Calculation of non-linear modes with the operator MODE_NON_LINE

Summary:

The operator MODE_NON_LINE allows to calculate the non-linear modes of an autonomous conservative linear system equipped with localised non-linearities of shock. One describes here the digital methods used for calculation of the non-linear modes.

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1 Introduction

The non-linear modes are a mathematical tool which makes it possible to interpret non-linear dynamic phenomena. They are an extension of the clean modes [R5.01.01] used to interpret the linear dynamic systems.

The operator `MODE_NON_LINE` allows to calculate the non-linear modes of dynamic systems with localised non-linearities of shocks.

One will point out the definition of the non-linear modes, then one will describe non-linearities of shocks supported by the operator (contact unilateral, bilateral and annular) and finally one will describe the algorithm employed. This last is a combination of the method of balancing harmonic (HEY) and digital asymptotic method (MAN), which one calls EHMAN (cf [Bib1]).

This documentation is not an exhaustive presentation of the non-linear modes. It is satisfied to point out the principal assumptions under which one can search non-linear modes using MODE_NON_LINE, as well as the principles adopted for the representation of nonthe linearity and the basic algorithmic elements. For a more complete presentation, and the details of the implementation, one returns the reader to the reference [Bib1].

2 Definitions of the non-linear modes

The first definition was given by Rosenberg in the years 1960, but it is at the beginning of years 1990 that the most solid definition theoretically of the non-linear modes was given by using the dynamic system theory. Thus a non-linear mode is defined like “an invariant variety of dimension 2 of the space of the phases, tangent at a point of steady balance of the linearized system” (cf [Bib2]).

For the conservative systems, one can define a non-linear mode as a “family of periodic solutions”. This definition is much gravitational because giving access to powerful digital tools like the methods of continuation. This is why it is this definition which was here selected, consequently `MODE_NON_LINE` the calculation of family of periodic solutions allows.

3 Modeling

It is supposed that in the absence of contacts, the studied structure has a linear behavior and that the equations governing its dynamic balance were discretized by finished differences or finite elements. One obtains a discrete system of differential equations of the second order with n degrees of freedom.

In a general way, these equations take the following shape:

$$\mathbf{M}.\ddot{\mathbf{U}}(t)+\mathbf{K}.\mathbf{U}(t)+\mathbf{F}^{\mathbf{nl}}(\mathbf{U}(t))=\mathbf{0},$$

- $\mathbf{M}$ is the matrix of mass of the system,
- $\mathbf{K}$ is the elastic matrix of rigidity of the system,
- $\mathbf{F}^{\mathbf{nl}}$ is the vector of the non-linear forces of shocks,

One places oneself within the framework of located non-linearities, i.e. the forces apply to a relatively low number of degrees of freedom. One will note I_{nl} the whole of the indices of these degrees of freedom, and I_l its complementary, i.e. $I_n=\{1,\dots,n\}=I_{nl}\cup I_l$. Size of the vector I_{nl} is equal to n_{nl}.

To be able to use the algorithm EHMAN which combines the method of balancing harmonic (HEY) and digital asymptotic method (MAN), it is necessary to adopt a certain formalism. This one consists in rewriting the problem in the form of a system of order one, having non-linearities with most quadratic, would be a system of unknown factor S form

$$A(\dot{S})=C+L(S)+Q(S,S), \tag{1}$$

where C is a constant, L is a linear form and Q a quadratic form. One in general gathers the terms of the member of right-hand side under name $R(S)$, the problem to then solve mets in condensed form

$$A(\dot{S})=R(S) \tag{2}$$

For that, one introduces two vectors of additional unknown factors $\mathbf{F}^{\mathbf{nl}}(t)$ and $\mathbf{Z}(t)$, as well as additional equations to define these vectors. The vector $\mathbf{F}^{\mathbf{nl}}(t)$ described the force of contact and the vector $\mathbf{Z}(t)$ vector of the auxiliary variables is called. The vector $\mathbf{Z}(t)$ is of size n_z. This relatively abstract rewriting, detailed by the means of examples, conduit to solve a system of the form

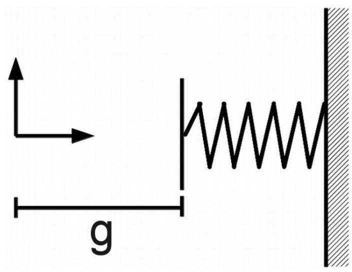
$$\begin{cases} \dot{\mathbf{U}}(t)=\mathbf{V}(t), \\ \mathbf{M}.\dot{\mathbf{V}}(t)=-\mathbf{K}\,\mathbf{U}(t)-\begin{pmatrix}\mathbf{0}\\ \mathbf{F}^{\mathbf{nl}}(\mathbf{U}(t))\end{pmatrix}, \\ \mathbf{0}=\mathbf{G}(\mathbf{U}^{\mathbf{nl}}(t),\mathbf{F}^{\mathbf{nl}}(t),\mathbf{Z}(t)), \end{cases} \tag{3}$$

