Discrete formulation of contact-friction

Summary :

One describes in this document the digital methods used to deal with the problems of contact/friction in great displacements in the operator STAT_NON_LINE or DYNA_NON_LINE using a discrete formulation. The algorithms available make it possible to deal with the problem in an exact way (in contact) or approximate (in contact and friction), without restrictive choice on the subjacent mechanical problem (kinematic or laws of non-linear behavior).

Contents

1 Introduction

Two distinct solids are known as in contact when they share a common surface which is part of their border. To treat the unilateral contact consists in preventing that one of the solids “does not cross” the other. Friction characterizes the tangential slip of a solid compared to the other. In *Code_Aster*, only the solid friction of Coulomb is available.

The contact in discrete formulation in *Code_Aster* has the following characteristics:
- The laws of contact and friction are established starting from already discretized quantities: displacements and nodal forces on the structures with a grid;
- The problem of contact/friction is solved by uncoupling it from the problem of the balance of the structure.

In this document one will present the various ingredients of the problem:
- Description of the laws of contact and friction;
- Geometrical pairing: it is the phase where one binds a node to another node or a node with a mesh to create a couple *potential* of contact, i.e. one locates the degrees of freedom for which one will write discrete relations of nonpenetration;
- Establishment of the problem of contact/friction;
- Description of the algorithms available to solve the problem of contact/friction.

This document is thus limited to the methods known as GCP, CONSTRAINT and PENALIZATION.

The formulation known as “continues” (FORMULATION= ' CONTINUE' in DEFI_CONTACT) fact the object of the document [R5.03.52].
The formulation associated with elements XFEM (FORMULATION= ' XFEM' in DEFI_CONTACT) is discussed in [R5.03.53].

2 Friction and contact mechanics

2.1 Definition of the problem and notations

One considers two solids being able to come into rubbing contact, the zone of contact is either specific, or linear, or surface. That is to say $\{n\}$ the outgoing normal on the surface of the one of the solids in contact and $\{u\}$ the vector displacement enters the two solids. Then $g=\langle u\rangle.\{n\}$ is project displacement on this normal, one will call it *game*. From the data of the constraint of Cauchy, one defines the pressure[1] $p=\langle n\rangle.[\sigma].\{n\}$ and the tangential shear stress $\{r\}=[\sigma].\{n\}-p.\{n\}$ as being exerted by one of surfaces on the other.

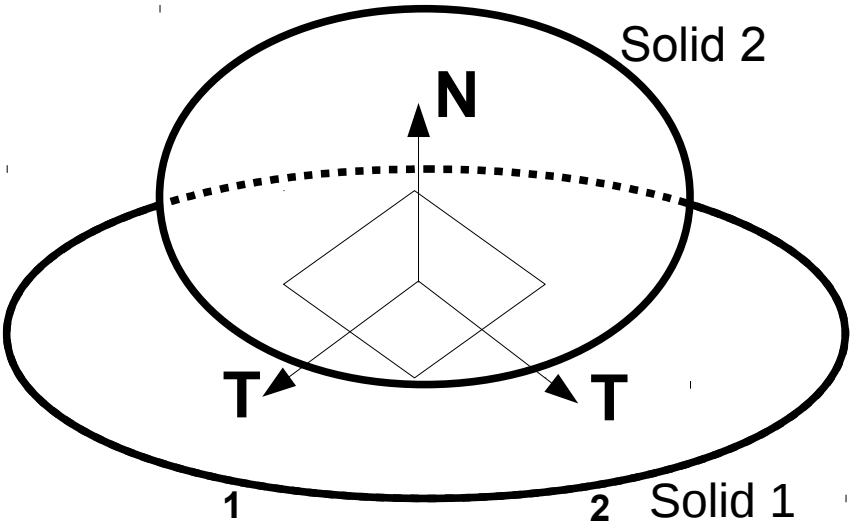

Figure 2.1-a : definition of the local reference mark of contact.

The force of shearing has as a direction in the zone of contact a vector $\{t\}$ located in the tangent plan $(\{t_1\},\{t_2\})$ indicated on the figure (2.1-a). The equation (1) defines the shear stress r exerted by the solid (2) on the solid (1) per unit of area of contact.

$$\{r\}=[\sigma].\{n\}-(\langle n\rangle.[\sigma].\{n\}).\{n\}=r.\{t\} \text{ with } r=\|\{r\}\| \quad (1)$$

2.2 Conditions of contact of Signorini

One introduced the two variables defining the contact:

- The distance signed between two surfaces or "gap" g ;
- The contact pressure p ;

The three conditions of contact of Hertz-Signorini-Moreau are then defined

- Geometrical condition: impenetrability of the matter (Signorini-Hertz).

$$g\geq 0\Leftrightarrow\begin{cases} g>0 & \text{pas de contact} \\ g=0 & \text{contact}\end{cases} \quad (2)$$

- Mechanical condition: intensility (Signorini-Hertz)

$$p\geq 0\Leftrightarrow\begin{cases} p>0 & \text{contact} \\ p=0 & \text{pas de contact}\end{cases} \quad (3)$$

- Energy condition: complementarity/exclusion (Moreau)

1 The term "pressure" is an abusive abbreviation of "density the effort of contact"

$$p\,.\,g=0 \Leftrightarrow \begin{cases} p=0 \text{ décollement} \\ g=0 \text{ contact} \end{cases} \quad (4)$$

Graphically, the three conditions are represented on the figure (2.2-a) where the shaded zones represent the excluded zones and the black white areas or features represent the authorized zones.

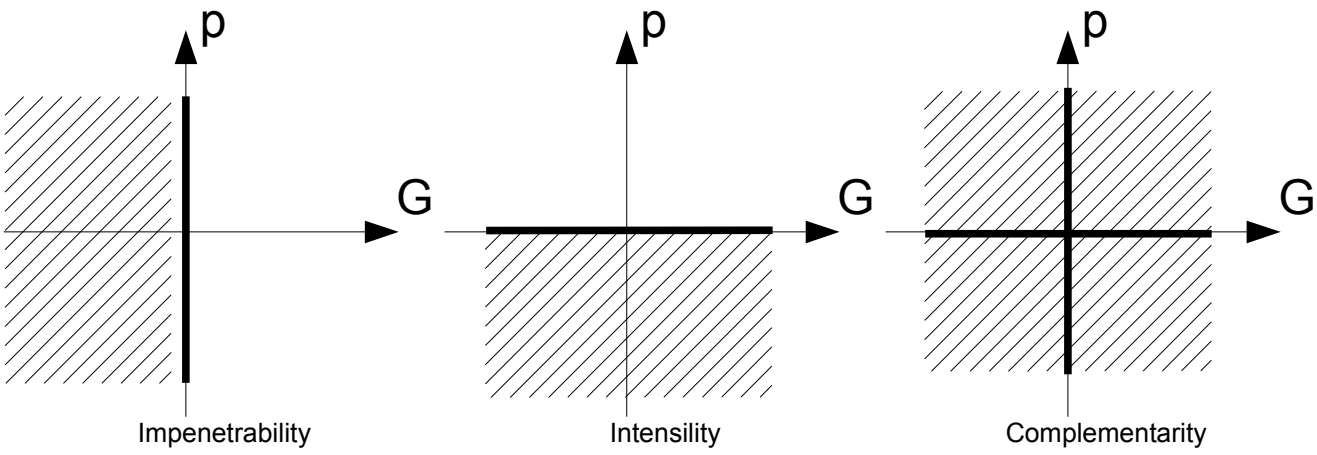

Figure 2.2-a: chart of the conditions of Hertz-Signorini-Moreau.

By combining the three conditions one obtains the graph of the figure (2.2-b).

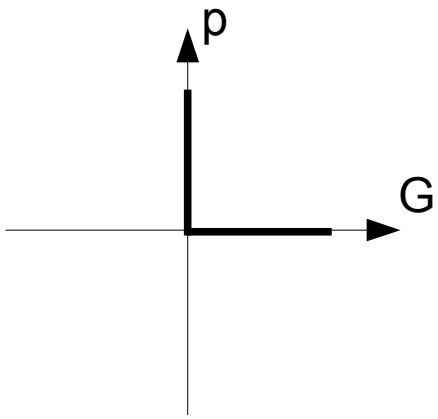

Figure 2.2-b : graph of the condition of unilateral contact.

The problem of contact thus posed introduced a relation *not-univocal* (p is not one *function* of g), *semi-definite positive* and *not-differentiable* in $p=g=0$. It is about a problem mathematically difficult to treat. The contact is a phenomenon reversible and conservative for which one can introduce a natural energy and whose result does not depend on the way of loading, it is similar to the elastoplastic model of Hencky. The law of contact of Signorini is written:

$$\begin{cases} g\geq 0 & (a) \\ p\geq 0 & (b) \\ p\,.\,g=0 & (c) \end{cases} \quad (5)$$

Note:

- The contact is supposed without adhesion thanks to the second condition (intensility) .
- The third condition makes it possible the problem of unilateral contact to be well posed to be soluble by classical techniques of optimization under constraints (introduction of the conditions of Kuhn & Tucker) like method of the active constraints.

2.3 Formulation of the problem of friction

2.3.1 Definitions

The selected criteria of friction are form:

$$h(\{r\})\leq 0 \quad (6)$$

where $h(\{r\})$ is a convex function. The field of nonslip is defined by the interior of the convex one. Two criteria of friction of the form $h(\{r\})\leq 0$ are particularly used: the criterion of Tresca and the criterion of Coulomb.

2.3.2 The criterion of Tresca

The criterion of Tresca is defined by the function $h(\{r\})$ following:

$$h(\{r\})=\|\{r\}\|-k\leq 0 \text{ with } k \text{ a constant data} \quad (7)$$

One notes C the convex disc of ray k centered at the origin defined by:

$$C=\{\{r\} \text{ tel que } \|\{r\}\|\leq k\} \quad (8)$$

The condition of nonslip is then defined by the membership of $\{r\}$ inside the disc C. In the event of slip, for $\{r\}$ located on the border of C, direction of slip $\{t\}$ of $\{\dot{u}\}$ is given by the normal to the criterion in $\{r\}$, as indicated on Figure 2.3.2-a.

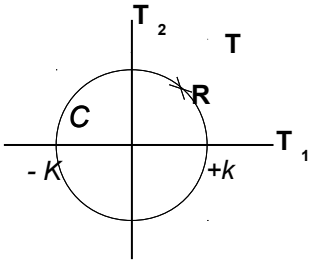

Figure 2.3.2-a : disc of friction for the criterion of Tresca.

What results in an expression the speed of slip similar to the plastic multiplier:

$$\{\dot{u}_t\}=\lambda.\{r\} \quad (9)$$

With $\lambda\geq 0$, i.e. the speed of slip is in the same direction as the tangential constraint.

2.3.3 The criterion of Coulomb

The criterion of Coulomb is defined by the function $h(\{r\})$ following:

$$h(\{r\},\mu,p)=\|\{r\}\|-k(\mu,p)\leq 0 \text{ and } k=\mu.|p| \quad (10)$$

The value of k depends on the contact pressure p and of the coefficient of friction of Coulomb μ Thus defined, h is a cone. In the event of slip, for $\{r\}$ located on the border of h, direction of slip $\{t\}$ of $\{\dot{u}\}$ is not given by the normal to the criterion in $\{r\}$, but by the normal with the convex disc C of ray $k=\mu.|p|$. The criterion of Tresca corresponds to a “slice” according to the orthogonal plan with the cone of Coulomb.

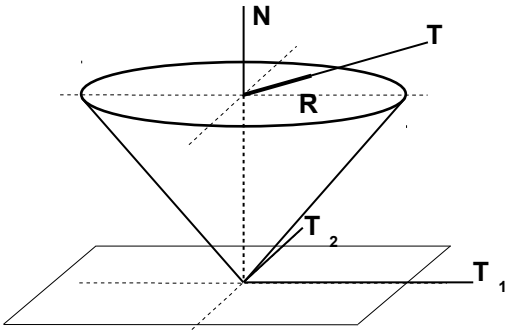

Figure 2.3.3-a : cone of friction for the criterion of Coulomb.

2.3.4 Application to the friction of Coulomb

Let us write the system of equations and inequations having to be checked by these sizes in the case of the criterion of friction of Coulomb:

$$\begin{cases} g \geq 0 & (a) \\ p \geq 0 & (b) \\ p \,.\, g = 0 & (c) \\ \| \{r\} \| - \mu \,.\, |p| \leq 0 & (d) \\ \{\dot{u}_t\} = \lambda \,.\, \{r\} & (e) \\ \lambda \,.\, (\| \{r\} \| - \mu \,.\, |p|) = 0 & (f) \\ \lambda \geq 0 & (g) \end{cases} \tag{11}$$

First set of equations and inequations (11a-c) corresponds to the management of the contact. The second batch (11d-g) corresponds to the description of friction obeying the criterion of Coulomb. It utilizes several fields and binds between them: normal pressure p, the shear stress $\{r\}$ and tangent speed $\{\dot{u}_t\}$. It can be understood as follows:

$$\begin{cases} \text{Si } \| \{r\} \| < \mu \,.\, |p| \text{ alors } \lambda = 0 \text{ et } \{\dot{u}_t\} = 0 & (a) \\ \text{Si } \| \{r\} \| = \mu \,.\, |p| \text{ alors } \lambda > 0 \text{ et } \{\dot{u}_t\} = \lambda \,.\, \{r\} & (b) \end{cases} \tag{12}$$

On the figure (2.3.4-a) the cone of Coulomb was represented. Within the space of constraints, the effort of rubbing contact can be only inside the cone of Coulomb: if it is strictly inside, the contact is adherent; if it is on the surface of the cone, the contact is slipping.

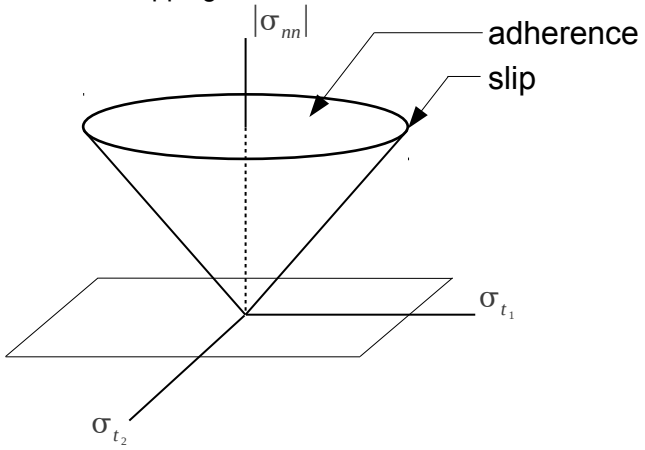

Figure 2.3.4-a : interpretation of the cone of friction.

One can give another representation of this criterion (see figure (2.3.4-b)).

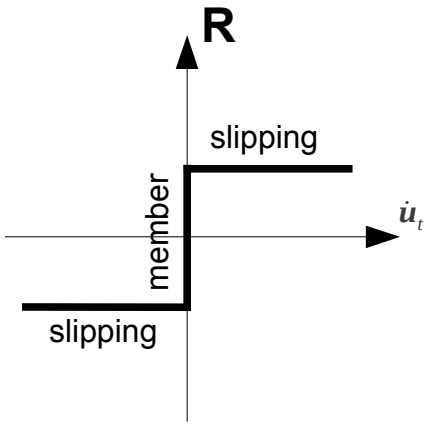

Figure 2.3.4-b : graph of the friction of Coulomb.

Friction induces the concept of threshold. The relation introduced by the friction of Coulomb is a not-differentiable not-univocal relation. Contrary to the contact, the relation is not associative[2] and cannot derive directly from a natural energy. It will be also noticed that friction binds *speed* relative of two surfaces and not displacement. However one can establish that in statics, if one solves the problem in incremental form (what is the case when one uses Newton as in *Code_Aster*), one can replace relative speed by the increment of tangential displacement.

2.4 Formulation by differential inclusions

The not-differentiable character of the relations of friction leads us to introduce the concept of differential under-inclusion.
One notes V the whole of displacements kinematically acceptable of the problem. The relation between the speed of relative slip $\{\dot{\boldsymbol{u}}\}$ and the shear stress $\boldsymbol{r}$ translated the two possible states of the system: not slip or relative slip following the normal direction in $\{\boldsymbol{r}\}$ with the convex disc C. For the three criteria presented, the function $\{\dot{\boldsymbol{u}}_t\}(\boldsymbol{r})$ and its reciprocal $\{\boldsymbol{r}(\dot{\boldsymbol{u}}_t)\}$ both to under - differential of two combined pseudopotentials belong $\Psi_c(\boldsymbol{r})$ and $\Psi_c^*(\dot{\boldsymbol{u}}_t)$, so that one can write:

$$\{\dot{\boldsymbol{u}}\}\in\partial\Psi_c(\boldsymbol{r}) \text{ and } \{\boldsymbol{r}\}\in\partial\Psi_c^*(\dot{\boldsymbol{u}}) \tag{13}$$

The appearance of differential inclusions comes from the not-differentiable character of the laws of contact - friction. Indeed, Ψ_c indicate the indicating function of the convex disc C of ray k, centered at the origin, previously definite. It is such as:

$$\Psi_c(\boldsymbol{r})=\begin{cases}0 \text{ si } \{\boldsymbol{r}\}\in C\\ +\infty \text{ sinon}\end{cases} \tag{14}$$

The under-differential $\partial\Psi_c(\boldsymbol{r})$ function Ψ_c in $\boldsymbol{r}$ merges with the normal external with C in $\{\boldsymbol{r}\}$. $\Psi_c^*(\dot{\boldsymbol{u}})=k\|\dot{\boldsymbol{u}}\|$, where k is the threshold of friction resistance, Ψ_c^* is combined of Fenchel of the indicating function Ψ_c. Ψ_c^* of degree one is positively homogeneous. This function is interpreted like the density of power dissipated in the slip. Using the concepts of under-differential, one can establish the following relations for $\{\dot{\boldsymbol{u}}\}$ and $\{\boldsymbol{r}\}$ associated:

2 On the other hand the friction of Tresca is well an associative law

$$\begin{cases} \dot{\boldsymbol{u}} \in \partial \Psi_c(\boldsymbol{r}) \Leftrightarrow \begin{cases} \Psi_c(\boldsymbol{r}) = 0 \\ \dot{\boldsymbol{u}} . (\boldsymbol{r}' - \boldsymbol{r}) \leq 0, \forall \boldsymbol{r}' \in C \end{cases} \\ \boldsymbol{r} \in \partial \Psi_c^*(\dot{\boldsymbol{u}}) \Leftrightarrow \begin{cases} \dot{\boldsymbol{u}} \in V \\ \boldsymbol{r} . (\dot{\boldsymbol{v}} - \dot{\boldsymbol{u}}) \leq \Psi_c^*(\dot{\boldsymbol{v}}) - \Psi_c^*(\dot{\boldsymbol{u}}), \forall \dot{\boldsymbol{v}} \in V \end{cases} \\ \Psi_c^*(\dot{\boldsymbol{u}}) + \Psi_c(\boldsymbol{r}) = \dot{\boldsymbol{u}} . \boldsymbol{r} = k \|\dot{\boldsymbol{u}}\| \end{cases} \tag{15}$$

Note:

- two combined pseudopotentials presented are not - differentiable.
- Once known the normal reaction for the criterion of Coulomb, one is brought back locally to a criterion of friction of Tresca whose threshold is worth $k = \mu . |p|$.
- Adopted local criteria having a circular form one from of deduced that $\dot{\boldsymbol{u}} \in \partial \Psi_c(\boldsymbol{r})$ imply that there exists λ positive reality such as $\dot{\boldsymbol{u}} = \lambda \boldsymbol{r}$.
- The formulation of the problem of speed suggests an incremental digital resolution of the problem of friction. The resolution of the problem of balance will thus be presented in incremental form.

2.5 Resolution of the problem of balance

Two solids of total volume are considered Ω whose surface of contact is Γ_c . To simplify, one will suppose the existence of a differentiable deformation energy to characterize the response of the two solids separated to external requests (in fact, one can show that the results given hereafter are independent of this assumption). One notes V the whole of the fields of displacement kinematically acceptable, constrained by the respect of the conditions of contact and friction on the interface. The balance of the two solids in the absence of friction is written:

To find $\boldsymbol{U}$ *field of displacement kinematically acceptable such as*

$$\left\{ \boldsymbol{U} = \underset{\boldsymbol{v} \in V}{\operatorname{argmin}} \left[\Phi(\varepsilon(\boldsymbol{v})) - W(\boldsymbol{v}) \right] \right\} \Leftrightarrow \left\{ \Phi(\boldsymbol{U}) - W(\boldsymbol{U}) \leq \Phi(\boldsymbol{v}) - W(\boldsymbol{v}), \forall \boldsymbol{v} \in V \right\} \tag{16}$$

In elasticity, $\Phi(\boldsymbol{v}) = \int_\Omega \varphi(\varepsilon(\boldsymbol{v})) . d\Omega$ is the deformation energy. The function $W(\boldsymbol{v})$ represent the work of the external forces. A requirement (which becomes sufficient if Φ is strictly convex) so that this balance is checked is that:

$$D\Phi(\boldsymbol{U}) - DW(\boldsymbol{U}) = D\Phi(\boldsymbol{U}) - \boldsymbol{L}^{\text{ext}} = 0 \tag{17}$$

where D is the operator derived from Cakes and $\boldsymbol{L}^{\text{ext}}$ is the linear form associated with the external forces.
With the introduction of friction, the problem must be approached in incremental form. One is led (see [2] and [3]) with the problem of minimization following on the unit $\bar{V}$ fields kinematically acceptable constrained by the respect of the conditions of contact and friction on the interface:

$\boldsymbol{U}$ *known, to find* $\Delta \boldsymbol{U} \in \bar{\boldsymbol{V}}$ *such as*

$$\boldsymbol{U} + \Delta \boldsymbol{U} = \underset{\Delta \boldsymbol{v} \in \bar{\boldsymbol{V}}}{\operatorname{argmin}} \left[\Phi(\varepsilon(\boldsymbol{U} + \Delta \boldsymbol{v})) + \Psi_c^*(\Delta \boldsymbol{v}_t) - W(\boldsymbol{U} + \Delta \boldsymbol{v}) \right] \tag{18}$$

$\boldsymbol{U} + \Delta \boldsymbol{U}$ is thus solution of:

$$\min_{\Delta \boldsymbol{v} \in \bar{V}} \left[\int_\Omega \varphi(\varepsilon(\boldsymbol{U} + \Delta \boldsymbol{v})) d\Omega + \int_{\Gamma_c} k |\Delta \boldsymbol{v}_t| d\Gamma_c - W(\boldsymbol{U} + \Delta \boldsymbol{v}) \right] \tag{19}$$

where $\Delta \boldsymbol{v}_t$ is the tangential component of the increment of relative displacement[3] solid 2 compared to the solid 1 along the surface of contact, with the conventions adopted with the §2.3.
By using the relations $\Psi_c^*(\Delta \boldsymbol{v}_t) = k |\Delta \boldsymbol{v}_t|$ and $\Psi_c^*(\Delta \boldsymbol{v}_t) \geq \boldsymbol{r} \Delta \boldsymbol{v}$ if $\boldsymbol{r} \in C$ one from of deduced that $\boldsymbol{U} + \Delta \boldsymbol{U}$ is solution of the problem of minmax according to, on space V fields kinematically acceptable:

3 $\Delta \boldsymbol{v}_t$ is the tangential component of the increment of relative displacement in quasi-static formulation, it becomes the relative speed of two surfaces in dynamic formulation.

$$\underset{\Delta v \in V}{\text{Min}}\ \underset{r}{\text{Max}}\ J(U+\Delta v, r) \tag{20}$$

where the functional calculus J is worth:

$$J(U+\Delta v, r)=\int_{\Omega} \phi(\varepsilon(U+\Delta v)).d\Omega+\int_{\Gamma_c}(r.\Delta v_t-\Psi_c(r)).d\Gamma_c-W(U+\Delta v) \tag{21}$$

The presence of the indicating function in this expression indicates that shearing r on the surface of contact Γ_c belongs to the convex disc of friction C.

