Integration of the relations of behavior elastoplastic of Von Mises

Summary :

This document describes the quantities calculated by the operator STAT_NON_LINE necessary to the implementation of the quasi static nonlinear algorithm describes in [R5.03.01] in the case of the elastoplastic behaviors. These quantities are calculated by the same subroutines in the operator DYNA_NON_LINE in the case of a dynamic stress [R5.05.05].

This description is presented according to the various keywords which make it possible to the user to choose the relation of behavior wished. The relations of behavior treated here are:

- the behavior of Von Mises with isotropic work hardening (linear or not linear)
- the behavior of Von Mises with linear kinematic work hardening (model of Prager)

The method of integration used is based on a direct implicit formulation. From the initial state, or as from the moment of preceding calculation, one calculates the stress field resulting from an increment of deformation. The tangent operator is also calculated.

Contents

1 Introduction

1.1 Relations of behaviors described in this document

In the operator STAT_NON_LINE [U4.51.03] (or DYNA_NON_LINE [U4.53.01]), two types of behaviors can be treated:

- the incremental behavior: keyword factor BEHAVIOR,
- the behavior in nonlinear elasticity: keyword factor BEHAVIOR.

For each behavior one can choose:

- the relation of behavior: keyword RELATION,
- way of calculating of the deformations: keyword DEFORMATION.

For more details, to consult the document [U4.51.03] user's manual, the behaviors described here not raising but of the keyword factor BEHAVIOR.

The relations treated in this document are:

VMIS_ISOT_LINE:	Von Mises with linear isotropic work hardening,
VMIS_ISOT_TRAC:	Von Mises with isotropic work hardening given by a traction diagram,
VMIS_ISOT_PUIS:	Von Mises with isotropic work hardening given by an analytical curve,
VMIS_JOHN_COOK:	Von Mises with isotropic work hardening of Johnson-Cook,
VMIS_CINE_LINE:	Von Mises with linear kinematic work hardening.

1.2 Goal of integration

To solve the nonlinear total problem posed on the structure, the document [R5.03.01] described the algorithm used in *Code_Aster* for nonlinear statics (operator STAT_NON_LINE) and the document [R5.05.05] described the method used for nonlinear dynamics (operator DYNA_NON_LINE).

These two algorithms are based on the calculation of local quantities (in each point of integration of each finite element) which result from the integration of the relations of behavior.

With each iteration n method of Newton [R5.03.01 § 2.2.2.2] one must calculate the nodal forces $\mathbf{R}(\mathbf{u}_i^n)=\mathbf{Q}^T\boldsymbol{\sigma}_i^n$ (options RAPH_MECA and FULL_MECA) constraints σ_i^n being calculated in each point of integration of each element starting from displacements u_i^n via the relation of behavior. One must also build the tangent operator to calculate $\mathbf{K}_i^n$ (option FULL_MECA).

Before the first iteration, for the phase of prediction, one calculates $\mathbf{K}_{i-1}$ (option RIGI_MECA_TANG). The calculation of $\mathbf{K}_{i-1}$, which is necessary to the phase of initialization [R5.03.01 § 2.2.2.2] corresponds to the calculation of the tangent operator deduced from the problem of speed.

This operator is not identical to that which is used to calculate $\mathbf{K}_i^n$ by the option FULL_MECA, during iterations of Newton. Indeed, this last operator is tangent with the discretized problem in an implicit way.

One describes here for the relations of behavior VMIS_ISOT_LINE, VMIS_ISOT_TRAC, VMIS_ISOT_PUIS, VMIS_JOHN_COOK and VMIS_CINE_LINE, the calculation of the tangent matrix of the phase of prediction, $\mathbf{K}_{i-1}$, then the calculation of the stress field starting from an increment of deformation, the calculation of the forces **nodal** R and of the tangent matrix $\mathbf{K}_i^n$.

2 General notations and assumptions on the deformations

All the quantities evaluated at the previous moment are subscripted by $^-$.

Quantities evaluated at the moment $t+\Delta t$ are not subscripted.

