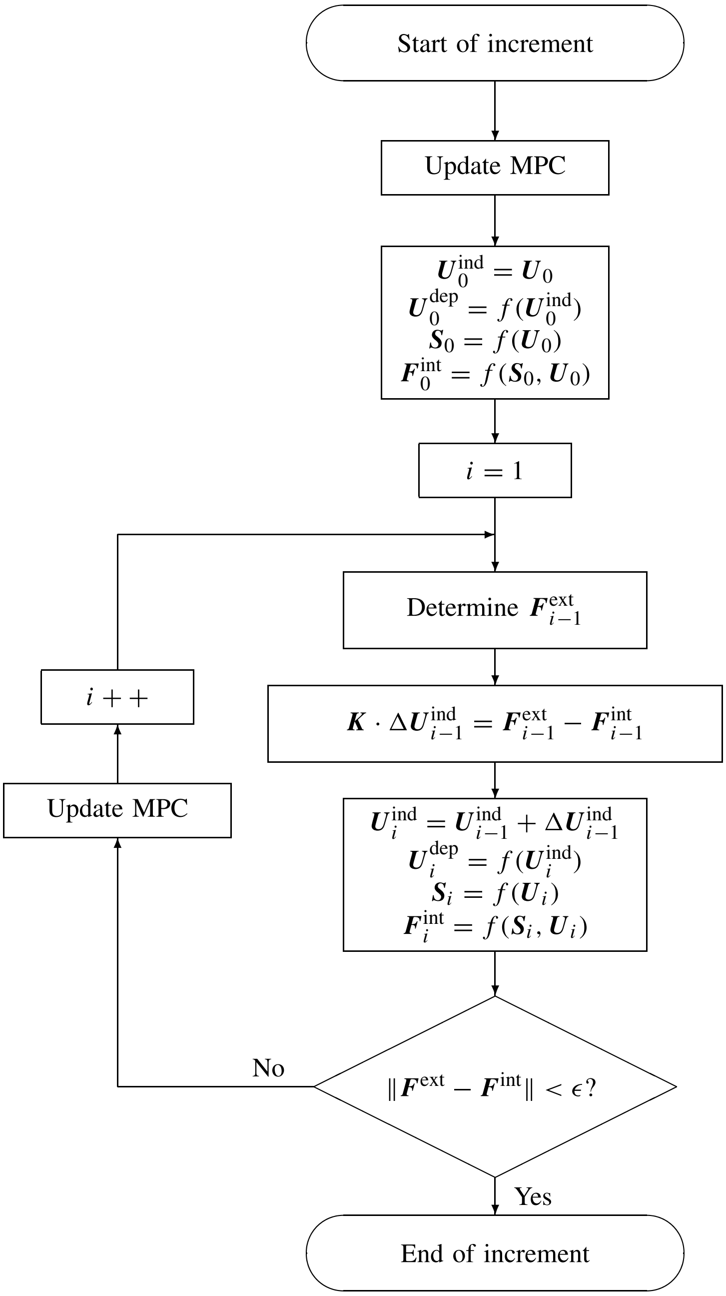

Figure 3.8 Stream chart of the nonlinear solution procedure
"dep" = dependent, "ind" = independent, "int" = internal, "ext" = external

For an infinitesimal angle $\mathrm{d}\theta$, the change $\mathrm{d}\boldsymbol{r}$ of $\boldsymbol{r}$ is perpendicular to $\boldsymbol{r}$ and satisfies

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\theta(\boldsymbol{n} \times \boldsymbol{r}) \tag{3.59}$$

in component notation:

$$\mathrm{d}r_i = \mathrm{d}\theta e_{ijk} n_j r_k. \tag{3.60}$$

Defining the matrix $\boldsymbol{S}$ by

$$S_{ik} := e_{ijk} n_j \tag{3.61}$$

one finds

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\theta \boldsymbol{S} \cdot \boldsymbol{r} \tag{3.62}$$

or

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\theta} = \boldsymbol{S} \cdot \boldsymbol{r}. \tag{3.63}$$