2.6 Variational formulation

If φ is convex, the problem of minmax to solve in an equivalent way in the form is put:

To find $\Delta U \in V$ and $r \in C$, together of independent variables such as

$$\delta J(U+\Delta U, r) \ni 0 \tag{22}$$

This amounts solving the system of equations to following balance:

$$\begin{cases} \int_{\Omega} \frac{\partial \varphi}{\partial \varepsilon}(\varepsilon(U+\Delta U)).\delta\varepsilon.d\Omega+\int_{\Gamma_c} r.\delta v_t.d\Gamma_c-L^{ext}.\delta v=0 & (a) \\ \int_{\Gamma_c}(\delta r.\Delta U_t.d\Gamma_c-\partial\Psi_c(r)) \ni 0 & (b) \end{cases} \tag{23}$$

or in an equivalent way:

$$\begin{cases} \int_{\Omega} \frac{\partial \varphi}{\partial \varepsilon}(\varepsilon(U+\Delta U)).\delta\varepsilon.d\Omega+\int_{\Gamma_c} r.\delta v_t.d\Gamma_c-L^{ext}.\delta v=0 & (a) \\ \Delta U_t \in \partial\Psi_c(r) & (b) \\ r=(\sigma_1.n).t \text{ sur } \Gamma_c & (c) \end{cases} \tag{24}$$

As in the preceding section, L^{ext} is the linear form associated with the external forces. The linear form L^{frot} is associated with the forces of shearing exerted by the solid 2 on the surface of contact of the solid 1. It will be also noted that the variational formulation makes it possible to find not only the equilibrium equations of the system but ausif itmembership of ΔU_t with the under-differential of Ψ_c.

2.7 Stages of the resolution of the problem of contact-friction

Method of discrete resolution established in *Code_Aster* is founded on a writing of the relations of interpenetration on the nodes of the grids in opposite, which implies:

1) A discrete description of surfaces of contact (grid);
2) The research of the minimal distance from projection and the position of this projection (operation known as of pairing);
3) The writing of the relations kinematics between the nodes;
4) Algorithms of resolution of the problem of contact/friction;

We will detail each one of these aspects in the continuation of the document.

For the formulations known as “discrete” (in opposition to the formulation known as “continuous”, to see [R5.03.52]), *Code_Aster* solves the problem of contact by methods which belong to the family of the methods called “method of the statutes” in the literature, with a decoupling of the total-room type.

The algorithms of contact/friction act as two times:

1) Geometrical problem of pairing
   - Location: definition of potential surfaces of contact (cf § 3.2) by the user [4] .

4 I L there not of mechanism automatic of location of surfaces in contact in *Code_Aster*

- Pairing: determination of the potential couples of contact (cf § 3.4) by a research method of the minimum clearance enters a node and a facet (for pairing node/facet) or of the node nearest to another node (for nodal pairing).
- Kinematics: writing of the relation of nonpenetration by the determination of the direction of projection and the evaluation of the coefficients (cf § 3.5). The relation is written between the main node slave and nodes.

2) Mechanical problem. Several types of algorithms are established in *Code_Aster* :
   - An algorithm based on the method of the active constraints [1] usable in contact without friction only. It is that which is used by default and which corresponds to ALGO_CONT = ' CONSTRAINT '.
   - An alternative of the method of the active constraints using an iterative resolution by gradient combined project. ALGO_CONT = ' GCP' . Method reserved exclusively for the problems of contact without friction.
   - An algorithm of resolution by regularization of the conditions of contact and/or friction, which one activates with ALGO_CONT= ' PENALISATION' and ALGO_FROT= ' PENALISATION' by choosing a coefficient of judicious penalization (what implies *a fortiori*, a parametric study on the value of this coefficient).

3)

3 Geometrical problem of pairing

3.1 Introduction

The first problem to treat the contact in the case of is to define in a way reliable and robust which parts of two surfaces are likely to make contact. It is the operation known as of pairing, commune with the mixed hybrid formulation known as “continuous” method presented in the document [R5.03.52].
There are two types of pairing available in *Code_Aster* :

- Nodal pairing;
- Master-slave pairing (or node-facet).

Nodal pairing (APPARIEMENT=' NODAL') impose that relative displacement between a node slave and the main node which is paired to him, project on the direction of the normal to the node slave, is lower than the initial game in this direction. The use of this formulation is disadvised because it requires to have compatible grids (nodes “opposite”) which remains it during the deformation (assumption of small slips), and for which the normals Master and slave are about colinéaires. Without these assumptions, the made approximation becomes hazardous and it is preferable to use the node-facet formulation.
The master-slave pairing, chosen by the keyword APPARIEMENT=' MAIT_ESCL', a role equivalent to two surfaces does not grant: surface (S_1) described under GROUP_MA_MAIT or MAILLE_MAIT is called surface Master and surface (S_2) is surface slave. The conditions of noninterpenetration express that nodes of surface slave (of stars on the figure (3.1-a)) do not penetrate in the meshs of surface Master. One can see, on the other hand, that it is possible that the main nodes (rounds) penetrate in surface slave.

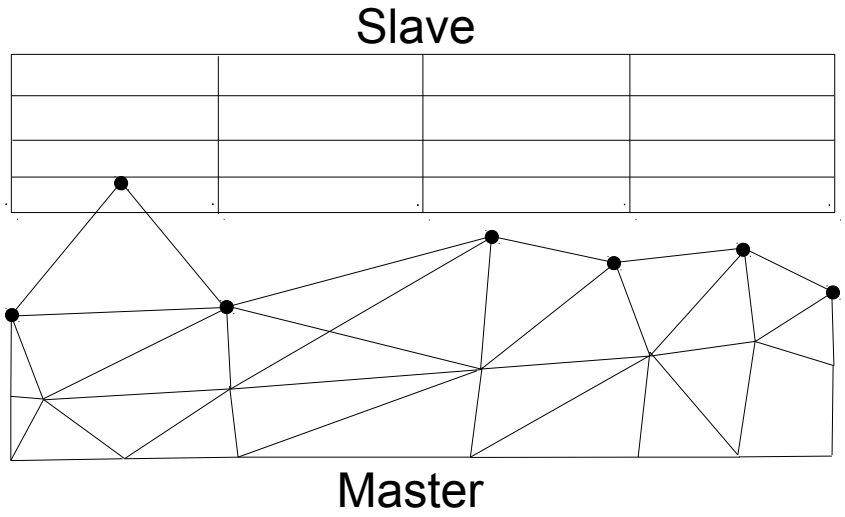

Figure 3.1-a : surface Master and surface slave.

Note:

- The nodes slaves are by default all the nodes belonging to the meshs of contact defining surface slave. Keywords SANS_NO and SANS_GROUP_NO allow to give, zone by zone, a list of the nodes which must be removed list of the nodes slaves. That makes it possible to remove the nodes subjected to boundary conditions of Dirichlet incompatible with the contact (see § 7.2).

3.2 Definition of the potential zones of contact

Code_Aster not having mechanism of automatic control of the potential zones of contact, it is thus to the user to define *a priori* the zones of which he predicts that they will make contact. These zones must be sufficiently broad not to observe interpenetration. It should be noted that the phase of pairing is an inexpensive operation in general (much less expensive than the resolution of the mechanical problem of contact) and than one can thus define sufficiently wide zones without being likely to penalize them performances, subject respecting some care to ensure, in particular, the unicity of projections.

One considers the three solids of the figure (3.2-a), represented in 2D. One defined three possible zones of interpenetration between the solids: a zone enters the solid A and the solid B, and two zones between the solid B and the solid C. The user, who defines these zones in the command file, supposes here that apart from these zones, there is no risk of interpenetration, taking into account the loading.

Figure 3.2-a : definition of three zones of contact.

Each zone of contact is defined in the operator DEFI_CONTACT, keyword factor ZONE. A zone is composed by definition of two surfaces which one seeks to prevent the interpenetration: first is defined under the keyword GROUP_MA_MAIT (or MAILLE_MAIT), the second under the keyword GROUP_MA_ESCL (or MAILLE_ESCL), i.e. by the data of the meshs **of edge** who constitute them. These meshs are SEG2 or of SEG3 for a grid 2D, TRIA3, TRIA6, QUAD4, QUAD8 or QUAD9 for a grid 3D.

Note:

- The meshs of edge necessary to the contact will not be created by *Code_Aster* starting from the voluminal elements and must thus already exist in the file of grid.
- The choice of surfaces which will be Masters or slaves is important. Information on this subject is available in documentation [U2.04.04].

3.2.1 Typical case of the contact for a cable or a beam in 3D

It is possible in 3D to treat the contact between a mesh SEG2 or SEG3 (modelling a cable or a beam) and a surface. In this case, it is imperatively necessary to use the method of pairing 'MAIT_ESCL' and to give the segments under the keyword GROUP_MA_ESCL (meshs slaves). The section of the beam can be then taken into account by the use of the keyword DIST_ESCL (cf §3.6).

3.2.2 Case of nodal pairing

One must choose to take as surface slave that which comprises less nodes (an error message will stop you if your surface Master contains less nodes than your surface slave), in order to maximize the chances to have an injective pairing (a main node is paired only with one node slave). The main node paired with each node slave is determined by a calculation moreover nearer close explained in the §3.4.2. One uses the normal with the main node to write the relation of noninterpenetration.

Even in the case of nodal pairing, surfaces of contact are defined in terms of meshs. The nodes slaves and Masters are then the nodes of the meshs thus defined .

3.3 Orientation of the normals

It is imperative that the meshs of contact are defined so that the normal is outgoing: the connectivity of the segments must be defined in the order AB , that of the triangles in the order ABC , and that of the quadrangles in the order $ABCD$, as indicated on the figure 3.3-a. For a better reading of the drawing, one a little drew aside here the mesh of edge serving in contact with the “face” of the voluminal element 2D or 3D on which it is based. One will be able to use for that the operator MODI_MAILLAGE, options ORIE_PEAU_2D, ORIE_PEAU_3D or ORIE_COQUE.

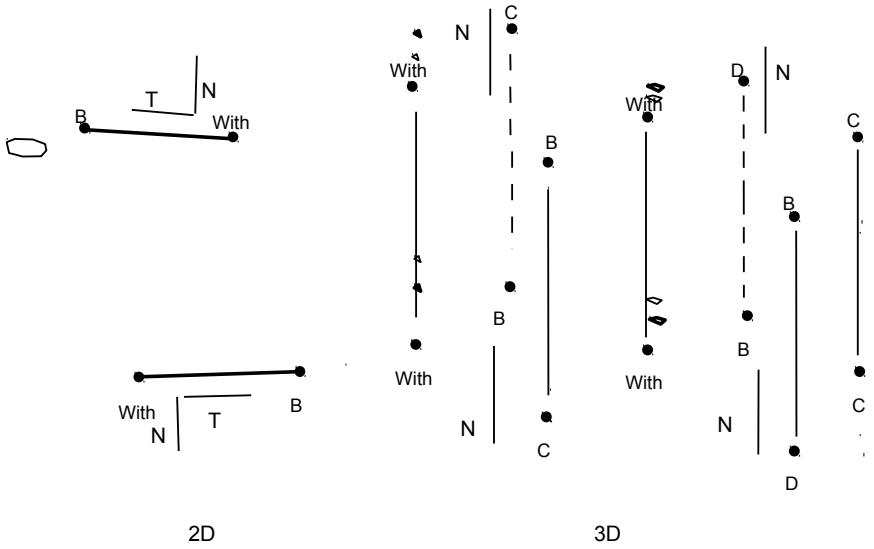

Figure 3.3-a : classification of the meshs of contact to have an outgoing normal.

3.4 Algorithm of pairing

3.4.1 Principle

The algorithm of pairing proceeds in two times:

1. For each node slave one searches the main node nearest;
2. One seeks then the mesh Master attached to the main node previously given which is closest to the node slave;

In the case of nodal pairing, one makes obviously only the first stage.

3.4.2 Research of nearest close to a node

This phase is common to both formulations: node/facet and node/node. The method used to search the main node nearest to a node slave is very simple: it is enough to calculate the distance (in brought up to date geometry, cf §3.7) between the main node slave and nodes candidates.

3.4.3 Research of the mesh Master paired

The algorithm to search the mesh Master which is paired with the node slave is the following:

- Knowing the main node nearest to the node slave P (see §3.4.2), one examines successively the meshs Masters containing this node.
- For each mesh thus located, one seeks the orthogonal project M node slave P on the mesh Master.
- The mesh minimizing the distance PM is selected to be paired with the node slave.

Note:

- If the node slave is projected apart from a mesh Master, one folds back his projection on this mesh under certain conditions (see § 3.4.5).
- So at least a projection takes place inside a mesh Master, then this one will be preferred with a projection leading to a folding back (whatever the measured distance)
- If all projections of the node slave take place apart from their respective mesh Master, the node slave is regarded as not paired and will thus be excluded from the phase of resolution

3.4.4 Calculation of the projection of the node slave on the mesh Master

3.4.4.1 Notations

Parts of surfaces $\partial\Omega^i$ likely to make contact at the time of the deformation of the two solids are noted γ_c^i. One supposes the existence of noted regular cards Θ^i describing surfaces γ_c^i. These cards are defined as follows:

$$\begin{aligned}\Theta^i:\quad & \omega^i \to \overline{\Omega^i} \\ & \left(\xi_1^i, \xi_2^i\right) \to \boldsymbol{p}^i = \Theta^i\left(\xi_1^i, \xi_2^i\right)\end{aligned} \tag{25}$$

where ω^i is a field limited (of reference) contents in $\mathbb{R}^2$ (it is the parametric space of reference of the finite element). In addition, one indicates by φ^i the transformation of the solid B^i, defined by:

$$\begin{aligned}\varphi^i:\quad & \overline{\Omega^i} \to \overline{\Omega_t^i} \\ & \left\{\boldsymbol{p}^i\right\} \to \left\{\boldsymbol{x}^i\right\}\end{aligned} \tag{26}$$

One always places oneself at a time t fixed. Surface Master is noted γ_c^1 and slave surfaces it is noted γ_c^2. One carries out pairing while searching, for any point $\left\{\boldsymbol{x}^i\right\} = \varphi^i\left(\boldsymbol{p}^i, t\right)$ border γ_c^1 the point $\left\{\tilde{\boldsymbol{x}}\right\}$ of γ_c^2 nearest. That amounts solving the problem of optimization under constraints according to. For any point $\left\{\boldsymbol{x}^1\right\} = \varphi^1\left(\Theta^1\left(\xi_1^1, \xi_2^1\right), t\right)$ with $\left(\xi_1^1, \xi_2^1\right) \in \omega^1$, and for all $t \geq 0$, to find $\tilde{\zeta}_t = \left(\xi_1^2, \xi_2^2\right) \in \omega^2$ such as:

$$\tilde{\zeta}_t = \underset{\left(\xi_1^2, \xi_2^2\right) \in \omega^2}{\operatorname{argmin}} \left\{ \frac{1}{2} . \left\| \varphi^1\left(\Theta^1\left(\xi_1^1, \xi_2^1\right), t\right) - \varphi^2\left(\Theta^2\left(\xi_1^2, \xi_2^2\right), t\right) \right\|^2 \right\} \tag{27}$$

The solution $\tilde{\zeta}_t$ is the position within the space of parametric reference of projection M node slave P on the mesh Master.

3.4.4.2 Formulation of the problem of minimization

To solve the non-linear problem (27), one proposes to use an algorithm of minimization of Newton. The problem is reformulated in the following way: for a node slave given one seeks the point (in parametric coordinates) which minimizes the distance to a mesh Master given (that defined in the paragraph §3.4.3).
One notes f the standard L_2 distance enters a node slave located in $\left\{\boldsymbol{x}_e\right\}$ and a point $\left\{\boldsymbol{x}_m\right\}$ belonging to the mesh Master in opposite:

$$f\left(\left\{\boldsymbol{x}_m\right\}\right) = \left\| \left\{\boldsymbol{x}_e\right\} - \left\{\boldsymbol{x}_m\right\} \right\|^2 \tag{28}$$

One writes the functional calculus in parametric space, $\left\{\boldsymbol{x}_m\right\}$ expressing itself from nb_{no} nodes $\left\{\boldsymbol{x}_{i,m}\right\}$ mesh Master paired:

$$f\left(\xi_1, \xi_2\right) = f\left(\left\{\boldsymbol{\zeta}\right\}\right) = \left\| \left\{\boldsymbol{x}_e\right\} - \sum_{i=1}^{nb_{no}} \Phi_i\left(\left\{\boldsymbol{\zeta}\right\}\right) . \left\{\boldsymbol{x}_{i,m}\right\} \right\|^2 \tag{29}$$

It is this quantity which it is necessary to minimize. One applies the algorithm of Newton to the conditions of stationnarity (Euler-Lagrange) of the problem of minimization thus stated, i.e. with the vectorial functional calculus $\left\{\nabla f\left(\left\{\boldsymbol{\zeta}\right\}\right)\right\}$, expression of the development of Taylor to the first order, the point $\left\{\boldsymbol{\zeta}_0\right\}$:

$$\left\{\nabla f\left(\left\{\boldsymbol{\zeta}\right\}\right)\right\} \underset{linéarisation}{=} \left\{\nabla f\left(\left\{\boldsymbol{\zeta}_0\right\}\right)\right\} + \left[\boldsymbol{H}\left(\boldsymbol{\zeta}_0\right)\right] . \left(\left\{\boldsymbol{\zeta}\right\} - \left\{\boldsymbol{\zeta}_0\right\}\right) \tag{30}$$

Where $\left[\boldsymbol{H}\right]$ is the Hessienne matrix of the derivative second of f and $\left\{\nabla f\left(\left\{\boldsymbol{\zeta}\right\}\right)\right\}$ is sound gradient. A minimum of f occurs when its gradient is null.

The iterative algorithm is written then:

1. To leave the initial point $\left\{\boldsymbol{\zeta}_{i=0}\right\}$ on the mesh Master. This starting point is simply selected in $\left\{\boldsymbol{\zeta}_{i=0}\right\} = \left\{0\right\}$;
2. To evaluate the gradient $\left\{\nabla f\left(\left\{\boldsymbol{\zeta}_i\right\}\right)\right\}$ and Hessienne stamps it $\left[\boldsymbol{H}\left(\boldsymbol{\zeta}_i\right)\right]$ in this point;
3. To calculate the direction of descent $\left\{\boldsymbol{d}\,\boldsymbol{\zeta}_i\right\} = -\left[\boldsymbol{H}\left(\boldsymbol{\zeta}_i\right)\right]^{-1} . \left\{\nabla f\left(\left\{\boldsymbol{\zeta}_i\right\}\right)\right\}$

4. To calculate the linear parameter of research α (see §3.4.4.4)
5. To calculate the following point such as $\{\zeta_{i+1}\}=\{\zeta_i\}+\alpha\{d\zeta_i\}$;
6. If the process converged, one stops, if not one buckles into 2.

Note:

- This problem is written without constraints;
- One cannot guarantee the unicity of the solution, the process of Newton will find the first point carrying out the conditions of stationnarity.
- The existence of the gradient and the Hessienne matrix is assured if the mesh is sufficiently regular, which is always the case on a grid too not distorted finite element.
- The iterative process stops by a criterion on the increment of displacement in parametric space $\varepsilon=\frac{\sqrt{\langle\zeta_{i+1}\rangle\{\zeta_{i+1}\}-\langle\zeta_i\rangle\{\zeta_i\}}}{\sqrt{\langle\zeta_{i+1}\rangle\{\zeta_{i+1}\}}}$ with $\varepsilon\leq 10^{-4}$ (nonmodifiable value by the user). This value being estimated compared to the parametric coordinates, it does not have there problems with the size of the meshs and the units used.
- The maximum number of iterations of Newton is fixed at 200 (nonmodifiable value by the user).
- If the maximum number of iterations is reached, one selects as reiterated project which minimized the distance between the node slave and the mesh Master (it is thus not perfectly the orthogonal project).
- There exists an alternative in which the direction of research is not estimated by the algorithm of Newton but a direction fixes given by the user, which can be useful in certain difficult cases (perfectly convex mesh which does not give a single projection for example). One activates it by the keyword `TYPE_APPA=' FIXE'` and the vector is given by `DIRE_APPA`.
- Projection on the elements segments in 3D (case of the beams or the cables) requires a definition of the normal by the user via the keywords `VECT_MAIT`/`VECT_ESCL`.