The increments are indicated by Δ . One has as follows:

$$\mathbf{Q}=\mathbf{Q}(t+\Delta t)=\mathbf{Q}(t)+\Delta\mathbf{Q}=\mathbf{Q}^-+\Delta\mathbf{Q}\,.$$

For the calculation of the derivative, one will note: $\dot{\mathbf{Q}}$ derived from $\mathbf{Q}$ compared to time

$\boldsymbol{\sigma}$	tensor of the constraints.
$\sim$	operator déviatoire: $\tilde{\boldsymbol{\sigma}}_{ij}=\boldsymbol{\sigma}-\frac{1}{3}\boldsymbol{\sigma}_{kk}\boldsymbol{\delta}_{ij}$.
$()_{eq}$	equivalent value of Von Mises: $\boldsymbol{\sigma}_{eq}=\sqrt{\frac{3}{2}\tilde{\boldsymbol{\sigma}}_{ij}\tilde{\boldsymbol{\sigma}}_{ij}}$
$\boldsymbol{\Delta}\,\varepsilon$	increment of deformation.
$\mathbf{A}$	tensor of elasticity.
λ,μ,E,ν,K	moduli of the isotropic elasticity, respectively: coefficients of Lamé, Young modulus, Poisson's ratio and module of compressibility.
$3K=3\lambda+2\mu$	module of compressibility
α	thermal dilation coefficient average.
t	time.
T	temperature.
$()_+$	positive part.

To calculate the tangent operators, one will adopt the convention of writing of the symmetrical tensors of order 2 in the form of vectors with 6 components. Thus, for a tensor a :

$$\vec{a}={}^t\left[a_{xx}\quad a_{yy}\quad a_{zz}\;\sqrt{2}\,a_{xy}\quad\sqrt{2}\,a_{xz}\quad\sqrt{2}\,a_{yz}\right]$$

The hydrostatic vector is introduced $\vec{\mathbf{1}}$ and stamps it deviatoric projection $\mathbf{P}$:

$$\vec{\mathbf{1}}={}^t\left[1\quad 1\quad 1\,0\quad 0\quad 0\right]$$
$$\mathbf{P}=\mathbf{Id}-\frac{1}{3}\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}$$

2.1 Partition of the deformations (small deformations)

One writes for any moment:

$$\boldsymbol{\varepsilon}(t)=\boldsymbol{\varepsilon}^{e}(t)+\boldsymbol{\varepsilon}^{th}(t)+\boldsymbol{\varepsilon}^{p}(t)$$

with

$$\boldsymbol{\varepsilon}^{e}(t)=\mathbf{A}^{-1}(T(t))\boldsymbol{\sigma}(t)$$

with

$$\boldsymbol{\varepsilon}^{th}(t)=\alpha(T(t))\left(T(t)-T_{ref}\right)\mathbf{Id}$$

or in a more general way:

$$\begin{aligned}\varepsilon^{th}(T) &= \alpha(T)\left(T-T_{def}\right)-\alpha\left(T_{ref}\right)\left(T_{ref}-T_{def}\right)\\ &= \hat{\alpha}(T)\left(T-T_{ref}\right)\\ \text{et}\quad \varepsilon^{th}\left(T_{ref}\right) &= 0\end{aligned}$$

A depends on the moment t via the temperature. The thermal dilation coefficient $\alpha(T(t))$ is an average dilation coefficient which can depend on the temperature T. The temperature T_{ref} is the temperature of reference, i.e. that for which thermal dilation is supposed to be worthless if the average dilation coefficient is not known compared to T_{ref}, one can use a temperature of definition of the average dilation coefficient T_{def} (defined by the keyword TEMP_DEF_ALPHA of DEFI_MATERIAU) different from the temperature of reference [R4.08.01].