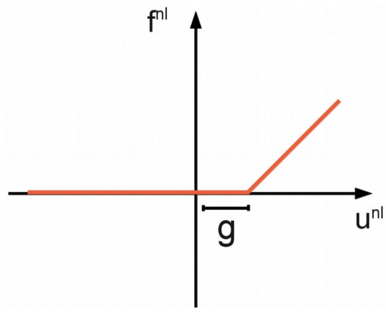
where $\mathbf{G}$ is a quadratic operator, who defines the implicit contact of way. $\mathbf{G}$ perhaps broken up in the following way

$$\mathbf{G}(\mathbf{S}(t))=\mathbf{G_c}+\mathbf{G_l}(\mathbf{S}(t))+\mathbf{G_q}(\mathbf{S}(t),\mathbf{S}(t)). \tag{4}$$

$\mathbf{G_c}$ is a constant application, $\mathbf{G_l}$ linear and $\mathbf{G_q}$ bilinear.

3.1 Unilateral contact





The non-linearity of unilateral contact, in its not-regular form, is written:

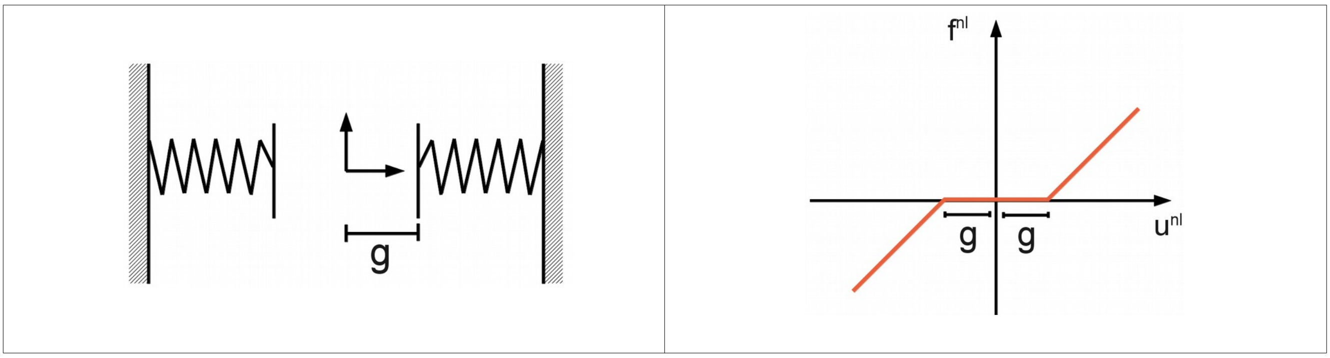
$$\mathbf{f}^{\mathbf{nl}}(t)=\begin{cases}\alpha(u^{nl}(t)-g)\ si\ u^{nl}(t)\geqslant g, \\ 0\ si\ u^{nl}(t)\leqslant g,\end{cases} \tag{5}$$

where α is the stiffness of contact, g the game enters the node of the structure and the node of contact. The regularization suggested is written in the implicit form

$$\mathbf{f}^{\mathbf{nl}}(t)(\mathbf{f}^{\mathbf{nl}}(t)-\alpha(u^{nl}(t)-g))-\alpha\eta=0, \tag{6}$$

where η represent the parameter of regularization.

3.2 Bilateral contact



The non-linearity of bilateral contact, in its not-regular form, is written

$$\mathbf{f}^{\mathbf{nl}}(t)=\begin{cases}\alpha(u^{nl}(t)-g)\ si\ u^{nl}(t)\geqslant g,\\ 0\ si\ |u^{nl}(t)|\leqslant g,\\ \alpha(u^{nl}(t)+g)\ si\ u^{nl}(t)\leqslant -g,\end{cases} \quad (7)$$