This is a linear homogeneous vector differential equation with the solution

$$\boldsymbol{r} = \mathrm{e}^{\boldsymbol{S}\theta} \cdot \boldsymbol{r}_0 \tag{3.64}$$

satisfying the initial condition $\boldsymbol{r}(0) = \boldsymbol{r}_0$. Equation (3.64) can be expanded into

$$\boldsymbol{r} = \left(\boldsymbol{I} + \theta \boldsymbol{S} + \frac{1}{2!}\theta^2 \boldsymbol{S}^2 + \frac{1}{3!}\theta^3 \boldsymbol{S}^3 + \cdots \right) \cdot \boldsymbol{r}_0. \tag{3.65}$$

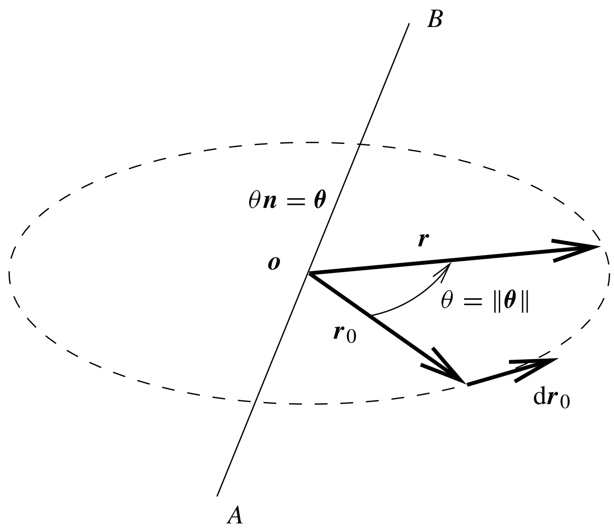

Figure 3.9 Large rotation about the axis AB

Since $\boldsymbol{S} \cdot \boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r}$ (Equations (3.59) and (3.62)) and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, one finds

$$\boldsymbol{S}^2 \cdot \boldsymbol{r} = \boldsymbol{S} \cdot (\boldsymbol{S} \cdot \boldsymbol{r}) = \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{r}) = (\boldsymbol{n} \cdot \boldsymbol{r})\boldsymbol{n} - \boldsymbol{r} \tag{3.66}$$

$$\boldsymbol{S}^3 \cdot \boldsymbol{r} = \boldsymbol{S} \cdot (\boldsymbol{S}^2 \cdot \boldsymbol{r}) = \boldsymbol{n} \times [(\boldsymbol{n} \cdot \boldsymbol{r})\boldsymbol{n} - \boldsymbol{r}] = -\boldsymbol{n} \times \boldsymbol{r} = -\boldsymbol{S} \cdot \boldsymbol{r} \tag{3.67}$$

from which one finds

$$\boldsymbol{S}^3 = -\boldsymbol{S}. \tag{3.68}$$

Accordingly, all powers of $\boldsymbol{S}$ exceeding 2 can be reduced to $\pm\boldsymbol{S}$ or $\pm\boldsymbol{S}^2$. Consequently,

$$\begin{aligned} \mathrm{e}^{\boldsymbol{S}\theta} &= \boldsymbol{I} + \boldsymbol{S}\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots\right) \\ &\quad + \boldsymbol{S}^2\left(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \frac{1}{6!}\theta^6 - \cdots\right) \\ &= \boldsymbol{I} + \sin\theta\boldsymbol{S} + (1-\cos\theta)\boldsymbol{S}^2. \end{aligned} \tag{3.69}$$

Hence,

$$\boldsymbol{r} = \left(\boldsymbol{I} + \sin\theta\boldsymbol{S} + (1-\cos\theta)\boldsymbol{S}^2\right) \cdot \boldsymbol{r}_0. \tag{3.70}$$