What leads to: $\dot{\boldsymbol{\varepsilon}}(t)=\overbrace{\mathbf{A}^{-1}(T(t))\boldsymbol{\sigma}(t)}^{\cdot}+\dot{\boldsymbol{\varepsilon}}^{th}(t)+\dot{\boldsymbol{\varepsilon}}^{p}(t)$

This choice is made by preoccupation with a coherence with elasticity: it is necessary to be able to find the same solution in elasticity (operator MECA_STATIQUE) and in elastoplasticity (operator STAT_NON_LINE) when the characteristics of material remain elastic. This choice leads to the discretization:

$$\Delta\varepsilon=\Delta\varepsilon^{p}+\Delta(A^{-1}\sigma)+\Delta\varepsilon^{th}$$

with:

$$\Delta(\mathbf{A}^{-1}\boldsymbol{\sigma})=\mathbf{A}^{-1}(t^{-}+\Delta t)(\boldsymbol{\sigma}^{-}+\Delta\boldsymbol{\sigma})-\mathbf{A}^{-1}(t^{-})\boldsymbol{\sigma}^{-}$$

and

$$\Delta\boldsymbol{\varepsilon}^{\mathbf{th}}=\left(\alpha(t^{-}+\Delta t)(T-T_{ref})-\alpha(t^{-})(T^{-}-T_{ref})\right)\mathbf{Id}$$

2.2 Reactualization

In `STAT_NON_LINE`, under the keyword factor `BEHAVIOR`, several ways of calculating of the deformations are possible:

- `'SMALL'`
- `'SIMO_MIEHE'` [R5.03.21] (which carries out calculation in great deformations for an isotropic work hardening)
- `'GDEF_HYPO_ELAS'` [R5.03.24] which carries out calculation in great deformations, but with a hypo-rubber band formulation, and which is usable for an unspecified work hardening)
- `'GROT_GDEP'` [R5.03.22] (which carries out calculation in great displacements and great rotations, but in small deformations)
- `'PETIT_REAC'` (which is a substitute with calculation in great deformations, valid for small increments of load, and for small rotations [bib2]).

This last possibility consists in reactualizing the geometry before calculating $\Delta\varepsilon$:
One writes $x=x_0+u_{i-1}+\Delta u_i^n$, the calculation of the gradients of Δu_i^n is thus made with the geometry x instead of the initial geometry x_0 .

2.3 Initial conditions

They are taken into account via $\boldsymbol{\sigma}^-$, p^- , $\mathbf{u}^-$.

In the event of continuation or resumption of a preceding calculation, there is directly the initial state $\boldsymbol{\sigma}^-$, p^- , $\mathbf{u}^-$ on the basis of $\boldsymbol{\sigma}$, p , $\mathbf{u}$ preceding calculation at the specified moment.

3 Relation of Von Mises with isotropic work hardening

3.1 Expression of the relations of behavior

These relations are obtained by the keywords `VMIS_ISOT_LINE`, `VMIS_ISOT_TRAC` and `VMIS_ISOT_PUIS`.

One describes here these relations into small deformation (`DEFORMATION=' PETIT')` :

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p=\dfrac{3}{2}\dot{p}\,.\dfrac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}=\dot{\boldsymbol{\varepsilon}}-\overbrace{A^{-1}}^{\cdot}\boldsymbol{\sigma}-\dot{\boldsymbol{\varepsilon}}^{th} \\ \sigma_{eq}-R(p)\le 0 \\ \begin{cases} \dot{p}=0 \text{ si } \sigma_{eq}-R(p)<0 \\ \dot{p}\ge 0 \text{ si } \sigma_{eq}-R(p)=0 \end{cases} \end{cases}$$

$\dot{\boldsymbol{\varepsilon}}^p$: vitesse de déformation plastique,
p : déformation plastique cumulée,
$\boldsymbol{\varepsilon}^{th}$: déformation d'origine thermique : $\boldsymbol{\varepsilon}^{th}=\alpha(T-T_{ref})\,\mathbf{Id}$

The function of work hardening $R(p)$ is deduced from a simple test tensile monotonous and isothermal

The user can choose a linear work hardening (relation VMIS_ISOT_LINE) or a given traction diagram is points by points (relation VMIS_ISOT_TRAC), that is to say by an analytical expression (relation VMIS_ISOT_PUIS).

The law of behavior VMIS_JOHN_COOK differ from the preceding ones in the direction where the function of work hardening depends on the speed of the cumulated plastic deformation and of the temperature.