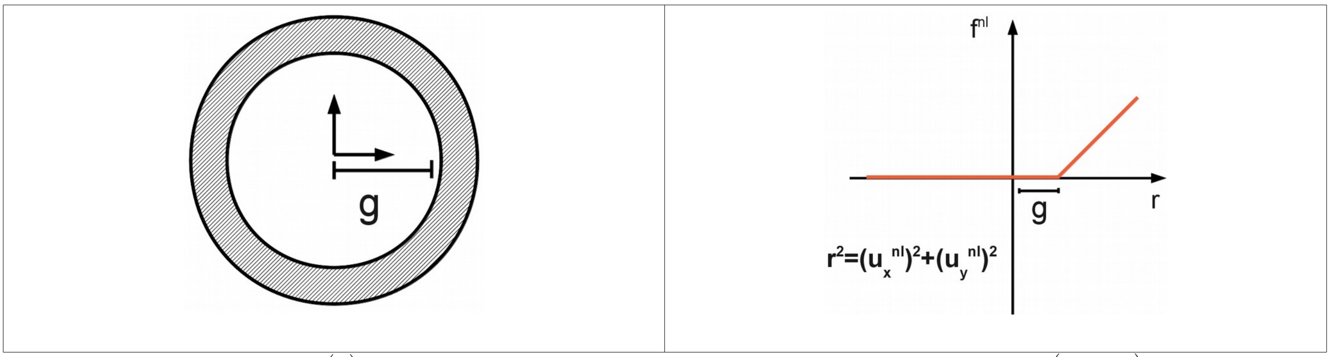
where α is the stiffness of contact, g the game enters the node of the structure and the node of contact. The regularization suggested is written in the following implicit form

$$f^{nl}(t)(f^{nl}(t)-\alpha(u^{nl}(t)-g))(f^{nl}(t)-\alpha(u^{nl}(t)+g))-\alpha^2\eta u^{nl}(t)=0, \quad (8)$$

where η represent the parameter of regularization. One can rewrite it in a quadratic form,

$$\begin{cases}\alpha^2 g^2 f^{nl}(t)-\eta\alpha^2 u^{nl}(t)-f^{nl}(t)z(t)=0,\\ z(t)-(f^{nl}(t)-\alpha u^{nl}(t))^2=0.\end{cases} \quad (9)$$

3.3 Annular contact



One introduces $z(t)$, the radial distance compared to the position of balance (e_x, e_y) node of shock, whose current position is (u_x^{nl}, u_y^{nl}) . $z(t)$ is written

$$z(t)=\sqrt{(u_x^{nl}(t)-e_x)^2+(u_y^{nl}(t)-e_y)^2}. \quad (10)$$

The annular non-linearity of contact, in its not-regular form, is written

$$\mathbf{f}^{\mathbf{nl}}(t)=\begin{cases}\alpha(z(t)-g)\ si\ z(t)>g,\\ 0\ si\ z(t)\leqslant g,\end{cases} \quad (11)$$

where g indicate the ray of the ring (in other words game) and α ($\alpha>0$) indicate the associated coefficient of stiffness. By introducing the components f_x^{nl} and f_y^{nl} non-linear force in the directions x and y , the regularization suggested is written in the following implicit form

$$\begin{cases} \mathbf{f^{nl}}(t)\big(\mathbf{f^{nl}}(t)-\alpha(z(t)-g)\big)-\alpha\eta=0, \\ z^2(t)-(u_x^{nl}(t)-e_x)^2-(u_y^{nl}(t)-e_y)^2=0, \\ f_x^{nl}(t)z(t)-\mathbf{f^{nl}}(t)u_x^{nl}(t)=0, \\ f_y^{nl}(t)z(t)-\mathbf{f^{nl}}(t)u_y^{nl}(t)=0, \end{cases} \quad (12)$$

where η represent the parameter of regularization.

4 Procedure of calculation of the non-linear modes

One seeks the periodic solutions of the system

$$\begin{cases} \mathbf{M}.\ddot{\mathbf{U}}(t)+\mathbf{K}\,\mathbf{U}(t)+\begin{pmatrix}\mathbf{0}\\ \mathbf{F^{nl}}(\mathbf{U}(t))\end{pmatrix}=\mathbf{0}, \\ \mathbf{G}(\mathbf{U^{nl}}(t),\mathbf{F^{nl}}(t),\mathbf{Z}(t))=\mathbf{0}, \end{cases} \quad (13)$$

by writing the vector $\mathbf{U}(t)$ using Fourier series whom one truncates with the order H_l (by supposing that the influence of the higher harmonics is negligible), that is to say

$$\mathbf{U}(t)=\mathbf{U_0}+\sum_{k=1}^{H_l}\mathbf{U_{Ck}}\cos(k\omega t)+\mathbf{U_{Sk}}\sin(k\omega t). \quad (14)$$

One carries out the same development for the vectors of unknown factors $\mathbf{F^{nl}}(t)$ and $\mathbf{Z}(t)$, but with a different order of truncation, which one calls H_{nl} . It comes then

$$\begin{cases} \mathbf{F^{nl}}(t)=\mathbf{F_0^{nl}}+\sum_{k=1}^{H_{nl}}\mathbf{F_{Ck}^{nl}}\cos(k\omega t)+\mathbf{F_{Sk}^{nl}}\sin(k\omega t), \\ \mathbf{Z}(t)=\mathbf{Z_0}+\sum_{k=1}^{H_{nl}}\mathbf{Z_{Ck}}\cos(k\omega t)+\mathbf{Z_{Sk}}\sin(k\omega t). \end{cases} \quad (15)$$