3.4.4.3 Form of the tangent matrix and the residue

One gives below the expressions of the intervening terms in the writing of the algorithm of Newton describes above. The functional calculus is pointed out:

$$f(\xi_1,\xi_2)=f(\{\zeta\})=\left\|\{x_e\}-\sum_{i=1}^{nb_{no}}\varphi_i(\{\zeta\}).\{x_{i,m}\}\right\|^2 \tag{31}$$

The gradient of this functional calculus:

$$\{\nabla f(\{\zeta\})\}=\begin{Bmatrix}\frac{\partial f}{\partial\xi_1}\\ \frac{\partial f}{\partial\xi_2}\end{Bmatrix} \tag{32}$$

The terms are written explicitly:

$$\frac{\partial f}{\partial\xi_j}(\xi_1,\xi_2)=2\left(\{x_e\}-\sum_{i=1}^{nb_{no}}\Phi_i(\{\zeta\}).\{x_{i,m}\}\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial\Phi_i}{\partial\xi_j}(\{\zeta\}).\{x_{i,m}\}\right) \tag{33}$$

The hessien of this functional calculus:

$$[H(\zeta)]=\begin{bmatrix}\frac{\partial^2 f}{\partial\xi_1^2} & \frac{\partial^2 f}{\partial\xi_1.\partial\xi_2}\\ \frac{\partial^2 f}{\partial\xi_1.\partial\xi_2} & \frac{\partial^2 f}{\partial\xi_2^2}\end{bmatrix} \tag{34}$$

The terms are written explicitly:

$$\frac{\partial^2 f}{\partial \xi_1^2}(\xi_1,\xi_2)=\left\|\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_j}(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right\|^2+2\left(\{\boldsymbol{x}_e\}-\sum_{i=1}^{nb_{no}}\phi_i(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial^2 \phi_i}{\partial \xi_j^2}(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right) \quad (35)$$

And for the cross terms:

$$\begin{aligned}\frac{\partial^2 f}{\partial \xi_j \partial \xi_k}(\xi_1,\xi_2) &= \left(\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_j}(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right).\left(\sum_{i=1}^{nb_{no}}\frac{\partial \phi_i}{\partial \xi_k}(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right)\\ &+2\left(\{\boldsymbol{x}_e\}-\sum_{i=1}^{nb_{no}}\phi_i(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right).\left(-\sum_{i=1}^{nb_{no}}\frac{\partial^2 \phi_i}{\partial \xi_j.\partial \xi_k}(\{\boldsymbol{\zeta}\}).\{\boldsymbol{x}_{i,m}\}\right)\end{aligned} \quad (36)$$

3.4.4.4 Linear research in the algorithm of projection

To improve the robustness of the algorithm of projection, a linear phase of research is added in the algorithm of Newton (cf §3.4.4.2). It acts, being given a direction of descent $\{\boldsymbol{d\zeta}_i\}$, to determine a parameter of projection α who minimizes a functional calculus F associated with f.
The linear algorithm of research used (linear research with graining and rule of Armijo) is the same one as for the implicit integration of the laws of behavior [R5.03.14].

The functional calculus F, associated with the functional calculus outdistances f defined in (31), is the following one:

$$F(\alpha)=\frac{1}{2}\left\|\nabla f\left(\{\boldsymbol{\zeta}\}+\alpha\{\boldsymbol{d\zeta}_i\}\right)\right\|^2 \quad (37)$$

To choose α, one does not go in fact not to seek to minimize the functional calculus exactly F but an approximation of this one (quadratic, cubic). To know which approximation to use, one is based on the rule of Armijo. For more details, one returns to [R5.03.14] and [14] who detail the implementation of such an algorithm.

For the linear research implemented in the algorithm of projection, the selected parameters (into hard) are the following:

Parameter of the rule of Armijo	$\omega=0,0001$
Limit min for the folding back	$\alpha_{min}=0,1$
Limit max for the folding back	$\alpha_{max}=0,5$
Maximum number of cubic interpolations	$it_{max}=2$

These parameters imply in particular that if the step by default ($\alpha=1$) the rule does not satisfy with Armijo, i.e. the direction of selected descent does not bring closer to the orthogonal project, then α will be with most equal to $0,5$. This choice makes it possible to support the robustness rather (to avoid following a direction of erroneous research) than the performance (values of α higher than 1).
For grids of poor quality, with distorted meshs, linear research showed its interest, finding the good project orthogonal on difficult cases where an algorithm without linear research failed.

3.4.5 Treatment of projections except mesh

There exist projections whose result is sensitive to purely digital parameters or whose existence and unicity mathematics are not guaranteed. Under certain conditions, one can detect contact between two surfaces whereas there is not. The problem comes initially from an incorrect and imperfect definition of surfaces likely to make contact. Let us take the case of the contact in 2D (surfaces of contact are thus segments) where a node slave must be projected apart from surface Master (see figure 3.4.5-a):

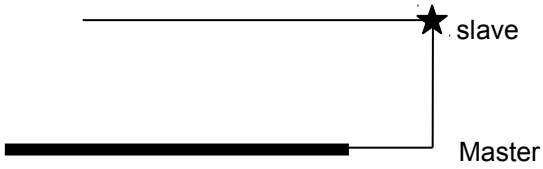

Figure 3.4.5-a : projection out of a mesh Master.

The user can choose “Re-to project” this node slave on the mesh Master prolonged or not to show it (figure 3.4.5-b).

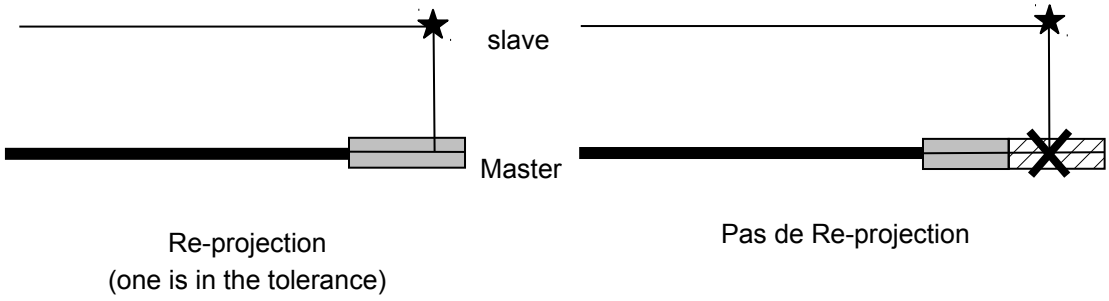

Figure 3.4.5-b : principle of the zone of tolerance for projection on a mesh Master.

The limiting value of this Re-projection is fixed by the keyword TOLE_PROJ_EXT who takes for argument the value (paid to the element of reference) of the extension of the mesh Master in which one authorizes Re-projection. By default, this value is fixed at 0.50 . What means that any node slave projecting himself at a distance higher than half the length of the mesh Master Re-will not be projected. To prohibit Re-projection completely, it is enough to fix TOLE_PROJ_EXT to zero. This operator is valid in 2D and in 3D. In the case 3D, it is the extension of a surface mesh of contact, and, as one reasons in parametric space, the curve of the edges of the elements is well taken into account.

In the same way if surfaces of contact are extended too much, any node slave located behind the mesh Master is paired. The concept of “in front of-behind” is given by the orientation of the normals given by the user (see §3.3). One can restrict this search for mesh paired with the parameter TOLE_APPA who specifies the maximum distance from research of the meshs “appariables” with the node slave.

3.4.6 Case of projection in a given direction

When the option is used DIRE_APPA, the user can force the algorithm of projection not to seek the minimal distance between a node slave and his orthogonal projection on surface Master, but with simply finding the point of surface main related to the node slave by a given vector. In this case the algorithm is the following:

1. To leave the initial point $\left\{\zeta_{i=0}\right\}$ on the mesh Master. This starting point is simply selected in $\left\{\zeta_{i=0}\right\}=\left\{0\right\}$;
2. To build the matrix of Newton $\left[\boldsymbol{H}\left(\zeta_i\right)\right]$ by the use of the direction given by DIRE_APPA ;
3. To build the second member of Newton $\left\{\boldsymbol{R}\left(\left\{\zeta_i\right\}\right)\right\}$ by the difference between the node slave and his project on the mesh Master;
4. To calculate the direction of descent $\left\{d\,\zeta_i\right\}=-\left[\boldsymbol{H}\left(\zeta_i\right)\right]^{-1}.\left\{\boldsymbol{R}\left(\left\{\zeta_i\right\}\right)\right\}$
5. To calculate the following point such as $\left\{\zeta_{i+1}\right\}=\left\{\zeta_i\right\}+\left\{d\,\zeta_i\right\}$;
6. If the process converged, one stops, if not one buckles into 2.

There is no linear algorithm of research contrary to the preceding case.

3.5 Relations kinematics

3.5.1 Definition of the matrix of contact

One carries out a idealized modeling of the phenomenon of contact, in the sense that it supposes the borders of the bodies perfectly defined by a line or a surface: one then writes a condition of nondiscrete and linearized interpenetration [2]. That is to say P a node slave, M its projection on the mesh Master which was given during pairing. In 2D, this mesh Master has two nodes (SEG2) or three nodes (SEG3). In 3D, it can have of them three, four, six, eight or nine (TRIA3, QUAD4, TRIA6, QUAD8, QUAD9). The displacement of the point M is a linear combination of displacements of the nodes of the finite element, with for coefficients the values of the functions of form Φ in M. We place if the mesh Master is one SEG2 to simplify the talk. One has then :

$$\{u_M\}=[\Phi_A(M)].\{u_A\}+[\Phi_B(M)].\{u_B\} \quad (38)$$

Initially, one chooses to take as direction $\{N\}$ the outgoing normal of the mesh Master (cf. Figure 3.5.1-a).

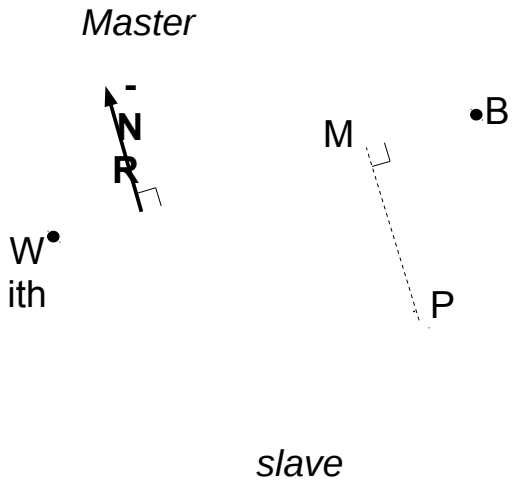

Figure 3.5.1-a : projection of a node slave on a mesh SEG2.

The normal game is written like the difference between the node slave P and its projection M on the mesh Master:

$$\langle N\rangle.\left[\{u_P\}-\sum_{j=1}^{n_m}[\Phi_{B_j}(M)].\{u_{B_j}\}\right]=d \quad (39)$$

If one writes such a relation for all the couples of contact, one obtains the geometrical conditions of nonpenetration in matric form:

$$[A^c].\{u\}=d \quad (40)$$

The matrix $[A^c]$, called matrix of contact, contains a line by couple of contact, and as many columns as of physical degrees of freedom of the problem. Let us suppose that one has two meshs of contact of the type SEG2 , according to the diagram of Figure 3.5.1-b.

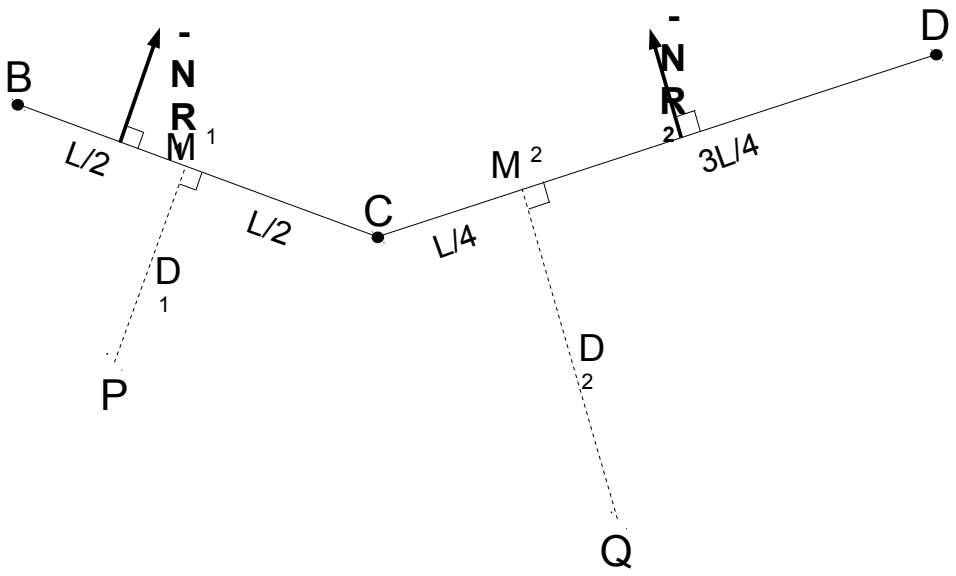

Figure 3.5.1-b : writing of the matrix of contact A on an example.

If one notes for example u_B the displacement of the node B according to the direction x, v_B the displacement of the node B according to the direction y, and d^1 and d^2 current games for the two couples:

$$\left[A^c\right].\begin{Bmatrix} u_P \\ v_P \\ u_Q \\ v_Q \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ v_D \end{Bmatrix}=\begin{Bmatrix} d^1 \\ d^2 \end{Bmatrix} \tag{41}$$

With the matrix of contact $\left[A^c\right]$:

$$\left[A^c\right]=\begin{bmatrix} N_x^1 & N_y^1 & 0 & 0 & -\frac{1}{2}.N_x^1 & -\frac{1}{2}.N_y^1 & -\frac{1}{2}.N_x^1 & -\frac{1}{2}.N_y^1 & 0 & 0 \\ 0 & 0 & N_x^2 & N_y^2 & 0 & 0 & -\frac{3}{4}.N_x^2 & -\frac{3}{4}.N_y^2 & -\frac{1}{4}.N_x^2 & -\frac{1}{4}.N_y^2 \end{bmatrix} \tag{42}$$

One considered here only the degrees of freedom of the nodes implied in the contact; the matrix $\left[A^c\right]$ should be hollower. But in practice, one always reduces the matrix of contact on the active degrees of freedom. Only the nonworthless coefficients are thus stored.

3.5.2 Definition of the matrix of friction

The concept of matrix of contact extends to the case from the tangential slips, on the tangent level. It is the matrix of the relations kinematics of friction $\left[A^f\right]$.

3.5.3 Choice of the normal

In the preceding paragraph, one chose to take as direction $\{N\}$ the outgoing normal of the mesh Master. It is the behavior by default in *Code_Aster*. However, it is possible to choose other normals:

- The normal slave NORMALE=' ESCL' ;
- An average enters the normal Master and the normal slave NORMALE=' MAIT_ESCL' ;

It is also possible to ask to use smoothed normals (LISSAGE=' OUI'), i.e. that instead of using the normal Master at the point of projection, one can take a normal resulting from the linear interpolation between the normals with the nodes of the mesh Master.
In the same way, the calculation of the normals is always done via the functions of form of the element, it is what one defines as “the true” normal or “automatic” normal. But it is possible to impose a normal on the mesh Master, the mesh slave or on both in a different way:

- Directly (VECT_MAIT or VECT_ESCL = ' FIXE')
- Indirectly by use of a trihedron (VECT_MAIT or VECT_ESCL=' VECT_Y'). In this last case, the normal used will be the vector resulting from the vector product between the tangent with the mesh and the vector VECT_Y given.

Note:

- As regards a pairing of the node-facet type, the normal slave is calculated itself by smoothing, the option of SMOOTHING thus does not have effect if one chooses NORMALE=' ESCL' ;
- The use of preset normals is obligatory in the case of the beams;
- The choice of a normal other than the mesh Master must be limited to exceptional cases like when the grid or the compatibility issues of the contact with the boundary conditions forced the user to coarsely take for Master a surface with a grid;
- The use of TYPE_APPA=' FIXE' for the research of the mesh Master nearest (see § 3.4.4) do not prejudge a choice of the normal in the writing of the relation of nonpenetration, which remains with the choice of the user. But it is more coherent to choose a fixed normal (VECT_MAIT=' FIXE').

3.5.4 Coefficients of the matrix of contact

3.5.4.1 Standard elements

Values of the functions of form $\Phi_{B_j}(M)$ main nodes at the point M for the various meshs of contact are standard (see [R1.01.01]).

3.5.4.2 Meshs QUAD8

The meshs of the quadrilateral type to eight nodes present a defect. Indeed, the classical functions of forms are not positive on all the field and lead to aberrant results when they are used in the contact. The principal symptom related to the use of the classical functions of form is the appearance of negative forces of contact not-physics which causes oscillations (between the nodes tops and the nodes mediums).
To avoid this phenomenon, *Code_Aster* carry out the modification of the element QUAD8 by imposing linear relations between nodes mediums and nodes tops and by finally using the functions of form of QUAD4.
In a general way, it is preferable to avoid using this kind of element and preferring complete quadratic elements to him like QUAD9 , because they are considered QUAD8 as linear elements and one introduces besides the linear relations which can be awkward.

3.5.4.3 Elements of COQUE_3D

Elements of the type COQUE_3D are nonisoparametric mixed finite elements. They are based on quadratic meshs of type QUAD9 (respectively TRIA7) but the node medium not carrying a degree of freedom of translation, one carries out a projection on one QUAD8 (respectively TRIA6) and one thus falls down on the defects evoked in the preceding paragraph. These finite elements are thus disadvised in the problems of contact.

3.6 Introduction of a fictitious game

One can want to model the contact between structures has certain characteristics (“hole” or “bump”) which one does not wish to net (see figure 3.6-a).

Figure 3.6-a : holes and bumps.

A solution consists in netting surface without these defects and adding a distance to it given by the user (see figure 3.6-b).

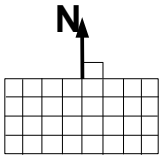

Figure 3.6-b : surface with a grid without defect.

The value of the game is corrected:

$$\langle \boldsymbol{N} \rangle . \left(\{ \boldsymbol{u}_P \} - \{ \boldsymbol{u}_M \} \right) . = d - \left(d_{\mathrm{e}} + d_{\mathrm{m}} \right) \quad (43)$$

where d_{e} and d_{m} are given by the user respectively under the keywords DIST_MAIT and DIST_ESCL for each zone of contact. These distances are signed: they represent the translation to be applied to the node of the grid in the direction of the normal $\{ \boldsymbol{n} \}$ outgoing to obtain the point of the real structure. These keywords make it possible to also give an account of the contact between hulls whose only average surfaces are with a grid: d_{e} and d_{m} are worth then the half-thickness of the hulls (positive values).

Note:

- If one uses DIST_MAIT and DIST_ESCL , it is necessary to take care of the visual interpretation of the results. If $\left(d_{\mathrm{e}} + d_{\mathrm{m}} \right) > 0$, the code will be able to announce contact whereas visualization shows a spacing of the two grids. If $\left(d_{\mathrm{e}} + d_{\mathrm{m}} \right) < 0$, the code will be able to announce contact whereas visualization shows two interpenetrated grids;
- To remember the signs, to think of:
  - $\left(d_{\mathrm{e}} + d_{\mathrm{m}} \right) > 0$: matter addition compared to the grid
  - $\left(d_{\mathrm{e}} + d_{\mathrm{m}} \right) < 0$: ablation of matter compared to the grid

Options DIST_POUTRE and DIST_COQUE are based on the elementary characteristics defined in the operator AFFE_CARA_ELEM to add the fictitious game corresponding to the thickness (in the case of hulls) or to the ray (in the case of beams with circular section).

3.7 Geometrical reactualization

Within the framework of the modeling of the contact in great displacements, the evolution of the geometry of surfaces plays a fundamental role. Indeed, it is it which potentially conditions the calculation of the normals to the faces in contact and thus which conditions pairing carried out.
The geometrical reactualization is defined by the keyword REAC_GEOM order DEFI_CONTACT. Its operation is the following:

- If REAC_GEOM= ' WITHOUT ' : there is no geometrical reactualization. All calculation is carried out on the initial configuration with initial pairing.
- If REAC_GEOM=' CONTRÔLE' : it is necessary to inform NB_ITER_GEOM . Within **even** pas de charges, one carries out NB_ITER_GEOM time the cycle *iteration until convergence, geometrical reactualization, pairing*.
- If REAC_GEOM=' AUTOMATIQUE' : the decision to remake a geometrical pairing is made automatically by the software. The criterion is the following :

$$\frac{\|\Delta \boldsymbol{u}^{+}-\Delta \boldsymbol{u}^{-}\|_{\infty}}{\|\Delta \boldsymbol{u}^{-}\|_{\infty}}<\text{RESI\_GEOM} \tag{44}$$

RESI_GEOM is worth 1% by default. If the infinite standard of displacement between two moments, divided by the infinite standard of the displacement obtained since balance is higher than RESI_GEOM, then one reactualizes. The infinite standard is defined by :

$$\|\Delta \boldsymbol{u}\|_{\infty}=\max \sqrt{u_x^2+u_y^2+u_z^2} \tag{45}$$

Note:

- With the first iteration or in the event of movements of rigid body (in dynamics), $\|\Delta \boldsymbol{u}^{-}\|_{\infty}=0$. To avoid dividing by zero, the reactualization is forced ;
- The infinite standard is evaluated on all the nodes of the grid (and not only on the nodes in contact) carrying degrees of freedom of displacement.