One describes here these relations into small deformation (DEFORMATION=' PETIT'):

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \frac{3}{2}\dot{p}.\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}} = \dot{\boldsymbol{\varepsilon}} - \dot{\overbrace{A^{-1}}}\boldsymbol{\sigma} - \dot{\boldsymbol{\varepsilon}}^{th} \\ \sigma_{eq} - R(p, \dot{p}, T) \le 0 \\ \begin{cases} \dot{p} = 0 \text{ si } \sigma_{eq} - R(p, \dot{p}, T) < 0 \\ \dot{p} \ge 0 \text{ si } \sigma_{eq} - R(p, \dot{p}, T) = 0 \end{cases} \end{cases}$$

$\dot{\boldsymbol{\varepsilon}}^p$: vitesse de déformation plastique,
p : déformation plastique cumulée,
$\dot{p}$: vitesse de déformation plastique cumulée,
$\boldsymbol{\varepsilon}^{th}$: déformation d'origine thermique : $\boldsymbol{\varepsilon}^{th} = \alpha(T - T_{ref})\mathbf{Id}$

The function of work hardening $R(p, \dot{p}, T)$ is deduced from a series of monotonous simple tensile tests at different speed of deformation and different temperature.

3.1.1 Relation VMIS_ISOT_LINE

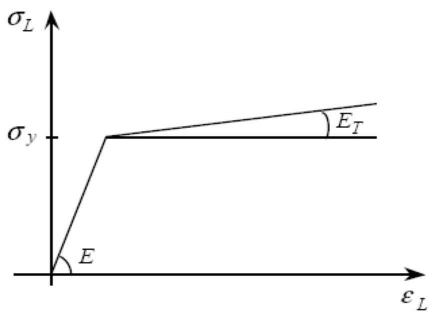
The data of the material characteristics are those provided under the keyword factor ECRO_LINE or ECRO_LINE_FO of the operator DEFI_MATERIAU [U4.43.01].

/ ECRO_LINE =_F (D_SIGM_EPSI = E_T , SY = σ_y)
/ ECRO_LINE_FO =_F (D_SIGM_EPSI = E_T , SY = σ_y)

ECRO_LINE_FO corresponds if E_T and σ_y depend on the temperature and are then calculated for the temperature of the point of current Gauss.

The Young modulus E and the Poisson's ratio ν are those provided under the keywords factors ELAS or ELAS_FO.

In this case the traction diagram is the following one:



I.e.:

$$\begin{cases} \sigma_L = E\,\varepsilon_L & \text{si } \varepsilon_L \le \dfrac{\sigma_y}{E} \\ \sigma_L = \sigma_y + E_T(\varepsilon_L - \dfrac{\sigma_y}{E}) & \text{si } \varepsilon_L \ge \dfrac{\sigma_y}{E} \end{cases}$$

Note:

σ_y is the elastic limit (the choice of σ_y fall to the user: it can correspond at the end of linearity of the real traction diagram, either to a lawful or conventional elastic limit. At all events, one uses here the single value defined under `ECRO_LINE`).

When the criterion is reached one a: $\sigma_{eq} - R(p) = 0$. To identify $R(p)$, the uniaxial state-owned properties of constraint are used:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \text{thus} \quad \begin{matrix} \sigma_{eq} = \sigma_L \\ p = \varepsilon_L^P = \varepsilon_L - \dfrac{\sigma_L}{E} \end{matrix} \quad \text{and the criterion is written: } \sigma_L - R(p) = 0$$

$$\sigma_L - \sigma_y = E_T(\varepsilon_L - \frac{\sigma_y}{E}) = E_T(\frac{\sigma_L}{E} + p - \frac{\sigma_y}{E})$$

thus

$$(\sigma_L - \sigma_y)\left(1 - \frac{E_T}{E}\right) = E_T\, p \quad \text{that is to say} \quad (\sigma_L - \sigma_y) = \frac{E_T . E}{(E - E_T)}\, p$$

from where the linear function of work hardening: $R(p) = \dfrac{E_T\, E}{E - E_T}\, p + \sigma_y$