Nota bene: The choice of a different nature rises directly from the sizes represented. For a little shocking system, in the vicinity of the linear mode, the answer $\mathbf{U}(t)$ will be close to a pure sine. One will thus choose a number of weak harmonic. One will increase H_l when one seeks to obtain solutions associated with high energy levels. On the other hand, even for forces of weak shock, the spectral contents can be important, in particular if the shock is stiff. It will thus be necessary to retain a significant number of harmonic, and one will thus have in general $H_{nl}\gg H_l$.

In practice, one will be able to start to carry out a calculation by choosing orders relatively weak, to have an idea of the principal behavior. On the other hand, with a number of weak harmonic, one will not be able to collect the junctions correctly. One will thus increase gradually the values of H_l and H_{nl} to reveal more complex behaviors.

After having carried out a harmonic balancing, one obtains a under-given algebraic system whose unknown factors are the coefficients of Fourier of $\mathbf{U}(t)$, $\mathbf{F^{nl}}(t)$ and $\mathbf{Z}(t)$ and the additional unknown factor, the own pulsation ω . One gathers these unknown factors in a single vector in order to put the system in the form (2). One can then use the MAN as indicated in the reference [Bib3]. For that, one develops in whole series the vector of unknown S according to a parameter of way a

$$S(a)=S_0+\sum_{k=1}^{N_{MAN}} a^k S_k, \quad (16)$$

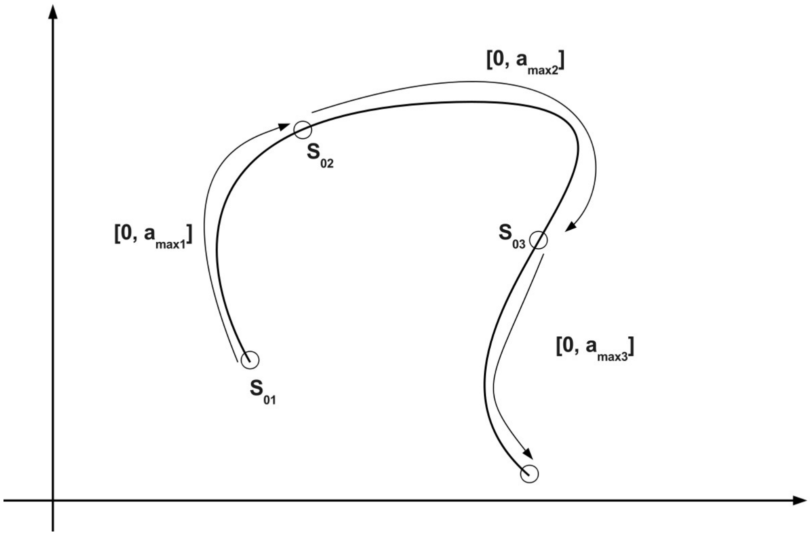
where S_0 corresponds to the vector of initialization of the algorithm, and S_k for $k=1,\ldots,N_{MAN}$ are the coefficients of the whole series which remain to be determined. Then, one develops in Taylor series the function R , in the vicinity of the vector of initialization S_0

$$0=R(S(a))=R(S_0)+\frac{dR}{dS}(S_0)\left(\sum_{k=1}^{N_{MAN}} a^k S_k\right)+\frac{d^2R}{dS^2}(S_0)\left(\sum_{k=1}^{N_{MAN}} a^k S_k\right)\left(\sum_{l=1}^{N_{MAN}} a^l S_l\right) \quad (17)$$

This relation is true some is a , one passes then to the resolution of a continuation of N_{MAN} linear systems having the same tangent matrix, depending the ones on the others in a recursive way. The resolution of these systems makes it possible to obtain the vectors S_k . It should be noted that one can calculate this matrix analytically because the function R is quadratic, that is to say

$$\frac{dR}{dS}(S_0)e_i=L(e_i)+Q(e_i,S_0)+Q(S_{0,}e_i), \quad (18)$$

where e_i is the vector of the canonical base. A branch of solutions thus is obtained $S(a)$ with $a\in[0,a_{max}]$, where a_{max} represent the field of validity of the whole series. Consequently, one can obtain the periodic branches of solutions which form the non-linear modes of the model.



The method also makes it possible to locate the simple junctions by the geometrical detection of series in the representation in whole series.

5 Bibliography

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