One can first of all notice that pairing is subjected to the geometrical phase of reactualization. Moreover, the fact of carrying out several times within the same step of load the cycle *iteration until convergence, geometrical reactualization, pairing* allows to follow the evolution of the geometry of the structure. It should indeed be stressed that this geometrical evolution is one of the nonlinear components of a calculation of contact in great displacements.

In practice, one can advise the following choices for the keyword REAC_GEOM :

- For a calculation in small displacements, REAC_GEOM= ' WITHOUT ' . One works on the initial configuration.
- For calculations in great displacements, that is to say to use REAC_GEOM=' AUTOMATIQUE' (value by default), that is to say to use REAC_GEOM=' CONTRÔLE' and a value for NB_ITER_GEOM depending on the importance of the geometrical evolution of surfaces.

If the user does not leave in Code_Aster the possibility of managing the geometrical reactualization automatically, the code will inform it by an alarm if the automatic criterion (being worth 1%) is not ensured because of the choice of the user.

4 Mechanical problem of contact/friction

The taking into account of contact-friction in a mechanical problem has two consequences:

- Modification of the equilibrium equation to take into account the reactions of contact-friction to the interfaces;
- The application of additional laws governing the contact and friction (see § 2) to calculate these reactions but also to impose conditions on kinematics.

For the contact without friction, the conditions of Signorini are brought back to a problem of optimization under classical constraint (Kuhn & Tucker). On the other hand, for the friction of Coulomb, one cannot write equivalent problem of optimization without making assumption (S) additional (S).

In *Code_Aster*, methods *discrete* of resolution of the problem of contact/friction are founded on an approach *uncoupled* between the balance and the law of contact/friction. After the resolution of the problem of mechanics *without* contact-friction, one corrects the solution (kinematic and reactions) by applying the law of Signorini-Coulomb. This strategy makes it possible not to make any other assumption on the nature of the mechanical problem, whether it is the kinematics or the relations of behavior. However, it is essential that the matrix of rigidity of the problem without contact is *symmetrical*.

Note:

- The expression “to do a calculation with contact” wants to say that one writes the relations of nonpenetration, but does not imply that there is effective contact for the loading considered. However it is the resolution of the problem of contact-friction which is most expensive.
- The contact acting like correction on the results resulting from a classical mechanical calculation, it is essential that the problem without contact is mechanically well posed and numerically soluble. In particular, the possible movements of rigid bodies must be presumedly eliminated except resolution of the problem of contact.

4.1 Mechanical problem without contact/friction

The resolution of a non-linear problem in the operator STAT_NON_LINE (or DYNA_NON_LINE) is described in detail in the document [R5.03.01]. With each step of time i, one seeks to check the total balance of the structure:

$$\begin{cases} \left\{\boldsymbol{L}_i^{\text{int}}(\boldsymbol{u}_i)\right\}+\left[\boldsymbol{B}\right]^T.\left\{\lambda_i\right\}=\left\{\boldsymbol{L}_i^{\text{méca}}(\boldsymbol{u}_i)\right\} \\ \left[\boldsymbol{B}\right].\left\{\boldsymbol{u}_i\right\}=\left\{\boldsymbol{u}_i^d\right\} \end{cases} \quad (46)$$

In order to prevent overloading the equations, one makes the assumption that the limiting conditions are eliminated and where one thus does not have matrix $\left[\boldsymbol{B}\right]$, one thus seeks to solve:

$$\left\{\boldsymbol{L}_i^{\text{int}}(\boldsymbol{u}_i)\right\}=\left\{\boldsymbol{L}_i^{\text{méca}}(\boldsymbol{u}_i)\right\} \quad (47)$$

This non-linear problem is solved by an iterative method of Newton-Raphson type which has as characteristics:

- A cutting *a priori* loading in “pas de times” (noted by the index i);
- A linearization of the problem of balance by the method of Newton (iterations being noted by the index n) .

The unknown factors are calculated in an incremental way. From $\left\{\boldsymbol{u}_{i-1}\right\}$, solution satisfying balance in t_{i-1}, one determines $\left\{\Delta\boldsymbol{u}_i\right\}$ who allows to obtain the solution in t_i :

$$\left\{\boldsymbol{u}_i\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i\right\} \quad (48)$$

The increment $\left\{\Delta\boldsymbol{u}_i\right\}$ is initially estimated by linearizing the problem compared to time around $\left(\left\{\boldsymbol{u}_{i-1}\right\},t_{i-1}\right)$ (phase of prediction or Euler). Then one uses a method of Newton or one of his alternatives to solve the equation (47) in an iterative way: a continuation is calculated $\left\{\delta\boldsymbol{u}_i^n\right\}$ where the exhibitor n is the number of the iteration. To simplify, one will not make pas de distinction enters the phase of prediction and the phase of correction of Newton . One writes finally:

$$\left\{\boldsymbol{u}_i^n\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^n\right\} \quad (49)$$

One places oneself at time t_i and with the iteration of Newton n . The following notations were used:

- $\left\{ \boldsymbol{u}_i^n \right\}$: displacements at the moment t_i and with the iteration of Newton n ;
- $\left\{ \boldsymbol{u}_{i-1} \right\}$: displacement at the moment t_{i-1} . This solution observes the equilibrium condition of the structure;
- $\left\{ \delta \boldsymbol{u}^n \right\}$: increment of displacements for the iteration of Newton n ;
- $\left\{ \Delta \boldsymbol{u}_i^{n-1} \right\}$: increment of displacements cumulated since the beginning of the step of time, before the iteration of Newton n ;

After linearization of (47) , for the iteration of Newton n , one introduces [5] the mechanical tangent matrix $\left[\boldsymbol{K}^{\mathrm{m},n-1} \right]$ and one must then find $\left\{ \delta \boldsymbol{u}^n \right\}$ such as:

$$\left[\boldsymbol{K}^{\mathrm{m},n-1} \right].\left\{ \delta \boldsymbol{u}^n \right\} = \left\{ \boldsymbol{L}_i^{\mathrm{méca},n-1} \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{int},n-1} \right\} \tag{50}$$

4.2 Modification of the equilibrium equation d' with contact/friction

To ensure the balance of the structure, it is necessary to introduce forces of contact/friction. In the most general possible way, one writes that the mechanical external loading is modified by the efforts of contact/friction:

$$\left\{ \boldsymbol{L}_i^{\mathrm{méca}}(\boldsymbol{u}_i) \right\} = \left\{ \boldsymbol{L}_i^{\mathrm{ext}}(\boldsymbol{u}_i) \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{c}}(\boldsymbol{u}_i) \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{f}}(\boldsymbol{u}_i) \right\} \tag{51}$$

Where:

- $\left\{ \boldsymbol{L}_i^{\mathrm{ext}}(\boldsymbol{u}_i) \right\}$ is the vector of the external forces (conditions of Neumann);
- $\left\{ \boldsymbol{L}_i^{\mathrm{c}}(\boldsymbol{u}_i) \right\}$ is the vector of the forces of contact;
- $\left\{ \boldsymbol{L}_i^{\mathrm{f}}(\boldsymbol{u}_i) \right\}$ is the vector of the forces of friction;

A priori, all these quantities are non-linear because they depend on the vector displacement $\left\{ \boldsymbol{u}_i \right\}$ (one speaks about “following” loadings). While injecting (51) in (47), ON thus seeks to solve:

$$\left\{ \boldsymbol{L}_i^{\mathrm{int}}(\boldsymbol{u}_i) \right\} = \left\{ \boldsymbol{L}_i^{\mathrm{ext}}(\boldsymbol{u}_i) \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{c}}(\boldsymbol{u}_i) \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{f}}(\boldsymbol{u}_i) \right\} \tag{52}$$

With each iteration of Newton n , the equation is linearized (52) compared to $\left\{ \boldsymbol{u}_i^n \right\}$. This process introduces the tangent matrix $\left[\boldsymbol{K}^{\mathrm{m},n-1} \right]$ who will contain the contributions resulting from the linearization of the interior and external efforts and the matrix $\left[\boldsymbol{K}^{\mathrm{cf},n-1} \right]$ who will contain the contributions resulting from the linearization from the forces from contact/friction , one must then find $\left\{ \delta \boldsymbol{u}^n \right\}$ such as:

$$\left(\left[\boldsymbol{K}^{\mathrm{m},n-1} \right] + \left[\boldsymbol{K}^{\mathrm{cf},n-1} \right] \right).\left\{ \delta \boldsymbol{u}^n \right\} = \left\{ \boldsymbol{L}_i^{\mathrm{ext},n-1} \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{int},n-1} \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{c},n-1} \right\} - \left\{ \boldsymbol{L}_i^{\mathrm{f},n-1} \right\} \tag{53}$$

4.3 Laws of contact/friction

The laws of Signorini and Coulomb imply inequalities and equalities. The discrete formulation available in *Code_Aster* consist in modifying the relations of inequality in relations of equality. With this intention, there are two possible methods:

1. Dualiser laws of contact friction;
2. To regularize the laws of contact friction.

4.3.1 Kinematics

4.3.1.1 Matrices of contact/friction

In the paragraph §3.5.1, we saw how to write the conditions kinematics of contact by introducing the matrix of contact $\left[\boldsymbol{A}^{\mathrm{c}} \right]$. Same manner, but by considering the conditions kinematics of friction (on the tangent level), one will evaluate the matrices of slip $\left[\boldsymbol{A}^{\mathrm{g}} \right]$ and of adherence $\left[\boldsymbol{A}^{\mathrm{a}} \right]$. These quasi-full and rectangular matrices are calculated by the evaluation of the relations kinematics between the node slave and the paired mesh Master.

5 There exist several manners of choosing and of calculating this matrix. For more details to see [R5.03.01]

The construction of the relations kinematics is done at the time of the geometrical procedure of pairing which intervenes at the beginning of each step of time and with each geometrical reactualization. For reasons of performances, one does not use and one builds only the under-part as of these matrices corresponding to the activation of the various thresholds (contact or slip), within the algorithms of resolution of the contact/friction.
These matrices depend on the geometrical reactualization.
At the total level, matrices not depending on the solution without contact of the mechanical problem.

4.3.1.2 Unilateral conditions of contact

The relation of nonpenetration consists in saying that relative displacement according to a given direction cannot exceed the initial game $\left\{d_{\text{ini}}^{\text{c}}\right\}$, measured on the grid, in this direction. The unilateral condition of contact is written (see § 3.5.1) :

$$\left[A^{\text{c}}\right].\left\{u_i\right\}\leq\left\{d_{\text{ini}}^{\text{c}}\right\} \tag{54}$$

This equation translates the fact that any movement of the structure must be made with the respect of the condition of nonpenetration, or that the displacement of the nodes of the surface of contact is lower than the initial game $\left\{d_{\text{ini}}^{\text{c}}\right\}$. After resolution of the mechanical problem by the method of Newton, the condition of nonpenetration becomes:

$$\left[A^{\text{c}}\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)\leq\left\{d_{\text{ini}}^{\text{c}}\right\} \tag{55}$$

This condition can be written in an iterative way:

$$\left[A^{\text{c}}\right].\left\{\delta u^n\right\}\leq\left\{d^{\text{c},n-1}\right\} \tag{56}$$

With $\left\{d^{\text{c},n-1}\right\}$ game evaluated before the current iteration of Newton n :

$$\left\{d^{\text{c},n-1}\right\}=\left\{d_{\text{ini}}^{\text{c}}\right\}-\left[A^{\text{c}}\right].\left(\left\{u_{i-1}\right\}+\left\{\Delta u_i^{n-1}\right\}\right) \tag{57}$$

4.3.1.3 Conditions of adherence

In adherence, the nodes do not move on the step of time, i.e.:

$$\left[A^{\text{a}}\right].\left(\left\{u_i\right\}-\left\{u_{i-1}\right\}\right)=0 \tag{58}$$

$\left[A^{\text{a}}\right]$ is the matrix of the nodes in adherent contact, i.e. the under-part of the matrix of friction $\left[A^{\text{f}}\right]$ (see § 3.5.1) applied to the nodes in adherent contact.

4.3.1.4 Conditions of slip

The shear stress $\left\{r_i\right\}$ is colinéaire with the tangent direction of slip, that is to say:

$$\left[A^{\text{g}}\right].\left(\left\{u_i\right\}-\left\{u_{i-1}\right\}\right)=\lambda.\left\{r_i\right\} \tag{59}$$

$\left[A^{\text{g}}\right]$ is the matrix of the nodes in slipping contact, i.e. the under-part of the matrix of friction $\left[A^{\text{f}}\right]$ (see § 3.5.1) applied to the nodes in slipping contact. One a:

$$\left[A^{\text{g}}\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)=\lambda.\left\{r_i\left(\left\{u_i\right\}\right)\right\} \tag{60}$$

The equation (60) depends on final displacement $\left\{u_i\right\}$. The conditions of slip are introduced with a multiplier of Lagrange $\left\{\mu^{\text{g}}\right\}$ such as:

$$\left\{\mu^{\text{g}}\left(u_i\right)\right\}=\mu.\left[\left|\mu^{\text{c}}\left(u_i\right)\right|\right].\left\{t^n\right\} \tag{61}$$

With t^n the unit vector of the direction of slip which is worth in 3D:

$$\left\{t\right\}=\frac{\left[A^{\text{g}}\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)}{\left\|\left[A^{\text{g}}\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)\right\|} \tag{62}$$

That is to say still:

$$\left\{\mu^{g}(u_i)\right\}=\left[k^{g}(u_i)\right].\left\{t^{n}\right\} \text{ with } \left[k^{g}(u_i)\right]=\mu.\left[\left|\mu^{c}(u_i)\right|\right] \quad (63)$$

The multiplier of Lagrange is nonlinear and depends on the solution $\{u_i\}$. In 2D, it is too non-linear but the multiplier does not depend on the direction of slip, one thus has:

$$\left\{\mu^{g}(u_i)\right\}_{2D}=\mu.\left[\left|\mu^{c}(u_i)\right|\right]=\left[k^{g}(u_i)\right] \quad (64)$$

4.3.2 Dualisation

In the case of the dualisation, one uses an algorithm test-error which applies a priori the state of a connection and which checks its state after application of the law of Signorini-Coulomb. To take into account the constraints (one speaks about the constraints in the direction of conditions of "restriction" and not in the mechanical direction of the term) bearing on the field of displacements or the efforts, one utilizes them in the equations through multipliers of Lagrange (as that can be made in *Code_Aster* for the boundary conditions kinematics). One introduces two whole of multipliers of Lagrange:

- $\{\mu^{c}\}$ relating to the conditions of contact;
- $\{\mu^{g}\}$ relating to the conditions of slip.

By writing the balance of the structure, one gives the following interpretation of the multipliers of Lagrange:

- $\left\{L_i^{c}(u_i)\right\}=[A^{c}]^{T}.\left\{\mu_i^{c}\right\}$ represent the nodal forces of contact;
- $\left\{L_i^{g}(u_i)\right\}=[A^{g}]^{T}.\left\{\mu_i^{g}\right\}$ represent the nodal forces of slip.

Even if the algorithm of friction uses the regularization, it is seen that the introduction of the "multipliers of Lagrange" is necessary to linearize the nodal forces of slip.

4.3.3 Regularization

The principle of the regularization is to amend the laws of contact-friction to obtain some easier to handle, for example, than the relations become univocal and derivable (to be able to apply the method of Newton for example). One thus introduces assumptions which make the model not-exact compared to the laws of Signorini-Coulomb.

4.3.3.1 Regularization of the conditions of contact

The principle of the regularization is to modify the graph of the law of contact in order to remove the not-univocal character of the relation of contact.

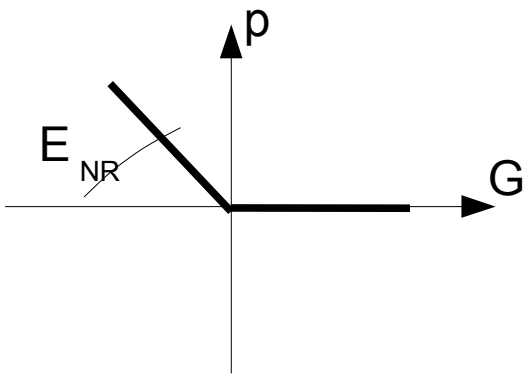

Figure 4.3.3.1-a : regularization of the condition of contact.

It is pointed out that the condition of not-interpenetration is written in iterative form:

$$\left[A^{c}\right].\left\{\delta u^{n}\right\}\leq\left\{d^{c,n-1}\right\} \quad (65)$$

The idea of the regularization is to penalize the situation for which there is interpenetration. Interpenetration $\{h^{n}\}$ is worth:

$$\left\{h^n\right\}=\left[A^c\right].\left\{\delta u^n\right\}-\left\{d^{c,n-1}\right\}>0 \quad (66)$$

In other words, it is written that the force of contact is all the more large as the interpenetration $\left\{h^n\right\}$ is important, from where the regularized form of the force of contact:

$$\left\{L_i^{c,n}\right\}_{regu}=E_N.\left[A^c\right]^T.\left\{h^n\right\}^+ \text{ with } E_N>0 \quad (67)$$

E_N is the coefficient of regularization (or penalization) contact and one notes $\square^+$ the positive part of a quantity. coefficient of penalization of contact be interpreted as the spring of stiffness E_N who opposes the penetration of the node slave in surface Master. The larger it is, the less there is interpenetration and the higher the back pulling force is.

4.3.3.2 Regularization condition D'adherence

The principle of the regularization of adherence is to modify the graph of the law of Coulomb . Initially, one remove not-univocal character of the law in the adherent part.

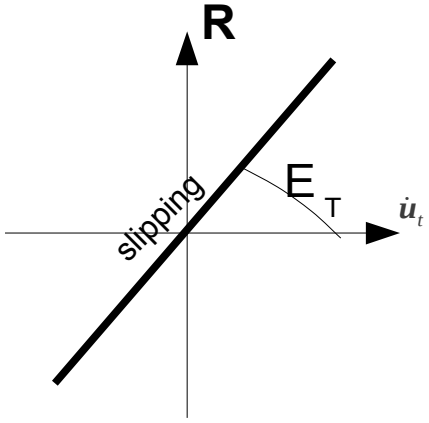

Figure 4.3.3.2-a : regularization of the condition of adherence – Model of Tresca.

One establishes the regularized form of the force of adherence:

$$\left\{L_i^{a,n}\right\}_{regu}=E_T.\left[A^a\right]^T.\left[A^a\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right) \text{ with } E_T>0 \quad (68)$$

E_T is the coefficient of regularization (or of penalization) friction . The concept of adherence strictly speaking thus goes to disappear, all the nodes slip . One sees on the figure (4.3.3.2-a) that this regularization is not satisfactory because one did nothing but transform the problem of Coulomb into problem of Tresca: the force is proportional to the relative slip . To approach the model of Coulomb, should be added an additional inequation:

$$\left\|E_T.\left[A^a\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right)\right\|\leq \mu.\left|p(u)\right| \quad (69)$$

This additional inequation modifies the graph (see figure (4.3.3.2-b)). It results from it a form approximate from the model of Coulomb, in which one will define the adherent nodes compared to the threshold of contact, product of the coefficient of Coulomb μ and of the contact pressure $p(u)$. The approximate character of the law of Coulomb results in the fact that the nodes known as "adherent" will slip more especially as the coefficient of regularization will be low. Physical interpretation is thus less direct than in the case of the contact, which mainly explains the difficulty of finding in practice a value satisfactory of this coefficient. However the made mistake ("distorts" detection of the threshold of slip) is often not very important compared to the assumptions of modeling. There remains a non-linearity which relates to the "slipping" part of the law. The treatment of this non-linearity is deferred to the § 4.3.4 .

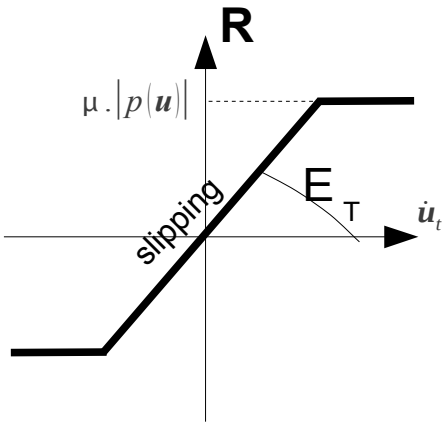

Figure 4.3.3.2-b : regularization of the condition of adherence – Model of Coulomb.

4.3.4 Modeling of the slip

The conditions of slip are always introduced with a multiplier of Lagrange $\{\mu^g\}$ who express yourself (in 3D) from (63):

$$\{\mu^g(u_i)\}=[k^g(u_i)].\{t^n\} \tag{70}$$

What reveals the two unknown factors which depend both on the solution: the contact pressure in the matrix of the thresholds of slip $[k^g]$ and direction of slip $\{t^n\}$. In practice, however, it is considered that the knowledge of the threshold of slip $[k^{g,n-1}]$ is acquired with the preceding iteration $(n-1)$, which amounts being brought back to a criterion of Tresca for each iteration. With convergence the threshold is obviously fixed: there are not thus more differences between the thresholds during iterations. $\{\mu^{g,n}\}$ is thus approximated compared to the state of the preceding iteration $(n-1)$:

$$\{\mu^{g,n}\}\approx[k^{g,n-1}].\frac{[A^g].(\{\Delta u_i^{n-1}\}+\{\delta u^n\})}{\left\|[A^g].(\{\Delta u_i^{n-1}\}+\{\delta u^n\})\right\|}=[k^{g,n-1}].\{t^n\} \tag{71}$$

There is also the threshold of slip to the preceding iteration:

$$[k^{g,n-1}]=\mu.[|\mu^{c,n-1}|] \tag{72}$$

4.3.5 Linearizations

To solve the non-linear problem, it is necessary to linearize the quantities which depend on $\{u_i^n\}$ in the equilibrium equation (52). The writing of the forces of contact/friction introduces non-linear quantities which it is thus advisable to linearize to be able to apply the algorithm of Newton. In the following table, one counts the cases where the linearization is necessary according to the type of algorithm.