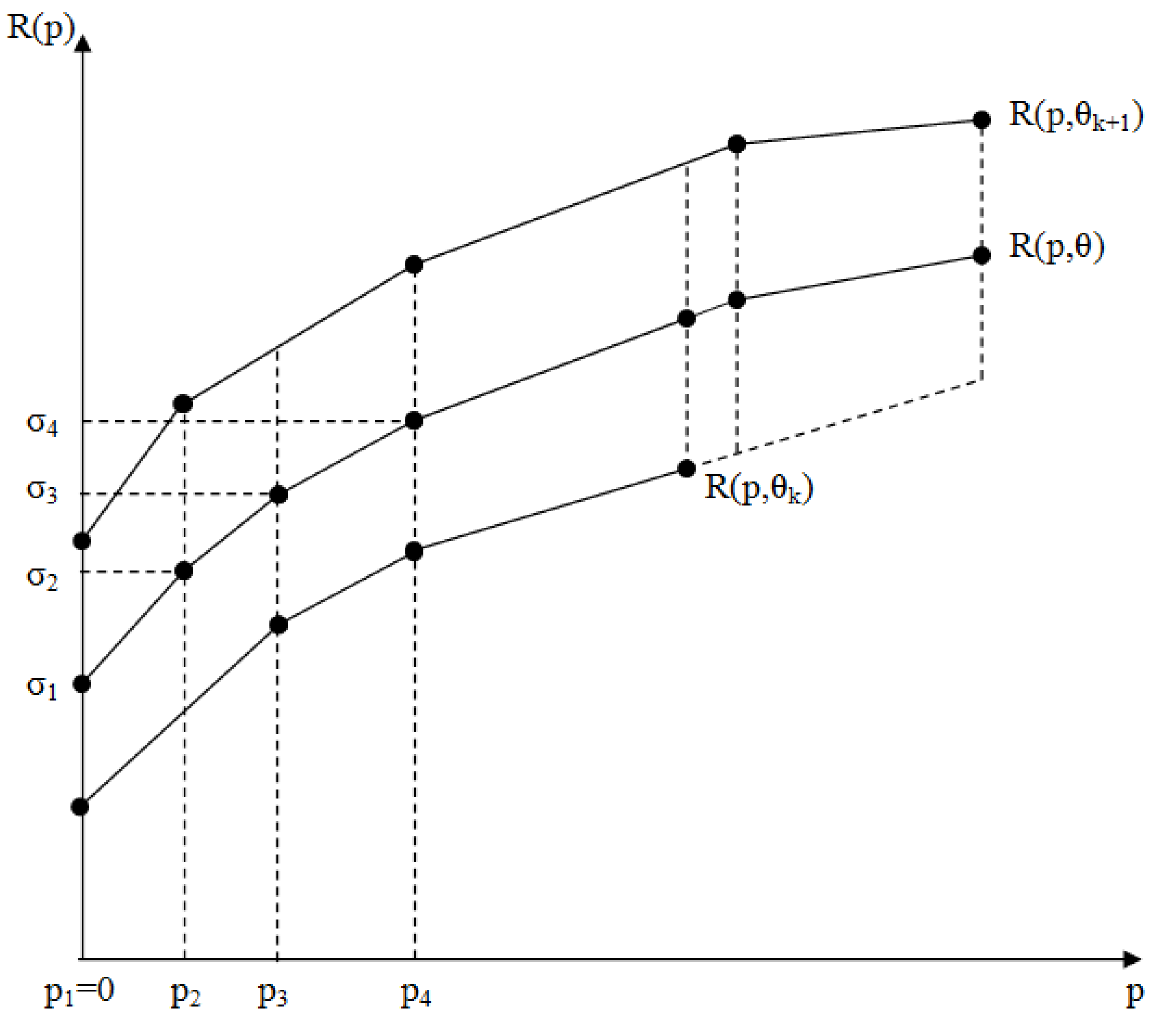
3.1.2 Relation `VMIS_ISOT_TRAC`

The data of material are those provided under the keyword factor `TRACTION =_F (SIGM = F)`, of the operator `DEFI_MATERIAU`.

`F` is a function with one or two variables representing the simple traction diagrams. The first variable is obligatorily the deformation, the second if it exists is the temperature (parameter of a tablecloth). For each temperature, the traction diagram must be such as:

- the X-coordinates (true deformations) are strictly increasing,
- the first point of the curve corresponds to $(\epsilon_{1,} \sigma_y)$ where $\epsilon_1 \ne 0$ and σ_y is the elastic limit,
- the slope between 2 successive points is lower than the elastic slope between 0 and the first point of the curve $(E = \sigma_y / \epsilon_1)$.