Since

$$\boldsymbol{S}^2 = \boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{I} \tag{3.71}$$

this also reduces to

$$\boldsymbol{r} = [\cos\theta\boldsymbol{I} + \sin\theta\boldsymbol{S} + (1-\cos\theta)\boldsymbol{n} \otimes \boldsymbol{n}] \cdot \boldsymbol{r}_0. \tag{3.72}$$

Defining

$$\hat{\boldsymbol{\theta}} = \theta\boldsymbol{S} \tag{3.73}$$

finally yields

$$\boldsymbol{r} = \boldsymbol{C} \cdot \boldsymbol{r}_0 \tag{3.74}$$

where

$$\boldsymbol{C} = \left[\cos\theta\boldsymbol{I} + \frac{\sin\theta}{\theta}\hat{\boldsymbol{\theta}} + (1-\cos\theta)\frac{\boldsymbol{\theta} \otimes \boldsymbol{\theta}}{\theta^2}\right] \tag{3.75}$$

or in component notation,

$$C_{ij} = \delta_{ij}\cos\theta + \frac{\sin\theta}{\theta}e_{ikj}\theta_k + \left(\frac{1-\cos\theta}{\theta^2}\right)\theta_i\theta_j. \tag{3.76}$$

Notice that this is a nonlinear relation in $\boldsymbol{\theta}$. Therefore, only a truly nonlinear calculation can take large rotations into account. In simple linear calculations, Equation (3.59) is sometimes used for finite rotations, yielding

$$\boldsymbol{r} = \boldsymbol{r}_0 + \theta(\boldsymbol{n} \times \boldsymbol{r}_0). \tag{3.77}$$

Using this relation amounts to the motion in Figure 3.10 and is only feasible for a small θ. The true angle α satisfies

$$\alpha = \arctan\theta \approx \theta - \frac{\theta^3}{3} + \cdots \tag{3.78}$$

and $\|\boldsymbol{r}\|$ satisfies

$$\|\boldsymbol{r}\| = r_0\sqrt{\theta^2 + 1} \approx r_0\left(1 + \frac{\theta^2}{2}\right). \tag{3.79}$$

3.5.2 Rigid body formulation

Defining a set of nodes to behave like a rigid body means that all degrees of freedom of the set are reduced to six degrees of freedom: three translations $\boldsymbol{w}$ of a point A and three rotations $\boldsymbol{\theta}$ about point A. Point A can be the center of gravity of the node set, but this does not have to be. Any point will do. Usually, we take an existing node belonging to the rigid node set to be point A. However, we can also generate an additional fictitious node to be point A. Hence, the motion $\boldsymbol{u}$ of a node at location $\boldsymbol{p}$ can be described as (Figure 3.11)

$$\boldsymbol{u} = \boldsymbol{w} + [\boldsymbol{C}(\boldsymbol{\theta}) - \boldsymbol{I}] \cdot (\boldsymbol{p} - \boldsymbol{q}) \tag{3.80}$$

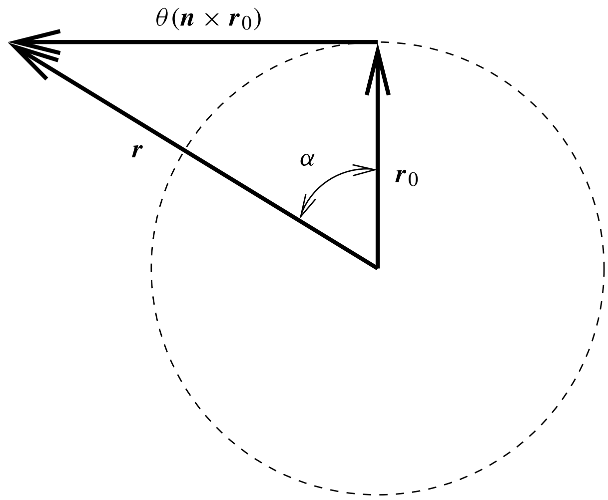

Figure 3.10 Linearized rotation