	Dualisation	Regularization
$\{L_i^{c,n}\}$	Pas de linearization	Linearization
$\{L_i^{a,n}\}$		Linearization
$\{L_i^{g,n}\}$		Linearization

4.3.5.1 Linearization of the forces of contact

One regularizes the condition of contact starting from the expression (67) force of contact, when one is in the phase of interpenetration, i.e. while supposing $\left\{h^n\right\}>0$:

$$\left\{L_i^{c,n}\right\}_{regu}=E_N.\left[A^c\right]^T.\left(\left[A^c\right].\left\{\delta u^n\right\}-\left\{d^{c,n-1}\right\}\right) \quad (73)$$

While linearizing, one obtains:

$$\left\{L_i^{c,n}\right\}_{regu} \underset{linéarisation}{\rightarrow} \left\{\hat{L}_i^{c,n}\right\}_{regu}=\left[K_i^{c,n-1}\right].\left\{\delta u^n\right\}+\left\{L_i^{c,n-1}\right\} \quad (74)$$

$\left[K_i^{c,n-1}\right]$ make a new contribution to the tangent matrix of the problem, it is the tangent matrix "of contact". It is worth obviously:

$$\left[K_i^{c,n-1}\right]=E_N.\left[A^{c,n-1}\right]^T.\left[A^{c,n-1}\right] \quad (75)$$

And $\left\{L_i^{c,n-1}\right\}$ contribute to the second member, he is worth:

$$\left\{L_i^{c,n-1}\right\}=-E_N.\left[A^{c,n-1}\right]^T.\left\{d^{c,n-1}\right\} \quad (76)$$

It is pointed out that the coefficient of penalization of contact be interpreted as the spring of stiffness E_N who opposes the penetration of the node slave in surface Master. The larger it is, the less there is interpenetration and the more the back pulling force is raised, but this coefficient also intervenes in the tangent matrix, which modifies its conditioning and returns the resolution of the more difficult linear system. When the contact is not activated (i.e. if $\left\{h^n\right\}\leq 0$), the matrix and the vector second member are worthless.

4.3.5.2 Linearization of the forces of adherence

ON regularizes the condition D'adherence from L'expression (68) DE the force D'adherence, when one is in the phase of adherence (in the direction defined in the §4.3.3.2), i.e. by supposing that the inequality (69) is strictly respected:

$$\left\{L_i^{a,n}\right\}_{regu}=E_T.\left[A^a\right]^T.\left[A^a\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta u^n\right\}\right) \quad (77)$$

While linearizing, one obtains:

$$\left\{L_i^{a,n}\right\}_{regu} \underset{linéarisation}{\rightarrow} \left\{\hat{L}_i^{a,n}\right\}_{regu}=\left[K_i^{a,n-1}\right].\left\{\delta u^n\right\}+\left\{L_i^{a,n-1}\right\} \quad (78)$$

$\left[K_i^{a,n-1}\right]$ make a new contribution to the tangent matrix of the problem, it is the tangent matrix "of adherence". It is worth obviously:

$$\left[K_i^{a,n-1}\right]=E_T.\left[A^{a,n-1}\right]^T.\left[A^{a,n-1}\right] \quad (79)$$

And $\left\{L_i^{a,n-1}\right\}$ contribute to the second member, he is worth:

$$\left\{L_i^{a,n-1}\right\}=E_T.\left[A_a^{n-1}\right]^T.\left\{\Delta u_i^{n-1}\right\} \quad (80)$$

It is pointed out that it coefficient of penalization of contact is interpreted as the spring of stiffness E_T who opposes the relative slip of two surfaces . More it is large, more the back pulling force is raised, but this coefficient also intervenes in the tangent matrix, which modifies its conditioning and returns the resolution of the more difficult linear system.

4.3.5.3 Linearization of the forces of slip

We saw in the § 4.3.4 that the forces of slip depended on final displacement $\left\{u_i\right\}$ by the threshold of Tresca, but which we made the assumption that information with the iteration $(n-1)$ were enough to solve the law of Coulomb [6] while writing:

$$\left\{\mu^{g,n}\right\}=\left[k^{g,n-1}\right].\left\{t^n\right\} \quad (81)$$

6 The idea to transform the problem of Coulomb into a succession of problems of Tresca is very largely justified in the literature (see [13] for example) and mainly used in the commercial codes. It is also the strategy retained in the version "not fixes" algorithm in continuous formulation (see [R5.03.52]).

Nevertheless, it is necessary to prepare what will occur to the following iteration $(n+1)$ if the threshold is not correct and will require a new iteration. With the next iteration of Newton, solution displacement will be:

$$\left\{\boldsymbol{u}_i^{n+1}\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n}\right\}+\left\{\delta\boldsymbol{u}^{n+1}\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}+\left\{\delta\boldsymbol{u}^{n+1}\right\} \tag{82}$$

The quantity thus should be linearized (81) in $(n+1)$:

$$\left\{\mu^{g,n+1}\right\}=\left[\boldsymbol{k}^{g,n}\right].\left\{\boldsymbol{t}^{n+1}\right\} \tag{83}$$

The equation (83) is form $\frac{\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n+1}\right)\right\}}{\left\|\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n+1}\right)\right\}\right\|}$ with $\left\{\boldsymbol{x}^{n+1}\right\}=\left\{\boldsymbol{x}^{n}\right\}+\left\{\delta\boldsymbol{x}^{n+1}\right\}$. The linearization is written:

$$\left[\boldsymbol{k}\right].\frac{\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n+1}\right)\right\}}{\left\|\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n+1}\right)\right\}\right\|}\underset{linéarisation}{\rightarrow}\left[\boldsymbol{k}\right].\left(\frac{\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}}{\left\|\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\right\|}+\frac{1}{\left\|\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\right\|}.\left(\left[\boldsymbol{I}\right]-\frac{\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\left\langle\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\right\rangle}{\left\|\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\right\|^{2}}\right).\left\{\delta\boldsymbol{x}^{n+1}\right\}\right) \tag{84}$$

One applies (84) on the expression (83) with:

$$\begin{gathered}\left\{\boldsymbol{x}^{n}\right\}\leftarrow\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}\text{ and }\left\{\boldsymbol{x}^{n+1}\right\}\leftarrow\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}+\left\{\delta\boldsymbol{u}^{n+1}\right\}\\ \left\{\delta\boldsymbol{x}^{n+1}\right\}\leftarrow\left\{\delta\boldsymbol{u}^{n+1}\right\}\\ \left\{\boldsymbol{h}\left(\boldsymbol{x}^{n}\right)\right\}\leftarrow\left[\boldsymbol{A}^{g}\right].\left(\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}\right)\text{ and }\left\{\boldsymbol{h}\left(\boldsymbol{x}^{n+1}\right)\right\}\leftarrow\left[\boldsymbol{A}^{g}\right].\left(\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}+\left\{\delta\boldsymbol{u}^{n+1}\right\}\right)\\ \left[\boldsymbol{k}\right]\leftarrow\left[\boldsymbol{k}^{g,n}\right]\end{gathered} \tag{85}$$

One linearizes (83) :

$$\left\{\mu^{g,n+1}\right\}\underset{\text{linéarisation}}{\rightarrow}\left\{\hat{\mu}^{g,n+1}\right\} \tag{86}$$

One starts by noting $\left\{\boldsymbol{g}^{t,n}\right\}$ tangential slip relating to the iteration n :

$$\left\{\boldsymbol{g}^{t,n}\right\}=\left[\boldsymbol{A}^{g}\right].\left(\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^{n}\right\}\right) \tag{87}$$

One obtains:

$$\left\{\hat{\mu}^{g,n+1}\right\}=\left[\boldsymbol{k}^{g,n}\right].\left(\frac{\left\{\boldsymbol{g}^{t,n}\right\}}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|}+\frac{1}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|}.\left(\left[\boldsymbol{I}\right]-\frac{\left\{\boldsymbol{g}^{t,n}\right\}\left\langle\boldsymbol{g}^{t,n}\right\rangle}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|^{2}}\right).\left\{\delta\boldsymbol{u}^{n+1}\right\}\right) \tag{88}$$

If one notes:

$$\left\{\rho^{g,n}\right\}=\left[\boldsymbol{k}^{g,n}\right].\frac{\left\{\boldsymbol{g}^{t,n}\right\}}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|} \tag{89}$$

And:

$$\left[\boldsymbol{B}^{g,n}\right]=\frac{\left[\boldsymbol{k}^{g,n}\right]}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|}.\left(\left[\boldsymbol{I}\right]-\frac{\left\{\boldsymbol{g}^{t,n}\right\}\left\langle\boldsymbol{g}^{t,n}\right\rangle}{\left\|\left\{\boldsymbol{g}^{t,n}\right\}\right\|^{2}}\right) \tag{90}$$

One can write (88) in the following condensed form:

$$\left\{\hat{\mu}^{g,n+1}\right\}=\left[\boldsymbol{B}^{g,n}\right].\left\{\delta\boldsymbol{u}^{n+1}\right\}+\left\{\rho^{g,n}\right\} \tag{91}$$

The force of slip is expressed according to the multiplier of Lagrange:

$$\left\{\boldsymbol{L}_i^{g,n+1}\right\}=\left[\boldsymbol{A}^{g}\right]^{T}.\left\{\mu^{g,n+1}\right\} \tag{92}$$

The matrix $\left[\boldsymbol{A}^{g}\right]$ not depending on displacement, the force of slip linearized can be written:

$$\left\{\boldsymbol{L}_i^{g,n+1}\right\}\underset{\text{linéarisation}}{\rightarrow}\left\{\hat{\boldsymbol{L}}_i^{g,n+1}\right\}=\left[\boldsymbol{A}^{g}\right]^{T}.\left\{\hat{\mu}^{g,n+1}\right\}=\left[\boldsymbol{K}^{g,n}\right].\left\{\delta\boldsymbol{u}^{n+1}\right\}+\left\{\boldsymbol{L}^{g,n}\right\} \tag{93}$$

With the matrix of slip $\left[\boldsymbol{K}^{g,n}\right]$ such as:

$$\left[\boldsymbol{K}^{g,n}\right]=\left[\boldsymbol{A}^{g}\right]^{T}.\left[\boldsymbol{B}^{g,n}\right] \tag{94}$$

And the vector:

$$\left\{L^{g,n}\right\}=\left[A^{g}\right]^{T}.\left\{\rho^{g,n}\right\} \tag{95}$$

$\left[K^{g,n}\right]$ make a new contribution to the tangent matrix of the problem, it is the tangent matrix "of slip". It is worth obviously:

$$\left[K^{g,n}\right]=\frac{\left[A^{g}\right]^{T}.\left[k^{g,n}\right]}{\left\|\left\{g^{t,n}\right\}\right\|}.\left([I]-\frac{\left\{g^{t,n}\right\}\left\langle g^{t,n}\right\rangle}{\left\|\left\{g^{t,n}\right\}\right\|^{2}}\right) \tag{96}$$

The second part of the expression is preceded by the sign $-$, the effect of this contribution is particularly destabilizing for the total behavior of the tangent matrix to the system, more particularly when one is far from balance and thus at the beginning of the resolution with each new step of time. One thus decides to take it into account only partially by affecting it of a coefficient $\theta\in[0,1]$ that one can modify via the keyword COEF_MATR_FROT:

$$\left[\tilde{K}_{\theta}^{g,n}\right]=\frac{\left[A^{g}\right]^{T}.\left[k^{g,n}\right]}{\left\|\left\{g^{t,n}\right\}\right\|}.\left([I]-\theta.\frac{\left\{g^{t,n}\right\}\left\langle g^{t,n}\right\rangle}{\left\|\left\{g^{t,n}\right\}\right\|^{2}}\right) \tag{97}$$

One advises to use an initial value of 0.5 for this coefficient and to decrease it if convergence is not obtained. If $\theta=0$ convergence always seems to be obtained but is particularly slow. When one is close to the solution, it is on the other hand very useful to have a value of this coefficient equalizes with 1 in order to accelerate convergence. That is done automatically in the code when the residue RESI_GLOB_RELA is lower than 10^{-3}. One replaces $\left[K^{g,n}\right]$ by $\left[\tilde{K}_{\theta}^{g,n}\right]$.
The second member is worth finally:

$$\left\{L^{g,n}\right\}=\left[A^{g}\right]^{T}.\left[k^{g,n}\right].\frac{\left\{g^{t,n}\right\}}{\left\|\left\{g^{t,n}\right\}\right\|} \tag{98}$$

In 2D, the multiplier of Lagrange for the slip is approximated by:

$$\left\{\mu^{g}\left(u_{i}\right)\right\}_{2D}=\mu.\left[\left|\mu^{c}\left(u_{i}\right)\right|\right]=\left[k^{g}\left(u_{i}\right)\right] \tag{99}$$

One can show $\left[K^{g,n}\right]_{2D}=0$. And thus:

$$\left\{\hat{L}_{i}^{g,n+1}\right\}_{2D}=\left[A^{g}\right]^{T}.\left\{\rho^{g,n}\right\} \tag{100}$$

4.3.6 Algorithms generals

The discrete methods of resolution of the problem of contact/friction are founded on an approach *uncoupled* between balance and the contact/friction. contact/friction (noted C in the second column) is treated after each iteration of Newton of the total problem (noted G in the second column) .
Without taking into account of the contact, one will note the vectors solutions with one $\sim$, for example:

$$\left\{\tilde{u}_{i}^{n}\right\}=\left\{u_{i-1}\right\}+\left\{\Delta u_{i}^{n-1}\right\}+\left\{\delta\tilde{u}^{n}\right\} \tag{101}$$

4.3.6.1 Case of the dualisation

General procedure for the dualized case is the following one:

With the iteration of Newton n			
1	G	Resolution of the problem of balance without contact, equation (53)	$\rightarrow\left\{\delta\tilde{u}^{n}\right\}$
2	G	Update of displacements without contact	$\left\{\tilde{u}_{i}^{n}\right\}=\left\{u_{i-1}\right\}+\left\{\Delta u_{i}\right\}+\left\{\delta\tilde{u}^{n}\right\}$
3	C	Modification of displacements to observe the conditions of	$\left\{\delta\tilde{u}^{n}\right\}\rightarrow\left\{\delta u^{n}\right\}$

		contact	
4	C	Update of displacements with taking into account of the contact	$\left\{\boldsymbol{u}_i^n\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}^n\right\}$
5	C	Calculation of the efforts of contact	$\left\{\boldsymbol{L}_i^{c,n}\right\}$ and $\left\{\boldsymbol{L}_i^{f,n}\right\}$
6	C	Calculation possible of matrices (case of the slip)	$\left[\boldsymbol{K}^{g,n+1}\right]$
7	G	Calculation of the interior and external efforts with modified displacements	$\left\{\boldsymbol{L}_i^{ext,n}\right\}$ and $\left\{\boldsymbol{L}_i^{int,n}\right\}$
8	G	Checking of balance	

The matrix $\left[\boldsymbol{K}^{cf,n+1}\right]$ only in the case of the slip is calculated. It will not necessarily be useful, all will depend on the convergence of Newton and the algorithm on contact/friction (threshold of Tresca converged on the problem of Coulomb).

4.3.6.2 Case of the regularization

General procedure for the regularized case (penalization) is the following one:

With the iteration of Newton n			
1	G	Resolution of the problem of balance without contact, equation (53)	$\rightarrow\left\{\delta\tilde{\boldsymbol{u}}^n\right\}$
2	G	Update of displacements without contact	$\left\{\tilde{\boldsymbol{u}}_i^n\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\tilde{\boldsymbol{u}}^n\right\}$
3	C	Calculation of the efforts of contact/friction	$\left\{\boldsymbol{L}_i^{c,n}\right\}$ and $\left\{\boldsymbol{L}_i^{f,n}\right\}$
4	C	Calculation of the modified matrices	$\left[\boldsymbol{K}^{cf,n+1}\right]$
5	G	Calculation of the interior and external efforts	$\left\{\boldsymbol{L}_i^{ext,n}\right\}$ and $\left\{\boldsymbol{L}_i^{int,n}\right\}$
6	G	Checking of balance	

The algorithm of contact/friction does not modify displacements, it is the total system which will modify them. What implies that for the penalized methods, one will always need at least two iterations of Newton, even in the elastic case: an iteration to solve the problem of balance without contact/friction and a second iteration to integrate the conditions of contact-friction who will modify total system.

5 Algorithmic resolution – Contact without friction

5.1 Connections of contact

Each node slave *potentially* in contact a statute has whose algorithms will have to determine nature. A node slave is called “potential connection of contact” or more simply “connection”. The connection term which alludes to the fact that the condition of contact is the result of the imposition of a kinematic relation on *degrees of freedom* of displacement of the node slave. These connections are joined together in various units:

- Ξ^{l} is the whole of the possible connections (active and nonactive);
- Ξ^{nc} is the whole of the nodes slaves which are not in contact (nonactive connections);
- Ξ^{c} is the whole of the nodes indeed in contact (active connections).

One thus has the following relations between the units:

- $\Xi^{\mathrm{c}} \cap \Xi^{\mathrm{nc}} = \emptyset$ because the nodes are in contact or not;
- $\Xi^{\mathrm{l}} = \Xi^{\mathrm{c}} \oplus \Xi^{\mathrm{nc}}$ because the nodes potentially in contact are it or not;

5.2 Methods dualized in contact

5.2.1 Balance of the structure in the presence of contact

It is pointed out that the equilibrium equation in the presence of contact is written :

$$\left\{\boldsymbol{L}_i^{\mathrm{int}}(\boldsymbol{u}_i)\right\} = \left\{\boldsymbol{L}_i^{\mathrm{ext}}(\boldsymbol{u}_i)\right\} - \left\{\boldsymbol{L}_i^{\mathrm{c}}\right\} \tag{102}$$

In the case of the dualized contact, the force of contact is written according to the multiplier of Lagrange of contact:

$$\left\{\boldsymbol{L}_i^{\mathrm{c}}\right\} = \left\{\boldsymbol{L}_i^{\mathrm{c}}\right\}_{\mathrm{dual}} = [\boldsymbol{A}^{\mathrm{c}}]^T . \left\{\mu_i^{\mathrm{c}}\right\} \tag{103}$$

After linearization of the equilibrium equation (102) , the tangent matrix is introduced $\left[\boldsymbol{K}^{\mathrm{m},n-1}\right]$ who will contain the contributions resulting from the linearization of the interior and external efforts and the matrix $\left[\boldsymbol{K}^{\mathrm{c},n-1}\right]$ for forces of contact , one finds $\left\{\delta \tilde{\boldsymbol{u}}^n\right\}$, increment of solution of the problem of balance to the iteration n but without application of the law of contact :

$$\left(\left[\boldsymbol{K}^{\mathrm{m},n-1}\right] + \left[\boldsymbol{K}^{\mathrm{c},n-1}\right]\right) . \left\{\delta \tilde{\boldsymbol{u}}^n\right\} = \left\{\boldsymbol{L}_i^{\mathrm{ext},n-1}\right\} - \left\{\boldsymbol{L}_i^{\mathrm{int},n-1}\right\} - \left\{\boldsymbol{L}_i^{\mathrm{c},n-1}\right\} \tag{104}$$

We saw that the forces of contact in the dualized case do not depend pas du displacement (see § 4.3.5) . One thus has $\left[\boldsymbol{K}^{\mathrm{c},n-1}\right] = 0$. What enables us to express the balance of the structure with taking into account of the forces of contact (while noting $[\boldsymbol{K}] = \left[\boldsymbol{K}^{\mathrm{m},n-1}\right]$ and $\{\boldsymbol{F}\} = \left\{\boldsymbol{L}_i^{\mathrm{ext},n-1}\right\} - \left\{\boldsymbol{L}_i^{\mathrm{int},n-1}\right\} - \left\{\boldsymbol{L}_i^{\mathrm{c},n-1}\right\}$ to reduce) :

$$[\boldsymbol{K}] . \left\{\delta \tilde{\boldsymbol{u}}^n\right\} = \{\boldsymbol{F}\} \tag{105}$$

The solution $\left\{\tilde{\boldsymbol{u}}_i^n\right\}$ is obtained after balance and front application of the conditions of contact. She is written:

$$\left\{\tilde{\boldsymbol{u}}_i^n\right\} = \left\{\boldsymbol{u}_{i-1}\right\} + \left\{\Delta \boldsymbol{u}_i^{n-1}\right\} + \left\{\delta \tilde{\boldsymbol{u}}^n\right\} \tag{106}$$

To obtain the solution $\left\{\tilde{\boldsymbol{u}}_i^n\right\}$, we did not apply the conditions of contact (law of Signorini) for the current iteration of Newton n . On the other hand, reactions of contact calculated with the preceding iteration of Newton $n-1$ are well taken into account in $\{\boldsymbol{F}\}$. To completely solve the problem of balance with contact/friction, with the iteration n , it is necessary to apply the law of Signorini, which is expressed by the following system :

$$\begin{cases} \left[\boldsymbol{A}^{\mathrm{c}}\right] . \left\{\boldsymbol{u}_i^n\right\} \leq \left\{\boldsymbol{d}_{\mathrm{ini}}^{\mathrm{c}}\right\} & (a) \\ \left\{\mu_i^{\mathrm{c}}\right\} \geq 0 & (b) \\ \left\langle \mu_i^{\mathrm{c}} \right\rangle . \left(\left[\boldsymbol{A}^{\mathrm{c}}\right] . \left\{\boldsymbol{u}_i^n\right\}\right) = 0 & (c) \end{cases} \tag{107}$$

One points out the interpretation of conditions of Signorini:

- The equation (107a) represent the geometrical conditions of noninterpenetration, the component inequality understanding itself by component (each line is relative to a potential couple of contact).

- The equation (107b) express the absence of opposition to separation (surfaces of contact can know only compressions), it is the condition known as of intensility.
- The equation (107c) is the condition of compatibility. When for a connection data the multiplier of Lagrange is nonnull, there is contact and thus the game is null. When the game is nonnull (two surfaces are not in contact), the associated multiplier must be null (not compression).