To interpolate compared to the temperature, *Code_Aster* transform initially all the curves $\sigma = f(\epsilon)$ data by the user in curves $\sigma(p) = R(p)$ in the following way: if E is the elastic slope between 0 and the first point of the curve $(\epsilon_{1,} \sigma_y)$ (where $\epsilon_1 \ne 0$ and σ_y is the elastic limit), any point of the curve (ϵ_i, σ_i) becomes the point (p_i, σ_i) with $p_i = \epsilon_i - \sigma_i / E$ (from where $p_1 = 0$). That is to say θ the temperature considered, if there exists k such as $\theta \in [\theta_k, \theta_{k+1}]$ where k indicate the index of the traction diagrams contained in the tablecloth, one builds the curve point by point $R(p, \theta)$ while interpolating from $R(p, \theta_k)$ and $R(p, \theta_{k+1})$ for all the values of p meeting of the values of the X-coordinates of the curves k and $k+1$ (if these two curves are prolonged linearly or by a constant function):



If n_k and n_{k+1} are the numbers of points of the curves k and $k+1$, the number of points n curve $R(p,\theta)$ is worth in the case general $n_k+n_{k+1}-1$ (case where all the nonworthless X-coordinates are distinct).

If θ is apart from the intervals of definition of the traction diagrams, one extrapolates in accordance with the prolongations specified by the user in DEFI_NAPPE [U4.21.03] and according to the preceding principle.

Note:
To avoid generating important errors of approximation or even obtaining by extrapolation of bad traction diagrams, it is not to better use linear prolongation in DEFI_NAPPE.

If the prolongation of the shortest curve is " EXCLUDED ", one stops the interpolation at this place and the prolongation of the interpolated curve is also " EXCLUDED ".

One thus obtains in all the cases a linear function of work hardening per pieces:

$$R(p,q)=s_i+\frac{s_{i+1}-s_i}{p_{i+1}-p_i}(p-p_i) \text{ for } p\in[p_i,p_{i+1}] \text{ for } i+1\leq n \text{, with } p_1=0$$

The Young modulus corresponding to the temperature θ is calculated in the following way:

$$E=E_k+\frac{q-q_k}{q_{k+1}-q_k}(E_{k+1}-E_k)$$

where, for $i=k$ or $i=k+1$, E_i is the elastic slope enters 0 and the first point of the curve $\sigma=f(\epsilon)$ corresponding to the temperature θ_i.

It is this Young modulus who is used in the integration of the relation of behavior.

Limit elastic at the temperature θ is worth:

$$\sigma_y=R(0,\theta)=\sigma_1$$

The user must also give the Poisson's ratio ν and a fictitious Young modulus (who is only used for to calculate the elastic matrix of rigidity if the keyword `NEWTON=_F (MATRICE=' ELASTIQUE')` is present in `STAT_NON_LINE`) by the keywords:

`/ ELAS =_F (NAKED =` ν `, E =` E `)`
`/ ELAS_FO =_F (NAKED =` ν `, E =` E `)`

3.1.3 Relation VMIS_ISOT_PUIS

The data of material are those provided under the keyword factor `ECRO_PUIS` or `ECRO_PUIS_FO` of the operator `DEFI_MATERIAU` [U4.43.01].

`ECRO_PUIS=_F (SY=` σ_y `, A_PUIS =a, N_PUIS =n)`

The curve of work hardening is deduced from the uniaxial curve connecting the deformations to the constraints, whose expression is:

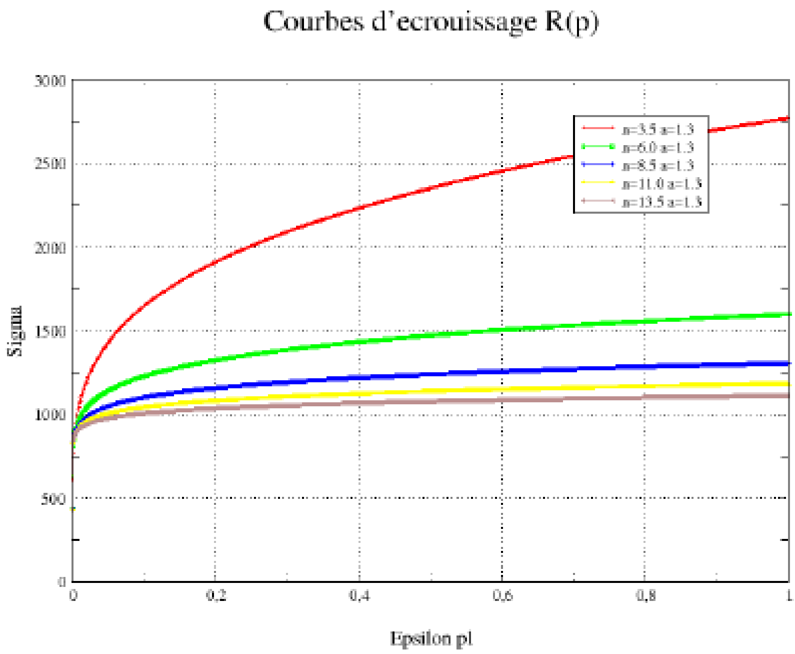
$$\varepsilon=\frac{\sigma}{E}+a\frac{\sigma_y}{E}\left(\frac{\sigma-\sigma_y}{\sigma_y}\right)^n \text{ for } \sigma>\sigma_y$$

what gives the curve of work hardening:

$$R(p)=\sigma_y+\sigma_y\left(\frac{E}{a\sigma_y}p\right)^{\frac{1}{n}}$$

The curve representative of such a function takes the following form, for various values of N:

The data of material are those provided under the keyword factor `ECRO_COOK` or `ECRO_COOK_FO` of the operator `DEFI_MATERIAU` [U4.43.01].



Caution: this law of work hardening **is not at all identical** with the law of Ramberg-Osgood (often used as assumption in the analyses simplified in breaking process). The relation stress-strains of the law of Ramberg-Osgood is:

$\varepsilon=\frac{\sigma}{E}+\alpha\frac{\sigma}{E}\left(\frac{\sigma}{\sigma_y}\right)^{n-1}$ for all σ .

3.1.4 Relation VMIS_JOHN_COOK

The data of material are those provided under the keyword factor ECRO_COOK or ECRO_COOK_FO of the operator DEFI_MATERIAU [U4.43.01].

```
ECRO_COOK =_F (A=A, B=B, C=C, N_PUIS=n, M_PUIS=m, EPSP0=epsp0, TROOM=troom, TMELT=tmelt,)
```

The curve of work hardening is written in the following way:

$$R(\ p,\dot{p},T)=(\ A+B\,p^n)\left(1+C\ln\left(\frac{\dot{p}}{\dot{p}_0}\right)\right)\left(1-\left(\frac{T-T_{room}}{T_{melt}-T_{room}}\right)^m\right)$$

or in a more concise way:

$$R(\ p,\dot{p},T)=(\ A+B\,p^n)(\ 1+C\dot{p}^*)(\ 1-T^{*m})$$

with $\dot{p}^*=\begin{cases}\frac{\dot{p}}{\dot{p}_0} & si\quad \dot{p}\geq\dot{p}_0\\ 1 & si\quad \dot{p}\leq\dot{p}_0\end{cases}$ and $T^*=\begin{cases}\frac{T-T_{room}}{T_{melt}-T_{room}} & si\quad T\geq T_{room}\\ 0 & si\quad T\leq T_{room}\end{cases}$

3.2 Tangent operator. Option RIGI_MECA_TANG

The goal of this paragraph is to calculate the tangent operator K_{i-1} (option of calculation RIGI_MECA_TANG called with the first iteration of a new increment of load) starting from the results known at the previous moment t_{i-1}.

For that, if the tensor of the constraints with t_{i-1} is on the border of the field of elasticity, one writes the condition:

$$\dot{f}=0$$

who must be checked (for the continuous problem in time) jointly in the condition:

$$f=0$$

with:

$$f(\sigma,p)=\sigma_{eq}-R(p)$$

So on the other hand the tensor of the constraints with t_{i-1} is inside the field, $f<0$, then the tangent operator is the operator of elasticity.