L 'application of the law of Signorini will modify the increment of displacement $\left\{\delta \tilde{\boldsymbol{u}}^{n}\right\}$ (which becomes $\left\{\delta \boldsymbol{u}^{n}\right\}$), from where the solution obtained after balance and *afterwards* application of the law of contact which is written:

$$\left\{\boldsymbol{u}_{i}^{n}\right\}=\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta \boldsymbol{u}_{i}^{n-1}\right\}+\left\{\delta \boldsymbol{u}^{n}\right\} \tag{108}$$

5.2.2 System reduced to the active connections

We will write the system allowing to completely solve the problem of balance with taking into account of the law of Signorini. The idea is to transform the inequalities of the system (107) in equalities. One starts by evaluating game given by the displacement calculated before the correction of the contact $\left\{\tilde{\boldsymbol{d}}^{c, n}\right\}$, for all the connections [7] :

$$\left\lfloor\left\{\tilde{\boldsymbol{d}}^{c, n}\right\}\right\rfloor_{\Xi^{1}}=\left\lfloor\left\{\boldsymbol{d}_{\mathrm{ini}}^{c}\right\}-\left[\boldsymbol{A}^{c}\right] .\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta \boldsymbol{u}_{i}^{n-1}\right\}+\left\{\delta \tilde{\boldsymbol{u}}^{n}\right\}\right)\right\rfloor_{\Xi^{1}} \tag{109}$$

It is said that a connection J is active if sound game $\left\lfloor\left\{\tilde{\boldsymbol{d}}^{c, n}\right\}\right\rfloor_{J \in \Xi^{1}}$ is negative, which indicates an interpenetration (the condition of contact is not observed) , this connection thus becomes a connection of contact and thus makes it possible to define the unit *initially* applied connections of contact Ξ^{c} :

$$\Xi^{c}=\left\{J \in \Xi^{1} \middle| \left\lfloor\left\{\tilde{\boldsymbol{d}}^{c, n}\right\}\right\rfloor_{J}<0\right\} \tag{110}$$

One applies that, for these active connections, the effective game will be null, and that thus the inequality $\left\lfloor\left[\boldsymbol{A}^{c}\right] .\left\{\boldsymbol{u}_{i}^{n}\right\}\right\rfloor_{\Xi^{1}} \leq\left\lfloor\left\{\boldsymbol{d}_{\mathrm{ini}}^{c}\right\}\right\rfloor_{\Xi^{1}}$ becomes an equality for the unit the active connections:

$$\left\lfloor\left[\boldsymbol{A}^{c}\right] .\left\{\boldsymbol{u}_{i}^{n}\right\}=\left\{\boldsymbol{d}_{\mathrm{ini}}^{c}\right\}\right\rfloor_{\Xi^{c}} \tag{111}$$

If one uses $\left\{\boldsymbol{d}^{c, n-1}\right\}$, game evaluated before the current iteration of Newton:

$$\left\{\boldsymbol{d}^{c, n-1}\right\}=\left\{\boldsymbol{d}_{\mathrm{ini}}^{c}\right\}-\left[\boldsymbol{A}^{c}\right] .\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta \boldsymbol{u}_{i}^{n-1}\right\}\right) \tag{112}$$

With:

$$\left\{\tilde{\boldsymbol{d}}^{c, n}\right\}=\left\{\boldsymbol{d}^{c, n-1}\right\}-\left[\boldsymbol{A}^{c}\right] .\left\{\delta \tilde{\boldsymbol{u}}^{n}\right\} \tag{113}$$

Equality (111) writing finally:

$$\left\lfloor\left[\boldsymbol{A}^{c}\right] .\left\{\delta \boldsymbol{u}^{n}\right\}=\left\{\boldsymbol{d}^{c, n-1}\right\}\right\rfloor_{\Xi^{c}} \tag{114}$$

The "mixed" system inequality/equality (containing the equilibrium equation and the conditions of Signorini) transforms itself finally into simple system which treats only equalities, on the basis of applied connection of contact Ξ^{c} :

$$\begin{cases}\left[\boldsymbol{K}\right] .\left\{\delta \boldsymbol{u}^{n}\right\}+\left[\boldsymbol{A}^{c}\right]^{T} .\left\{\mu_{i}^{c}\right\}=\left\{\boldsymbol{F}\right\} \\ \left\lfloor\left[\boldsymbol{A}^{c}\right] .\left\{\delta \boldsymbol{u}^{n}\right\}=\left\{\boldsymbol{d}^{c, n-1}\right\}\right\rfloor_{\Xi^{c}}\end{cases} \tag{115}$$

The unknown factors are sought $\left\{\delta \boldsymbol{u}^{n}\right\}$ and $\left\{\mu_{i}^{c}\right\}$ who are solutions of the following system:

$$\begin{bmatrix}\left[\boldsymbol{K}\right] & \left[\boldsymbol{A}^{c}\right]^{T} \\ \left[\boldsymbol{A}^{c}\right] & [\boldsymbol{0}]\end{bmatrix} .\begin{Bmatrix}\left\{\delta \boldsymbol{u}^{n}\right\} \\ \left\{\mu_{i}^{c}\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{\boldsymbol{F}\right\} \\ \left\{\boldsymbol{d}^{c, n-1}\right\}\end{Bmatrix} \Leftrightarrow\left[\boldsymbol{K}^{c}\right] .\begin{Bmatrix}\left\{\delta \boldsymbol{u}^{n}\right\} \\ \left\{\mu_{i}^{c}\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{\boldsymbol{F}\right\} \\ \left\{\boldsymbol{d}^{c, n-1}\right\}\end{Bmatrix} \tag{116}$$

It will be noticed that the dualized formulation of the condition of not-interpenetration translates the stationnarity of the Lagrangian one L thus defined:

7 One notes by $\lfloor\rfloor_{\square}$ operations that one makes on a subset connections such as defined in the § 5.1 .

$$L\left(\left\{\delta \boldsymbol{u}^{n}\right\},\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}\right)=\frac{1}{2}.\left\langle\delta \boldsymbol{u}^{n}\right\rangle.\left[\boldsymbol{K}\right].\left\{\delta \boldsymbol{u}^{n}\right\}-\left\langle\delta \boldsymbol{u}^{n}\right\rangle.\left\{\boldsymbol{F}\right\}+\left\langle\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\rangle.\left(\left[\boldsymbol{A}^{\mathrm{c}}\right].\left\{\delta \boldsymbol{u}^{n}\right\}-\left\{\boldsymbol{d}^{\mathrm{c},n-1}\right\}\right) \tag{117}$$

This stationnarity is expressed in the form of the problem of point-saddles:

$$\min_{\left\{\delta \boldsymbol{u}^{n}\right\}}\max_{\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}\geq 0} L\left(\left\{\delta \boldsymbol{u}^{n}\right\},\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}\right) \tag{118}$$

5.2.3 Method of the active constraints – ALGO_CONT=' CONTRAINTE'

It is a question of generalizing the approach of the preceding paragraph by adopting an iterative algorithm which works in three phases:

1. To make assumptions on the state of the connections (transformations inequalities → equality);
2. To solve the new system thus created;
3. To check the initial assumptions and to buckle (possibly) in 1.

One will be able to find a description complete of the method with the theoretical justifications necessary in [1] and [2]. The principle is the following: a set of constraints known as active are applied, which correspond to a null game (the relation inequality becomes an equality); one solves the system of equations obtained in this subspace, and one looks at if the starting postulate were justified. If the selected unit were too small (active connections had been forgotten), one adds with the unit the connection violating more the condition of not-interpenetration; if the selected unit were too large (presumedly active connections are not it in fact not), one removes from the unit the most improbable connection *i.e.* that whose multiplier of Lagrange violates more the condition of intensity. The fact of removing or of adding only one connection with each iteration of the method guarantees convergence in a finished number iterations inferior or equal to twice the maximum number of connections.
In elasticity, at the end of the iterations of active constraints, there is a result converged within the meaning of Newton. In plasticity or if the geometry is reactualized, it is not the case because several iterations of Newton are necessary to obtain balance. After each iteration of Newton, one launches the algorithm of active constraints to satisfy the conditions with contact. Thus, in elasticity, one will necessarily converge for each step in an iteration if REAC_GEOM= 'WITHOUT'.

5.2.3.1 Writing of the iterative problem

One leaves the increment obtained without treating the contact $\left\{\delta \tilde{\boldsymbol{u}}^{n}\right\}$ and one carries out the iterations of active constraints k until clean convergence of this algorithm. Convergence within the meaning of the active constraints is obtained when no connection violates the kinematic condition (inequality (107 has)) and when the associated multipliers of Lagrange are all positive (inequality (107 B)) .
One notes k iterations of active constraints. Lhas starting solution without correction of the contact is $\left\{\delta \tilde{\boldsymbol{u}}^{n}\right\}$ and the increment added by the new iteration of contact is $\left\{\delta_{k}\right\}$. One notes:

$$\begin{aligned}\left\{\delta u_{0}^{n}\right\}&=\left\{\delta \tilde{u}^{n}\right\}\\ \left\{\delta_{0}\right\}&=\left\{0\right\}\\ \left\{\delta u_{k}^{n}\right\}&=\left\{\delta u_{k-1}^{n}\right\}+\left\{\delta_{k}\right\}\end{aligned} \tag{119}$$

One places oneself at the iteration of contact k . One seeks to solve the system (115):

$$\begin{cases}\left[\boldsymbol{K}\right].\left\{\delta \boldsymbol{u}_{k}^{n}\right\}+\left[\boldsymbol{A}_{k}^{\mathrm{c}}\right]^{T}.\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}=\left\{\boldsymbol{F}\right\}\\ \left\{\left[\boldsymbol{A}_{k}^{\mathrm{c}}\right].\left\{\delta \boldsymbol{u}_{k}^{n}\right\}=\left\{\boldsymbol{d}^{\mathrm{c},n-1}\right\}\right\}_{\Xi_{k}^{\mathrm{c}}}\end{cases} \tag{120}$$

It is noted that the multiplier of Lagrange of the contact $\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}$ is not solved in an incremental way but in a total way (on the step of time). While injecting (119) in this system, one obtains:

$$\begin{cases}\left[\boldsymbol{K}\right].\left\{\delta \boldsymbol{u}_{k-1}^{n}\right\}+\left[\boldsymbol{K}\right].\left\{\delta_{k}\right\}+\left[\boldsymbol{A}_{k}^{\mathrm{c}}\right]^{T}.\left\{\boldsymbol{\mu}_{i}^{\mathrm{c}}\right\}=\left\{\boldsymbol{F}\right\}\\ \left\{\left[\boldsymbol{A}_{k}^{\mathrm{c}}\right].\left(\left\{\delta \boldsymbol{u}_{k-1}^{n}\right\}+\left\{\delta_{k}\right\}\right)=\left\{\boldsymbol{d}^{\mathrm{c},n-1}\right\}\right\}_{\Xi_{k}^{\mathrm{c}}}\end{cases} \tag{121}$$

The first equation can be rewritten:

$$\left\{\delta_k\right\}=\left[\boldsymbol{K}\right]^{-1}.\left\{\boldsymbol{F}\right\}-\left\{\delta\,\boldsymbol{u}_{k-1}^{n}\right\}-\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_k^{\mathrm{c}}\right]^{T}.\left\{\mu_i^{\mathrm{c}}\right\} \tag{122}$$

By taking account that $\left\{\delta\,\tilde{\boldsymbol{u}}^{n}\right\}=\left[\boldsymbol{K}\right]^{-1}.\left\{\boldsymbol{F}\right\}$, one simplifies:

$$\left\{\delta_k\right\}=\left\{\delta\,\tilde{\boldsymbol{u}}^{n}\right\}-\left\{\delta\,\boldsymbol{u}_{k-1}^{n}\right\}-\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_{\boldsymbol{k}}^{\boldsymbol{c}}\right]^{T}.\left\{\mu_{\boldsymbol{i}}^{\boldsymbol{c}}\right\} \tag{123}$$

From the applied state of the active connections:

$$\left[\left[\boldsymbol{A}_k^{c}\right].\left(\left\{\delta\,\boldsymbol{u}_{k-1}^{n}\right\}+\left\{\delta_k\right\}\right)=\left\{\boldsymbol{d}^{c,n-1}\right\}\right|_{\Xi_k^{c}} \tag{124}$$

And by taking again the expression of $\left\{\boldsymbol{d}^{c,n-1}\right\}$ data by (57):

$$\left[\left[\boldsymbol{A}_k^{c}\right].\left\{\delta\,\boldsymbol{u}_{k-1}^{n}\right\}+\left[\boldsymbol{A}_k^{c}\right].\left\{\delta_k\right\}\right|_{\Xi_c^k}=\left[\left(\boldsymbol{d}_{ini}^{c}\right)-\left[\boldsymbol{A}_k^{c}\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\,\boldsymbol{u}_i^{n-1}\right\}\right)\right|_{\Xi_k^{c}} \tag{125}$$

One uses the expression of $\left\{\delta_k\right\}$ data by (122):

$$\left[\left[\boldsymbol{A}_k^{c}\right].\left\{\delta\,\tilde{\boldsymbol{u}}^{n}\right\}-\left[\boldsymbol{A}_k^{c}\right].\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_k^{c}\right]^{T}.\left\{\mu_i^{c}\right\}\right|_{\Xi_k^{c}}=\left[\left(\boldsymbol{d}_{ini}^{c}\right)-\left[\boldsymbol{A}_k^{c}\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\,\boldsymbol{u}_i^{n-1}\right\}\right)\right|_{\Xi_k^{c}} \tag{126}$$

With final, multipliers of Lagrange $\left\{\mu_i^{\mathrm{c}}\right\}$ for the connections of contact are solutions of the following system:

$$\left[-\left[\boldsymbol{A}_k^{c}\right].\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_k^{c}\right]^{T}.\left\{\mu_i^{c}\right\}=\left\{\tilde{\boldsymbol{d}}^{c,n}\right\}\right|_{\Xi_k^{c}} \tag{127}$$

It is noticed that (127) is the complement of Schur $\left[\boldsymbol{K}\right]_{\mathrm{schur}}=\left[\boldsymbol{A}_k^{\mathrm{c}}\right].\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_k^{\mathrm{c}}\right]^{T}$ matrix $\left[\boldsymbol{K}^{\mathrm{c}}\right]$. One can finally calculate the increments of displacement $\left\{\delta_k\right\}$ with:

$$\left\{\delta_k\right\}=\sum_{l=0}^{k-1}\left\{\delta_l\right\}-\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_{\boldsymbol{k}}^{\boldsymbol{c}}\right]^{T}.\left\{\mu_{\boldsymbol{i}}\right\}^{c} \tag{128}$$

The resolution of (127) is the most expensive part in time calculation of the algorithm. All the effectiveness of the strategy consists in using the fact that one solves this system only on the whole of the connections active Ξ_k^{c}, two properties are thus used.

- The calculation of the complement of Schur $\left[\boldsymbol{K}\right]_{\mathrm{schur}}$ fact always call to a factorization of the type LDL^{T} , which makes it possible to save time, because such a factorization has the property remarkable to be incremental, i.e. the addition of a connection do not oblige to rebuild factorized since the beginning but only the part which one modifies .
- The resolution (gone up descent/) is done also only on the whole of the active connections Ξ_k^{c} .

5.2.3.2 Validity of the whole of active connections selected

At the end of each iteration of active constraint, it is advisable to check if the unit Ξ_k^{c} is quite correct. That is to say the connection $J\in\Xi^{1}$, three situations are possible:

1) Relative displacement compensates for the initial game $\left[\left[\boldsymbol{A}_k^{\mathrm{c}}\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\,\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\,\boldsymbol{u}_k^{n}\right\}\right)\right|_{J\in\Xi^{1}}=\left|\left\{\boldsymbol{d}_{\mathrm{ini}}^{\mathrm{c}}\right\}\right|_{J\in\Xi^{1}}$;
2) Relative displacement is lower than the initial game $\left[\left[\boldsymbol{A}_k^{\mathrm{c}}\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\,\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\,\boldsymbol{u}_k^{n}\right\}\right)\right|_{J\in\Xi^{1}}<\left|\left\{\boldsymbol{d}_{\mathrm{ini}}^{\mathrm{c}}\right\}\right|_{J\in\Xi^{1}}$;
3) Relative displacement is higher than the initial game $\left[\left[\boldsymbol{A}_k^{\mathrm{c}}\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\,\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\,\boldsymbol{u}_k^{n}\right\}\right)\right|_{J\in\Xi^{1}}>\left|\left\{\boldsymbol{d}_{\mathrm{ini}}^{\mathrm{c}}\right\}\right|_{J\in\Xi^{1}}$.

The situation (3) is prohibited because it corresponds to a violation of the condition of not-interpenetration. The situation (1) corresponds to a connection known as active, the situation (2) with a connection not - active.
At the beginning of the iteration k algorithm, one had applied a set of active connections Ξ_k^{c} . One found an increment possible $\left\{\delta_k\right\}$ unknown factors under these assumptions and one now will check that this increment is compatible with the assumptions. In practice, that consists in making two checks:

1) The unit Ξ_k^c is it too small? It is checked that nonactive supposed connections do not violate the condition of not-interpenetration, if not one activates one of them, that which violates more the condition.
2) The unit Ξ_k^c is it too large? It is checked that the presumedly active connections are associated with multipliers of contact $\left\{\mu_i^c\right\}$ positive or worthless, if not one disables of it that which violates more the condition.

The unit Ξ_k^c is it too small?

To check that the unit Ξ_k^c is not too small, one will calculate for all the presumedly inactive connections, the following quantity:

$$\rho_J=\left|\frac{\left\{\boldsymbol{d}_{\text{ini}}^c\right\}-\left[\boldsymbol{A}_k^c\right].\left(\left\{\boldsymbol{u}_{i-1}\right\}+\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\boldsymbol{u}_k^n\right\}\right)}{\left[\boldsymbol{A}_k^c\right].\left\{\delta_k\right\}}\right|_{J\in\Xi_k^{nc}}=\left|\frac{\left\{\boldsymbol{d}^{c,n-1}\right\}-\left[\boldsymbol{A}_k^c\right].\left\{\delta\boldsymbol{u}_k^n\right\}}{\left[\boldsymbol{A}_k^c\right].\left\{\delta_k\right\}}\right|_{J\in\Xi_k^{nc}} \quad (129)$$

There are two cases:

1) If $\left|\left[\boldsymbol{A}_k^c\right].\left\{\delta_k\right\}\right|_{J\in\Xi_k^{nc}}<0$, game for the connection J will increase, and thus the presumedly inactive connection remains in this state, ρ_J is strictly higher than 1 ;
2) S I $\left|\left[\boldsymbol{A}_k^c\right].\left\{\delta_k\right\}\right|_{J\in\Xi_k^{nc}}>0$, game for the connection J will decrease , and thus the connection supposed not - active will be activated, ρ_J is lower or equal to 1 ;

One thus examines $\bar{\rho}=\underset{J}{Min}\,\rho_j$ on the whole of the connections J declared nonactive. If $\bar{\rho}<1$, that indicates that a connection at least is violated (situation (3)): one then adds with the list of the active connections the number of the connection whose interpenetration is largest, i.e. that which realizes the minimum of ρ_J and one writes $\left\{\delta\boldsymbol{u}_{k+1}^n\right\}=\left\{\delta\boldsymbol{u}_k^n\right\}+\bar{\rho}.\left\{\delta_k\right\}$ (what corresponds to a null game for the added connection).

The algorithm used is presented:

If $\Xi_k^c=\Xi^1$			
	$\bar{\rho}=1$		
If not			
	$\bar{\rho}=1$		
	Buckle on the connections $J\in\Xi_k^{nc}$		
		Calculation of $\alpha_J=\left\|\left[\boldsymbol{A}_k^c\right].\left\{\delta_k\right\}\right\|_J$	
		If $\alpha_J<0$	
			$\rho_J=\dfrac{\left\|\left\{\boldsymbol{d}^{c,n-1}\right\}-\left[\boldsymbol{A}_k^c\right].\left\{\delta\boldsymbol{u}_k^n\right\}\right\|_J}{\alpha_J}$
			$\bar{\rho}=\min\left(\rho_J,\bar{\rho}\right)$
			$J_{min}=J$
	Fin Boucle		

At exit of this algorithm, there will be the number of the connection which violates more the condition of nonpenetration J_{min} and the value $\bar{\rho}$: $\left(J_{min},\bar{\rho}\right)=IsPetit\left(\Xi_k^c\right)$

The unit Ξ_k^c is it too large?

The second checking consists in wondering whether the whole of the active connections is too large. One places oneself now if $\bar{\rho} \geq 1$, i.e. one knows that Ξ_k^c is not too small. Then:

- If no connection is active, the method converged towards a state without contact;
- If there are presumedly active connections :
  - If all multipliers of Lagrange $\left\{\mu_i^c\right\}$ are positive or worthless, one also converged towards a state with effective contact;
  - If there exist multipliers of Lagrange $\left\{\mu_i^c\right\}$ negative, the corresponding connections should not be active : one withdraws from the whole of the active connections the connection whose negative multiplier is largest in absolute value.

At exit of this algorithm, there will be the number of the connection which violates more the condition of intensity: $\left\{J_{max}\right\} = IsGrand\left(\Xi_k^c\right)$.

5.2.3.3 Algorithm

With final, the algorithm for the active constraints is the following:

Ini	$\delta u_0^n = \delta \tilde{u}^n$ Calculation of $\left\{\tilde{d}^{c,n}\right\}_{\Xi^1}$ Evaluation of $\Xi_0^c = \left\{J \in \Xi^1 \middle\| \left\{\tilde{d}^{c,n}\right\}_J < 0\right\}$ and $\left[A_0^c\right]$		
B_k	Buckle on the active constraints $k = 1, Iter_{max}$		
	$\left\{\delta_{k-1}\right\} = \left\{\delta \tilde{u}^n\right\} - \left\{\delta u_{k-1}^n\right\}$ $\Xi_k^c \leftarrow \Xi_{k-1}^c$ and $\left[A_k^c\right] \leftarrow \left[A_{k-1}^c\right]$		
	If $\Xi_k^c \neq \emptyset$		
		Calculation of $\left[K_k\right]_{schur} = \left[A_k^c\right].\left[K\right]^{-1}.\left[A_k^c\right]^T$	
		Factorization of $\left\|\left[K_k\right]_{schur}\right\|_{\Xi_k^c}$	
		Resolution of $\left\|\left[K_k\right]_{schur}.\left\{\mu_i^c\right\} = \left\{\tilde{d}^{c,n}\right\}\right\|_{\Xi_k^c} \rightarrow \left\|\left\{\mu_i^c\right\}\right\|_{\Xi_k^c}$	
		$\left\{\delta_k\right\} = \left\{\delta_{k-1}\right\} - \left[K\right]^{-1}.\left[A_k^c\right]^T.\left\{\mu_i^c\right\}$	
	$\left\{J_{min}, \bar{\rho}\right\} = IsPetit\left(\Xi_k^c\right)$		
	$\left\{\delta u_k^n\right\} = \left\{\delta u_{k-1}^n\right\} + \bar{\rho}.\left\{\delta_k\right\}$		
	If $\bar{\rho} < 1$		
		$\Xi_k^c \leftarrow \Xi_k^c + \left\{J_{min}\right\}$	
	If not		
		If $\Xi_k^c = \emptyset$	
			Fine Goto F
		$\left\{J_{max}\right\} = IsGrand\left(\Xi_k^c\right)$	
		If $\left\{J_{max}\right\} = 0$	

				Fine Goto F
			If not	
				$\Xi_k^c \leftarrow \Xi_k^c - \{J_{max}\}$
B_k	Buckle $k=k+1$			
F	END			
	Calculation of $\{L_i^c\}=[A^c]^T.\{\mu_i^c\}$			

5.2.4 Method of the Gradient Combined Project – ALGO_CONT=' GCP'

The method of resolution presented in this part is an application of the algorithm of the Gradient Combined Project. It is very precisely the iterative version of the method of the Active Constraints presented in the preceding part.

5.2.4.1 Reformulation of the problem of contact

One recalls that the system to be solved with each iteration of Newton n is the following:

$$\begin{cases} [K].\{\delta u^n\}+[A^c]^T.\{\mu_i^c\}=\{F\} \\ [A^c].\{\delta u^n\}\leq\{d^{c,n-1}\} \end{cases} \quad (130)$$

It comes from the dualisation of the conditions of contact and it translates the stationnarity of the Lagrangian one L thus defined:

$$L(\{\delta u^n\},\{\mu_i^c\})=\frac{1}{2}.\langle\delta u^n\rangle.[K].\{\delta u^n\}-\langle\delta u^n\rangle.\{F\}+\langle\mu_i^c\rangle.([A^c].\{\delta u^n\}-\{d^{c,n-1}\}) \quad (131)$$

This stationnarity can be expressed in the form of the problem of point-saddles:

$$\min_{\{\delta u^n\}}\max_{\{\mu_i^c\}\geq 0} L(\{\delta u^n\},\{\mu_i^c\}) \quad (132)$$

However the minimum of L compared to δu_i is known; it has as an expression:

$$\{\delta u^n\}=[K]^{-1}.(\{F\}-[A^c]^T.\{\mu_i^c\}) \text{ and } \langle\delta u^n\rangle=(\langle F\rangle-\langle\mu_i^c\rangle.[A^c]).[K]^{-1} \quad (133)$$

One does not have thus any more but to make the maximization of L compared to $\{\mu_i^c\}\geq 0$. Knowing that to maximize a functional calculus J is equivalent minimizing $-J$, one is reduced to the problem of minimization according to:

$$\min_{\{\mu_i^c\}\geq 0} H(\{\mu_i^c\}) \quad (134)$$

Where the functional calculus H is written:

$$H(\{\mu_i^c\})=\frac{1}{2}.\langle\mu_i^c\rangle.[A^c].[K]^{-1}.[A^c]^T.\{\mu_i^c\}+\langle\mu_i^c\rangle.(\{d^{c,n-1}\}-[A^c].\{\delta\tilde{u}^n\})+\frac{1}{2}.\langle F\rangle.[K]^{-1}.\{F\} \quad (135)$$

This expression is the dual form of the problem of contact: it utilizes the field of multipliers of Lagrange $\{\mu_i^c\}$ and any more the field of displacement $\{\delta u^n\}$. The problem to be solved is now a minimization under constraint of positivity of the unknown factor. The method of the Gradient Combined Project is a simple and effective method for this kind of problem. One will note:

- $\{Z\}$ direction of research
- $\{r\}$ the under-gradient and $\{r^p\}$ its packaged version

5.2.4.2 Linear research

In an algorithm of combined gradient, it is necessary to estimate a step of advance. Two alternatives of linear research are available, acceptable or not. They are represented graphically on the figure 5.2.4.2-a . The acceptable alternative, which forces to remain in the convex field acceptable armature a single resolution by iteration; it leads to a rather regular convergence. The not-acceptable alternative, which authorize to leave then the acceptable field for there reprojeter, induced two resolutions by iteration; it leads to a rather erratic convergence but generally faster than the acceptable method.

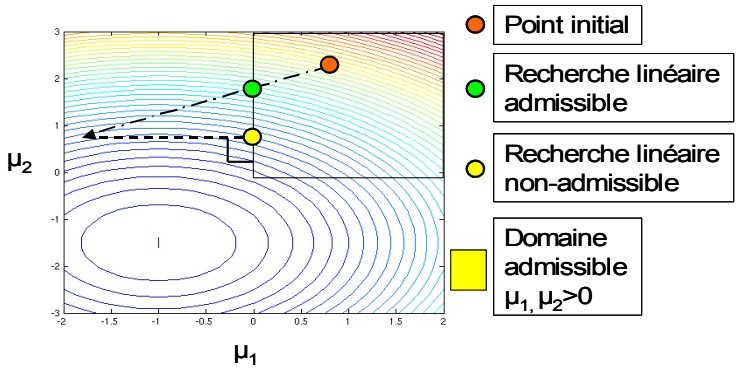

Figure 5.2.4.2-a: alternatives of linear research.

Here the algorithm of linear research:

RechLine		
Ini	State of contact $\left[A^c\right] \leftarrow \left[A_k^c\right]$ and $\Xi^c \leftarrow \Xi_k^c$	
	Calculation of the second member $\{F\} = \left[A^c\right]^T . \{Z\}$	
	Resolution of $[K] . \{\delta a\} = \{F\} \rightarrow \{\delta a\}$	
	Calculation of $\alpha = \frac{\{r\} . \{r^p\}}{\{\delta a\} . \{F\}}$	
	If ACCEPTABLE	
		$\forall J \in \Xi^1 : \text{Si } \{Z\}_J < 0 \text{ alors } \alpha = \min_{J \in \Xi^1} \left(\alpha, -\frac{\{\mu^c\}_J}{\{Z\}_J} \right)$
		$\{\mu_{k+1}^c\} = \{\mu_k^c\} + \alpha . \{Z\}$
		$\{\delta u_{k+1}\} = \{\delta u_k\} - \alpha . \{\delta a\}$
	If not	
		$\{\mu_{k+1}^c\} = \{\mu_k^c\} + \alpha . \{Z\}$
		$\{\mu_{k+1}^c\} \leftarrow \max\left(\{\mu_{k+1}^c\}, 0\right)$
		Calculation of the second member $\{F\} = \left[A^c\right]^T . \{\mu^c\}$
		Resolution of $[K] . \{\delta a\} = \{F\} \rightarrow \{\delta a\}$
		$\{\delta u_{k+1}\} = \{\delta \tilde{u}^n\} - \{\delta a\}$
F	END	

At exit of this algorithm, there will be the displacement and the multipliers of Lagrange after calculation of the step of advance: $\left(\{\delta u_{k+1}\}, \{\mu_{k+1}^c\}\right) = RechLine\left(\{Z\}, \{r\}, \{r^p\}\right)$. This algorithm understands two resolutions but

by using the matrix $[K]$ already factorized (it is the total matrix of the problem of balance), only one descent-increase is thus made, which is little expensive.
The not-acceptable alternative is obliged to recompute an increment of displacement compatible with the constraint of positivity on the multipliers of contact.

5.2.4.3 Prepacking

In the algorithm of the GCP, it is mentioned an optional call to a preconditionnor. The goal of this preconditionnor is to accelerate the convergence of the method. Its definition comes from the following considerations of functional analysis: in the phase of update $\{\mu_{k+1}^c\}=\{\mu_k^c\}+\alpha.\{Z_k\}$ and if one did not call upon the preconditionnor, the field $\{\mu_k^c\}$ belongs to $H^{-1/2}(\Gamma_c)$ while $\{Z_k\}$ belongs to $H^{1/2}(\Gamma_c)$. This sum thus does not have a mathematical direction. For him to give one of them, it is necessary to send $\{Z_k\}$ in $H^{-1/2}(\Gamma_c)$, it is the operation of pre-packaging . Knowing that $\{Z_k\}$ is obtained by the expression $\{Z_k\}=\{r_k^p\}+\gamma_k.\{Z_{k-1}\}$, it is on $\{r_k^p\}$ that one will operate. It is what is made by the preconditionnor by solving the following auxiliary problem:

$$\begin{cases}\left|[K].\{a\}+[A_k^c]^T.\{r_k^p\}\right|_{\Xi_k^c}=0\\ \left|[A_k^c].\{a\}=\{r_k^p\}\right|_{\Xi_k^c}\end{cases} \quad (136)$$

One thus solves a problem of displacement imposed on the part where the contact is effective and one recovers the reactions to embedding $\{r_k^p\}$ who belong well to $H^{-1/2}(\Gamma_c)$. In the terminology of the decomposition of field, it is a preconditionnor of Dirichlet.
To solve this auxiliary problem, one uses also an algorithm of combined gradient but without projection because no constraint of positivity appears there. This iterative approach authorizes an approximate resolution so as to save computing time.

- $\{X\}$ direction of research;
- $\{s\}$ the under-gradient.

Here the algorithm of the pre-conditioner:

PreCond		
Ini	$\{a_0\}=0$ State of contact $[A^c]\leftarrow[A_k^c]$ and $\Xi^c\leftarrow\Xi_k^c$	
	If $\Xi^c=\emptyset$	
		Fine Goto F
B_p	Buckle on $p=1, Iter_{max}$	
		Calculation of the gradient $\{s_p\}=[A^c].\{a_{p-1}\}-\{r\}$
		Calculation of the residue $\varepsilon=\max_{J\in\Xi^c}\{s_p\}$
		If $\varepsilon<\varepsilon_{precond}$
		Fine Goto F
		If $p=1$ or reactualization
		$\{X_p\}=\{s_p\}$
		If not, conjugation

			$\beta=\frac{\langle \boldsymbol{s}_p\rangle.\{\boldsymbol{s}_p\}}{\langle \boldsymbol{s}_{p-1}\rangle.\{\boldsymbol{s}_{p-1}\}}$
			$\{\boldsymbol{X}_p\}=\{\boldsymbol{s}_p\}+\beta.\{\boldsymbol{X}_{p-1}\}$
		Calculation of the second member $\{\boldsymbol{F}\}=\left\{\left[\boldsymbol{A}^{\mathrm{c}}\right]^T.\{\boldsymbol{X}_p\}\right\}_{\Xi^c}$	
		Resolution of $[\boldsymbol{K}].\{\delta\boldsymbol{a}\}=\{\boldsymbol{F}\}\rightarrow\{\delta\boldsymbol{a}\}$	
		Calculation of the complement of Schur $\{\delta\boldsymbol{a}\}_{schur}=\left\{\left[\boldsymbol{A}^c\right].\{\delta\boldsymbol{a}\}\right\}_{\Xi^c}$	
		Calculation of the step of advance $\alpha=\frac{\langle \boldsymbol{s}_p\rangle.\{\boldsymbol{s}_p\}}{\langle \boldsymbol{F}\rangle.\{\delta\boldsymbol{a}\}_{\mathrm{schur}}}$	
		New under-gradient $\{\boldsymbol{s}_p\}=\{\boldsymbol{s}_{p-1}\}+\alpha.\{\boldsymbol{X}_p\}$	
		New displacement $\{\boldsymbol{a}_p\}=\{\boldsymbol{a}_{p-1}\}-\alpha.\{\delta\boldsymbol{a}\}$	
B_k	Fin Boucle $k=k+1$		
F	END		
	$\{\boldsymbol{r}_k^p\}=\{\boldsymbol{s}_p\}$ Projection: $(max_J(r_k^p,0)\,si\,\mu_J<0)$		

At exit of this algorithm, there will be packaged under-gradient: $\{\boldsymbol{r}_k^p\}=PreCond(\{\boldsymbol{r}_k\})$.

5.2.4.4 Algorithm

Here the total algorithm for method GCP:

Ini	If $\langle\mu_{i-1}^{\mathrm{c}}\rangle\cdot\{\mu_{i-1}^{\mathrm{c}}\}\neq 0$ then $[\boldsymbol{K}].\{\boldsymbol{v}\}=[\boldsymbol{A}_{i-1}^{\mathrm{c}}]^T.\{\mu_{i-1}^{\mathrm{c}}\}\rightarrow\{\boldsymbol{v}\}$ If $\langle\mu_{i-1}^{\mathrm{c}}\rangle\cdot\{\mu_{i-1}^{\mathrm{c}}\}=0$ then $\{\boldsymbol{v}\}=\{\boldsymbol{0}\}$ Calculation of $\delta u_0^n=\delta\tilde{u}^n-v$ Evaluation of $\Xi_0^{\mathrm{c}}=\left\{J\in\Xi^1\middle\vert\left\{\tilde{\boldsymbol{d}}^{\mathrm{c},n}\right\}_J<0\right\}$		
B_k	Buckle on $k=1,Iter_{max}$		
		$\Xi_k^{\mathrm{c}}\leftarrow\Xi_{k-1}^{\mathrm{c}}$	
		Calculation of the under-gradient $\{\boldsymbol{r}_k\}=\left(\left[\boldsymbol{A}_k^{\mathrm{c}}\right].\{\delta\boldsymbol{u}_{k-1}^n\}-\{\boldsymbol{d}^{\mathrm{c},n-1}\}\right)$	
		Projection of the under-gradient $\{\boldsymbol{r}_k\}=\underset{J\in\Xi^1}{max}(\{\boldsymbol{r}_k\},0)$	
		Calculation of the residue $\varepsilon=\underset{J\in\Xi^1}{max}\{\boldsymbol{r}_k\}$	
		Pre-packaging $\{\boldsymbol{r}_k^p\}=PreCond(\{\boldsymbol{r}_k\})$	
		Conjugation	
			Calculation of the coefficient $\gamma^k=\frac{\langle \boldsymbol{r}_k\rangle.\{\boldsymbol{r}_k^p\}-\langle \boldsymbol{r}_k\rangle.\{\boldsymbol{r}_{k-1}^p\}}{\langle \boldsymbol{r}_{k-1}\rangle.\{\boldsymbol{r}_{k-1}^p\}}$
			$\{\boldsymbol{Z}_k\}=\{\boldsymbol{r}_k\}+\gamma^k.\{\boldsymbol{Z}_{k-1}\}$

		Linear research $\left\{\left\{\delta \boldsymbol{u}_{k+1}\right\},\left\{\mu^{c}\right\}\right\}=RechLine\left(\left\{\boldsymbol{Z}\right\},\left\{\boldsymbol{r}\right\},\left\{\boldsymbol{r}^{p}\right\}\right)$
B_k	Buckle $k=k+1$	
F	END	

One can make the following comments on the algorithm:

- At the time of linear research, the resolution of the system $\left[\boldsymbol{K}\right].\left\{\delta \boldsymbol{a}\right\}=\left[\boldsymbol{A}^{c}\right]^{T}.\left\{\boldsymbol{Z}\right\}$ and the calculation of the term $\left\langle\delta \boldsymbol{a}\right\rangle.\left[\boldsymbol{A}^{c}\right]^{T}.\left\{\boldsymbol{Z}\right\}$, one perceives that the algorithm presented is the iterative version of the method of the active constraints. Indeed, if the term is clarified $\left\langle\delta \boldsymbol{a}\right\rangle.\left[\boldsymbol{A}^{c}\right]^{T}.\left\{\boldsymbol{Z}\right\}$, one obtains $\left\langle\boldsymbol{Z}\right\rangle.\left(\left[\boldsymbol{A}_{c}\right].\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}_{c}\right]^{T}\right).\left\{\boldsymbol{Z}\right\}$. One finds the complement of Schur $\left[\boldsymbol{K}\right]_{schur}=\left[\boldsymbol{A}^{c}\right].\left[\boldsymbol{K}\right]^{-1}.\left[\boldsymbol{A}^{c}\right]^{T}$ who is explicitly built in the method of the active constraints. As in all the iterative methods, the Gradient Combined Project does not build this operator but uses his effect on a vector (here on $\left\{\boldsymbol{Z}\right\}$);
- To be really effective, this algorithm must be used with a direct solvor or an iterative solvor prepacked by the method `LDLT_SP`.
  - With a direct solvor, the matrix of the system being already factorized, each resolution is summarized with an descent-increase:
  - With an iterative solvor prepacked by the method `LDLT_SP`, each resolution is summarized with some iterations of the iterative solvor.
- One combines with each iteration.

5.3 Method penalized in contact – ALGO_CONT=' PENALISATION'

5.3.1 Balance of the structure in the presence of contact

It is pointed out that the equilibrium equation in the presence of contact is written :

$$\left\{\boldsymbol{L}_{i}^{int}(\boldsymbol{u}_{i})\right\}=\left\{\boldsymbol{L}_{i}^{ext}(\boldsymbol{u}_{i})\right\}-\left\{\boldsymbol{L}_{i}^{c}\right\} \tag{137}$$

In the case of the penalized contact, the force of contact is written:

$$\left\{\boldsymbol{L}_{i}^{c}\right\}=\left\{\boldsymbol{L}_{i}^{c,n}\right\}_{regu}=E_{N}.\left[\boldsymbol{A}^{c}\right]^{T}.\left(\left[\boldsymbol{A}^{c}\right].\left\{\delta \boldsymbol{u}^{n}\right\}-\left\{\boldsymbol{d}^{c,n-1}\right\}\right) \tag{138}$$

After linearization of the equilibrium equation (137) , the tangent matrix is introduced $\left[\boldsymbol{K}^{m,n-1}\right]$ who will contain the contributions resulting from the linearization of the interior and external efforts and the matrix $\left[\boldsymbol{K}^{c,n-1}\right]$ for forces of contact:

$$\left[\boldsymbol{K}_{i}^{c,n-1}\right]=E_{N}.\left[\boldsymbol{A}^{c,n-1}\right]^{T}.\left[\boldsymbol{A}^{c,n-1}\right] \tag{139}$$

The second being worth member:

$$\left\{\boldsymbol{L}_{i}^{c,n-1}\right\}=-E_{N}.\left[\boldsymbol{A}^{c,n-1}\right]^{T}.\left\{\boldsymbol{d}^{c,n-1}\right\} \tag{140}$$

Concerning the algorithm, we saw with the § 4.3.6.2 that it acts as correction of the problem of balance without contact.

When we are in the algorithm of contact, with the iteration of Newton n , we evaluate $\left[\boldsymbol{K}_{i}^{c,n+1}\right]$ and $\left\{\boldsymbol{L}_{i}^{c,n}\right\}$, which will be used only with the following iteration of Newton. There is a shift of an iteration: it is necessarily necessary at least two iterations of Newton to solve a problem with contact penalized.

5.3.2 Algorithm

Ini	Calculation of $\left\{\left\{\tilde{\boldsymbol{d}}^{c,n}\right\}\right\}_{\Xi^{1}}$
	Evaluation of $\Xi^{c}=\left\{J\in\Xi^{1}\middle\vert\left\{\left\{\tilde{\boldsymbol{d}}^{c,n}\right\}\right\}_{J}<0\right\}$ and $\left[\boldsymbol{A}^{c,n}\right]$
	Contact pressures $\left\{\left\{\mu^{c,n}\right\}\right\}_{\Xi^{c}}=-E_{N}.\left\{\left\{\tilde{\boldsymbol{d}}^{c,n}\right\}\right\}_{\Xi^{c}}$

	Second member $\left\{L_i^{c,n}\right\}=\left[A^{c,n}\right]^T.\left\{\mu^{c,n}\right\}_{\Xi^c}$
	Matrix $\left[K^{c,n+1}\right]=E_N.\left[A^{c,n}\right]^T.\left[A^{c,n}\right]$

Note:

- Quantities $\left[K_i^c\right]$ and $\left\{L_i^c\right\}$ are evaluated only for the active connections. For the rest of the unknown factors, one initializes to zero;
- The modification of the system does not introduce new variables compared to the problem without contact;
- There is no index k in the various quantities (in particular the whole of the active connections or the matrix of contact) because it is not an iterative algorithm;
- For the calculation of the matrix, one proceeds in two times: evaluations of the elementary matrices, then assembly. But these two operations are specific to the discrete contact, one does not use the mechanism elementary calculations/generic assembly of Code_Aster.
- The choice of the coefficient of penalization is crucial: too much weak, of the interpenetrations will be observed, too extremely, the conditioning of the tangent matrix is degraded.

6 Algorithmic resolution – Contact with friction

6.1 Connections of friction

One adds two whole compared to those defined in the §5.1 to take into account the statute of friction of a node slave:

- Ξ^a is the whole of the adherent nodes of contact;
- Ξ^g is the whole of the slipping nodes of contact.

One thus has the following relations between the units:

- $\Xi^c = \Xi^a \oplus \Xi^g$ because the nodes in contact are either slipping, or adherent;
- $\Xi^a \subseteq \Xi^c$ and $\Xi^g \subseteq \Xi^c$ because only the nodes in contact can be adherent or slipping.

6.2 Kinematics and tangent game

We saw in the § 4.3.1 that friction uses kinematics quantities similar to the case of the contact alone, by introducing the concept of "tangent" game $\left\{\tilde{\boldsymbol{g}}^{t,n}\right\}$ (relative slip of surfaces in the case of non-adhesion), this tangent game is defined by:

$$\left\{\tilde{\boldsymbol{g}}^{t,n}\right\} = \left[\boldsymbol{A}^f\right].\left(\left\{\Delta \boldsymbol{u}_i^{n-1}\right\} + \left\{\delta \tilde{\boldsymbol{u}}^n\right\}\right) \quad (141)$$

The matrix of friction is used $\left[\boldsymbol{A}^f\right]$ (see § 3.5.2). The use of the quantity $\left\{\delta \tilde{\boldsymbol{u}}^n\right\}$ instead of $\left\{\delta \boldsymbol{u}^n\right\}$ justifies itself owing to the fact that one solves a problem of Tresca, therefore without modifying the game during the process of resolution of the problem of slip (see § 4.3.4).

Note:

- In the algorithms presented in the continuation of the document, one generally does not make distinction between the slipping part and the adherent part of the nodes on the level them writings, to reduce the notations.

6.3 Method penalized in contact and in friction

6.3.1 Balance of the structure in the presence of contact

It is pointed out that the equilibrium equation in the presence of contact is written :

$$\left\{\boldsymbol{L}_i^{int}(\boldsymbol{u}_i)\right\} = \left\{\boldsymbol{L}_i^{ext}(\boldsymbol{u}_i)\right\} - \left\{\boldsymbol{L}_i^c\right\} \quad (142)$$

In the case of the penalized contact, the force of contact is written:

$$\left\{\boldsymbol{L}_i^c\right\} = \left\{\boldsymbol{L}_i^{c,n}\right\}_{regu} = E_N.\left[\boldsymbol{A}^c\right]^T.\left(\left[\boldsymbol{A}^c\right].\left\{\delta \boldsymbol{u}^n\right\} - \left\{\boldsymbol{d}^{c,n-1}\right\}\right) \quad (143)$$

After linearization of the equilibrium equation (142) , the tangent matrix is introduced $\left[\boldsymbol{K}^{m,n-1}\right]$ who will contain the contributions resulting from the linearization of the interior and external efforts and the matrix $\left[\boldsymbol{K}^{c,n-1}\right]$ for forces of contact:

$$\left[\boldsymbol{K}_i^{c,n-1}\right] = E_N.\left[\boldsymbol{A}^{c,n-1}\right]^T.\left[\boldsymbol{A}^{c,n-1}\right] \quad (144)$$

The second being worth member:

$$\left\{\boldsymbol{L}_i^{c,n-1}\right\} = -E_N.\left[\boldsymbol{A}^{c,n-1}\right]^T.\left\{\boldsymbol{d}^{c,n-1}\right\} \quad (145)$$

Concerning the algorithm, we saw with the § 4.3.6.2 that it acts as correction of the problem of balance without contact.

When we are in the algorithm of contact, with the iteration of Newton n , we evaluate $\left[\boldsymbol{K}_i^{c,n+1}\right]$ and $\left\{\boldsymbol{L}_i^{c,n}\right\}$, which will be used only with the following iteration of Newton. There is a shift of an iteration: it is necessarily necessary at least two iterations of Newton to solve a problem with contact penalized.

6.3.2 Algorithm

This method is simplest to implement. We saw in §4.3.5.1 that the regularization of the conditions of contact adds two new contributions:

- $\left[\boldsymbol{K}_i^{\mathrm{c}}\right]$ for the total tangent matrix;
- $\left\{\boldsymbol{L}_i^{\mathrm{c}}\right\}$ for the second member.

For friction, it is the same for the conditions of adherence (§ 4.3.5.2):

- $\left[\boldsymbol{K}_i^{\mathrm{a}}\right]$ for the total tangent matrix;
- $\left\{\boldsymbol{L}_i^{\mathrm{a}}\right\}$ for the second member.

But also for the conditions of slip (§ 4.3.5.3):

- $\left[\tilde{\boldsymbol{K}}_{\theta}^{\mathrm{g}}\right]$ for the total tangent matrix;
- $\left\{\boldsymbol{L}_i^{\mathrm{g}}\right\}$ for the second member.

Concerning the algorithm, we saw with the § 4.3.6.2 that it acts as correction of the problem of balance without contact friction. When we are in the algorithm of contact friction, with the iteration of Newton n , we evaluate the quantities imposing the conditions of contact/friction which will be used that with the following iteration of Newton. What gives us the following algorithm, with n the iteration of Newton:

Ini	Calculation of $\left\{\tilde{\boldsymbol{d}}^{\mathrm{c},n}\right\}_{\Xi^{\mathrm{l}}}$ and $\left\{\tilde{\boldsymbol{g}}^{\mathrm{t},n}\right\}_{\Xi^{\mathrm{l}}}$	
	Evaluation of $\Xi^{\mathrm{c}}=\left\{J\in\Xi^{\mathrm{l}}\middle\vert\left\{\tilde{\boldsymbol{d}}^{\mathrm{c},n}\right\}_J<0\right\}$	
	Contact pressures $\left\{\boldsymbol{\mu}^{\mathrm{c},n}\right\}_{\Xi^{\mathrm{c}}}=-E_N.\left\{\tilde{\boldsymbol{d}}^{\mathrm{c},n}\right\}_{\Xi^{\mathrm{c}}}$	
	Second member $\left\{\boldsymbol{L}_i^{\mathrm{c},n}\right\}=\left[\boldsymbol{A}^{\mathrm{c},n}\right]^T.\left\{\boldsymbol{\mu}^{\mathrm{c},n}\right\}_{\Xi^{\mathrm{c}}}$	
	If $\Xi^{\mathrm{c}}\neq\emptyset$	
		Matrix of contact $\left[\boldsymbol{K}^{\mathrm{c},n+1}\right]=E_N.\left[\boldsymbol{A}^{\mathrm{c},n}\right]^T.\left[\boldsymbol{A}^{\mathrm{c},n}\right]$
		Calculation of the standard of the slip $\left\{\left\Vert\left\{\tilde{\boldsymbol{g}}^{\mathrm{t},n}\right\}\right\Vert\right\}_{\Xi^{\mathrm{c}}}$
		Calculation of $\left\{\lambda^{\mathrm{f}}\right\}_{\Xi^{\mathrm{c}}}=max\left(0,\mu.\left\{\boldsymbol{\mu}^{\mathrm{c},n}\right\}_{\Xi^{\mathrm{c}}}\right)$
		Calculation of $\left\{\boldsymbol{a}\right\}_{\Xi^{\mathrm{c}}}=\left\lfloor\left(\left\Vert\left\{\tilde{\boldsymbol{g}}^{\mathrm{t},n}\right\}\right\Vert-\frac{1}{E_T}.\left\{\lambda^{\mathrm{f}}\right\}\right)\right\rfloor_{\Xi^{\mathrm{c}}}$
		Evaluation of $\Xi^{\mathrm{a}}=\left\{J\in\Xi^{\mathrm{c}}\middle\vert\left\{\boldsymbol{a}\right\}_J\leq 0\right\}$
		Evaluation of $\Xi^{\mathrm{g}}=\left\{J\in\Xi^{\mathrm{c}}\middle\vert\left\{\boldsymbol{a}\right\}_J>0\right\}$
		For the adherent nodes $\left\{\boldsymbol{\mu}^{\mathrm{f},n}\right\}_{J\in\Xi^{\mathrm{a}}}=\sqrt{E_T}$
		For the slipping nodes $\left\{\boldsymbol{\mu}^{\mathrm{f},n}\right\}_{J\in\Xi^{\mathrm{g}}}=\sqrt{\frac{\left\{\lambda^{\mathrm{f}}\right\}_J}{\left\{\left\Vert\left\{\tilde{\boldsymbol{g}}^{\mathrm{t},n}\right\}\right\Vert\right\}_J}}$
		Calculation vector $\left\{\boldsymbol{v}\right\}_{\Xi^{\mathrm{c}}}=\left\{\left[\boldsymbol{A}^{\mathrm{f},n}\right]^T.\left\{\boldsymbol{\mu}^{\mathrm{f},n}\right\}_{\Xi^{\mathrm{c}}}\right.$ and $\left\{\boldsymbol{v}\right\}_{\Xi^{\mathrm{nc}}}=0$
		Calculation first left matrix friction $\left[\boldsymbol{K}^{\mathrm{f_l}}\right]=\left\{\boldsymbol{v}\right\}.\left\langle\boldsymbol{v}\right\rangle$
		Calculation of the second member of friction $\left\{\boldsymbol{L}_i^{\mathrm{f},n}\right\}=\left[\boldsymbol{K}^{\mathrm{f_l}}\right].\left(\left\{\Delta\boldsymbol{u}_i^{n-1}\right\}+\left\{\delta\tilde{\boldsymbol{u}}^{n}\right\}\right)$
		Calculation of the standard of the slip $\left\{\left\Vert\left\{\tilde{\boldsymbol{g}}^{\mathrm{t},n}\right\}\right\Vert\right\}_{\Xi^{\mathrm{c}}}$

		Calculation of $\left\{\lambda^{f}\right\}_{\Xi^{c}}=max\left(0,\mu.\left\{\mu^{c,n}\right\}_{\Xi^{c}}\right)$
		Calculation of $\left\{a\right\}_{\Xi^{c}}=\left[\left(\left\|\left\{\tilde{g}^{t,n}\right\}\right\|-\frac{1}{E_T}.\left\{\lambda^{f}\right\}\right)\right]_{\Xi^{c}}$
		Evaluation of $\Xi^{a}=\left\{J\in\Xi^{c}\left\|\left\{a\right\}_{J}\leq 0\right.\right\}$
		Evaluation of $\Xi^{g}=\left\{J\in\Xi^{c}\left\|\left\{a\right\}_{J}>0\right.\right\}$
		For the adherent nodes $\beta_{J\in\Xi^{a}}=0$
		For the slipping nodes $\beta_{J\in\Xi^{c}}=\sqrt{\frac{1}{\left[\left\{\lambda^{f}\right\}.\left\|\left\{\tilde{g}^{t,n}\right\}\right\|\right]_{J\in\Xi^{c}}}}$
		Choice of θ
		Calculation vector $\left\{w\right\}_{\Xi^{c}}=\sqrt{\theta}.\left[\beta_{J\in\Xi^{c}}.\left[A^{f,n}\right].\left\{L_i^{f,n}\right\}\right]_{\Xi^{c}}$ and $\left\{w\right\}_{\Xi^{nc}}=0$
		Calculation second left matrix friction $\left[K^{f_2}\right]=\left\{w\right\}.\left\langle w\right\rangle$
		Resulting matrix $\left[K^{f}\right]=\left[K^{f_1}\right]-\left[K^{f_2}\right]$

One uses not bad tricks of calculation in this algorithm. One gives a value different from $\left\{\mu^{f}\right\}$ for adherence and the slip but one uses the same vector $\left\{v\right\}$, which makes it possible to find the two matrices (slip and adherence). Indeed, for adherence, one a:

$$\left[\left[K^{f_1}\right]\right]_{\Xi^{a}}=\left\{v\right\}.\left\langle v\right\rangle=\left[A^{f}\right]^{T}.\sqrt{E_T}.\sqrt{E_T}.\left[A^{f}\right] \tag{146}$$

One thus finds the form of the matrix of adherence (79) for the adherent nodes:

$$\left[\left[K^{f_1}\right]\right]_{\Xi^{a}}=\left[K_i^{a}\right]=E_T.\left[A^{a}\right]^{T}.\left[A^{a}\right] \tag{147}$$

For the slip:

$$\left[\left[K^{f_1}\right]\right]_{\Xi^{g}}=\left\{v\right\}.\left\langle v\right\rangle=\left[A^{f}\right]^{T}.\sqrt{\frac{\left\{\lambda^{f}\right\}}{\left\|\left\{\tilde{g}^{t,n}\right\}\right\|}}.\sqrt{\frac{\left\langle\lambda^{f}\right\rangle}{\left\|\left\{\tilde{g}^{t,n}\right\}\right\|}}.\left[A^{f}\right] \tag{148}$$

One finds the expression of the first part of the matrix of slip (97) for the slipping nodes:

$$\left[\left[K^{f_1}\right]\right]_{\Xi^{g}}=\frac{\left[A^{g}\right]^{T}.\left[k^{g,n}\right]}{\left\|\left\{g^{t,n}\right\}\right\|} \tag{149}$$

With the matrix of the thresholds of Tresca:

$$\left[k^{g}\right]=\left\{\lambda^{f}\right\}.\left\langle\lambda^{f}\right\rangle \tag{150}$$

To find to it second part of the matrix of slip, one uses the expression D U vector $\left\{w\right\}$:

$$\left\{w\right\}=\sqrt{\theta}.\beta.\left[A^{f}\right].\left[K^{f_1}\right].\left(\left\{\Delta u_i^{n-1}\right\}+\left\{\delta\tilde{u}^{n}\right\}\right) \text{ with } \beta=\sqrt{\frac{1}{\left\{\lambda^{f}\right\}.\left\|\left\{\tilde{g}^{t,n}\right\}\right\|}} \tag{151}$$

With the form of the matrix $\left[\left[K^{f_1}\right]\right]_{\Xi^{g}}$ in (148) and the definition of the tangential slip in (87), the product (tensorial) of $\left\{w\right\}$ by him even allows to find the second part of the matrix of slip (97) for the slipping nodes:

$$\left[\left[K^{f_2}\right]\right]_{\Xi^{g}}=\left\{w\right\}.\left\langle w\right\rangle=\theta.\frac{\left[A^{g}\right]^{T}.\left[k^{g,n}\right]}{\left\|\left\{g^{t,n}\right\}\right\|}.\frac{\left\{g^{t,n}\right\}\left\langle g^{t,n}\right\rangle}{\left\|\left\{g^{t,n}\right\}\right\|^{2}} \tag{152}$$

One used the product (150), which explains the multiplier β used in the vector $\left\{w\right\}$.

Note:

- The modification of the system does not introduce new variables compared to the problem without contact/friction;
- There is no index k in the various quantities (in particular the whole of the active connections or them matrices of contact friction) because it is not an iterative algorithm;
- For the choice of θ , it is the user who chooses it (parameter COEF_MATR_FROT), but this coefficient is put at zero as long as the residue of balance (RESI_GLOB_RELA) is lower than 10^{-3} ;

7 Theoretical results of convergence

For the problems without friction, one will find in [1] a demonstration of convergence for the method of the active constraints.

For the problems with friction, of the results of convergence with unicity of the solution to the discretized problem are established in [1] for low values of the coefficient of friction of Coulomb. The results are got by using an algorithm of point fixes associated with a method of multipliers of Lagrange. For each problem of solved contact, one studies the problem of associated friction. Once this one solved, one solves a new problem of contact and so on. These methods are however different from those presented here and one cannot thus have results of theoretical convergence for these last.

7.1 Recutting of the step of time

On the theoretical level, the convergence of the method of the active constraints is ensured in a finished number of iterations. In practice, certain digital artifacts can make this convergence delicate. Also a strategy it was developed to ensure the robustness of the algorithm.

During calculations of contact, in particular if the steps of load carried out are too large, of the undesirable phenomena can appear:

- The matrix of contact is singular,
- Oscillation of the method of the active constraints: a node is detected alternatively “stuck” then “taken off”.

To mitigate these difficulties, the following strategy was adopted. If:

- The matrix of contact is singular,
- The iteration count of active constraints is higher than a limit which depends amongst potential connections. This number is fixed at twice the full number of nodes slaves for the method of the active constraints, and with ITER_CONT_MULT time the full number of nodes slaves for the other methods.

Then one redécoupe the step of time *i.e.* one returns to the preceding step of load and instead of trying to reach the level of loading following in a step as one has just done it, one makes some several (For more precise details on this functionality of the operator STAT_NON_LINE , to see documentation [U4.51.03]).

7.2 Compatibility with the boundary conditions of Dirichlet

In the case of the methods with multipliers of Lagrange, one can observe incompatibilities with the fact of imposing boundary conditions of the Dirichlet type. Indeed, it is necessary that physically the problem has a direction. One cannot deal with a problem of contact in the direction of the axis z if all the points have a following null displacement z . As we will see it, to deal with such a problem led to a singularity of the matrices of the type $\left[A^{c}\right].\left[K\right]^{-1}.\left[A^{c}\right]^{T}$ with the treatment of the boundary conditions of Dirichlet by double lagrange of *Code_Aster*.

7.2.1 Writing of the boundary conditions

While taking as a starting point the reference material [R5.03.01] by STAT_NON_LINE, the dualisation of the boundary conditions of Dirichlet leads to the system of equations following to solve:

$$\begin{cases}\left[K\right].\left(\delta u\right)+\left[B\right]^{T}.\left(\delta\lambda\right)=\left(L_{i}^{\text{int}}\right)-\left(L_{i}^{\text{ext}}\right)\\ \left[B\right].\left(u_{i}\right)=\left(u_{i}^{d}\right)-\left[B\right].\left(u_{i-1}\right)\end{cases} \tag{153}$$

One notes then $\left[C\right]$ the matrix of rigidity of the system such as:

$$\left[C\right]=\begin{bmatrix}\left[K\right] & \left[B\right]^{T}\\ \left[B\right] & \left[0\right]\end{bmatrix} \tag{154}$$

This matrix has a reverse of the form:

$$\left[C\right]^{-1}=\begin{bmatrix}\left[E\right] & \left[F\right]\\ \left[F\right]^{T} & \left[G\right]\end{bmatrix} \quad (155)$$

such as: $\left[E\right].\left[B\right]^{T}=\left[0\right]$. One checks thus that for each boundary condition l there is the property $\left[E\right].\left[B_{l}\right]^{T}=\left[0\right]$.

7.2.2 Return to the problem of contact

The matrix $\left[A^{c}\right].\left[K\right]^{-1}.\left[A^{c}\right]^{T}$ can be also written $\left[A^{c}\right].\left[E\right].\left[A^{c}\right]^{T}$ since vectors of connection $\left[A\right]$ only the degrees of freedom of displacement utilize.

- It from of results whereas if a vector from connection J matrix $\left[A\right]$ of the boundary conditions of the Dirichlet type it is a linear combination checks the following property: $\left[E\right].\left[A_{J}\right]^{T}=\left[0\right]$. The matrix $\left[A^{c}\right].\left[E\right].\left[A^{c}\right]^{T}$ is then singular because it has a column of zeros. In practice, without particular treatment, one finishes in the code on a message of stop of the type `STOP ON SINGULAR MATRIX OF CONTACT-FRICTION`. The detection of these singular columns was put in work in the code in order to eliminate from the relations of contact-friction this kind of relations and to avoid the stop previously described.
- It from of results whereas if a vector from connection J matrix $\left[A\right]$ a linear combination of the boundary conditions of the Dirichlet type contains and is written $\left[A_{J}\right]=\sum \alpha_{i}.\left[B_{i}\right]+\left[\bar{A}_{J}\right]$, it checks the following property: $\left[E\right].\left[A_{J}\right]^{T}=\left[E\right].\left[\bar{A}_{J}\right]^{T}$. One can then have a matrix $\left[A^{c}\right].\left[E\right].\left[A^{c}\right]^{T}$ singular because it has two identical lines. This detection is not for the moment not available in the code and one finishes in the code on a message of stop of the standard stop on singular matrix of contact - friction.

Note:

This compatibility issue between contact-friction and the boundary conditions does not appear with the regularized methods insofar as one adds rigidity with total rigidity and that one does not make elimination as in the calculation of the lagranges.

8 Conclusion

Discrete modelings of contact-friction with 2D and slip surfaces 1D were established in *Code_Aster*. These modelings usable with STAT_NON_LINE and DYNA_NON_LINE are accessible under DEFI_CONTACT.

Modelings suggested are based on the grids of surfaces coming in contact and make it possible to retranscribe node with node the conditions of contact friction between surfaces after discretization of the corresponding variational formulation. The method extends then without difficulty of small displacements to the case from great displacements. Indeed, the absence of use of finite elements, between surfaces being able to come in contact, avoids the great distortion of the latter, in the case of great displacements. One can then use either of the conditions of direct connections nodes to nodes for initially compatible grids, or conditions of connections nodes to nodes balanced according to an approach by projection of the master-slave type for incompatible grids.

In the case of slip surfaces 1D one could develop an algorithm using only multipliers of Lagrange. The finished convergence of this kind of algorithm is proven for the unilateral contact without friction and in the case with friction for low values of the coefficient of friction of Coulomb In the case of slip surfaces 2D, the rubbing contact is treated by regularization.

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History of the versions of the document

This document is resulting from the fusion of the documents [R5.03.50] and [R5.03.51].

[R5.03.50] (Old version) unilateral Contact by conditions kinematics

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
5.0	N.Tardieu, I.Vautier	
8.5	M.Abbas, N.Tardieu EDF-R&D/AMA	Extension to dynamics, addition of the penalization.
9.4	M.Abbas N.Tardieu	Addition of the description of the algorithm of the GCP Addition of the description of the method of projection

[R5.03.51] discrete Contact-Friction in 2D and 3D

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
5.0	P. Massin	Friction of Coulomb or Tresca, method derived from the active constraints in 2D, penalization for friction in 3D, great displacements
7.1	N.Tardieu, P. Massin	Extension of the methods 2D to the situations 3D
8.5	Mr. Abbas	Addition of the reactualization geometrical (§7.2) and put up to date of the document

[R5.03.50] (New version) discrete Formulation of contact-friction

Version Aster	Author (S) or contributor (S), organization	Description of the modifications
10.1	M.Abbas, T. De Soza	Fusion of [R5.03.50] and [R5.03.51]
10.5	M.Abbas	Various corrections, modifications of the references for the new order DEFI_CONTACT
11.0	M.Abbas	Simplifications of the equations, details on the initialization of the algorithms, rewriting of the algorithms, removal of the advices (see U2 document)
11.3	M.Abbas, S.Mercier	Corrections of shells in the algorithms
11.4	M.Abbas, T. De Soza	Addition of linear research in the algorithm of projection
13.2	M.Abbas	Suppression of the “Lagrangian” methods