The quantities intervening in this expression are calculated at the previous moment t_{i-1}, which is the only known ones at the time of the phase of prediction. One thus obtains:

$$\dot{f}=\frac{\partial f}{\partial\boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial\boldsymbol{\sigma}}\dot{\tilde{\boldsymbol{\sigma}}}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial\boldsymbol{\sigma}}\left(2\mu\dot{\tilde{\boldsymbol{\varepsilon}}}-2\mu\dot{\boldsymbol{\varepsilon}}^P\right)+\frac{\partial f}{\partial p}\dot{p}$$
$$=\frac{\partial f}{\partial\boldsymbol{\sigma}}\left(2\mu\dot{\boldsymbol{\varepsilon}}-2\mu\dot{\boldsymbol{\varepsilon}}^P\right)+\frac{\partial f}{\partial p}\dot{p},$$

because $\frac{\partial f}{\partial\boldsymbol{\sigma}}$ is deviative.

With

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^-=\boldsymbol{\sigma}\left(t_{i-1}\right),\ \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^-=\boldsymbol{\varepsilon}\left(t_{i-1}\right),\ \boldsymbol{\varepsilon}^p=\boldsymbol{\varepsilon}^{p^-}=\boldsymbol{\varepsilon}^p\left(t_{i-1}\right)\text{ and }p=p^-=p\left(t_{i-1}\right)$$

Note:

One does not take account in this expression of the variation of the elastic coefficients with the temperature. It is an approximation, without important consequence, since this operator is only used to initialize the iterations of Newton. On the other hand, the dependence of the tangent operator compared to the thermal deformations is well taken into account on the level of the total algorithm [R5.03.01].

One has then: $\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}\left(2\mu\dot{\boldsymbol{\varepsilon}}-2\mu\dot{p}\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}\right)-R'(p)\dot{p}=0$

what leads to: $\dot{p}=\frac{(3\mu)}{\sigma_{eq}}\frac{\left(\tilde{\boldsymbol{\sigma}}.\dot{\tilde{\boldsymbol{\varepsilon}}}\right)}{3\mu+R'(p)}$ thus

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \begin{cases} \frac{9\mu}{2} \frac{(\tilde{\boldsymbol{\sigma}} . \dot{\tilde{\boldsymbol{\varepsilon}}})}{3\mu + R'(p)} \frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}^2}, \text{si } f(\sigma, p) = \sigma_{eq} - R(p) = 0 \\ 0, \qquad \text{si } \sigma_{eq} - R(p) < 0 \end{cases} \\ \dot{\sigma}_{ij} = K \dot{\varepsilon}_{kk} \boldsymbol{\delta}_{ij} + 2\mu \left(\dot{\tilde{\varepsilon}}_{ij} - \dot{\varepsilon}_{ij}^p \right) \end{cases}$$

Note:

Information $\boldsymbol{\sigma}_{eq}^- - R(p^-) = 0$ is preserved in the form of an internal variable ξ who is worth 1 in this case and 0 if $\boldsymbol{\sigma}_{eq}^- < R(p^-)$.

The tangent operator binds the vector of virtual deformations $\boldsymbol{\varepsilon}^*$ with a vector of virtual constraints $\boldsymbol{\sigma}^*$.

The matrix of tangent rigidity is written for an elastic behavior:

$$\boldsymbol{\sigma}^* = \left(K \vec{\mathbf{1}} \otimes \vec{\mathbf{1}} + 2\mu \mathbf{P} \right) \boldsymbol{\varepsilon}^*$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^* = \left(K \vec{\mathbf{1}} \otimes \vec{\mathbf{1}} + 2\mu \mathbf{P} - C_p \mathbf{s} \otimes \mathbf{s} \right) \boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of the deviatoric constraints associated with $\boldsymbol{\sigma}^-$ defined by:

$$\mathbf{s}^T = \left(\tilde{\sigma}_{11}^-, \tilde{\sigma}_{22}^-, \tilde{\sigma}_{33}^-, \sqrt{2}\, \tilde{\sigma}_{12}^-, \sqrt{2}\, \tilde{\sigma}_{23}^-, \sqrt{2}\, \tilde{\sigma}_{31}^- \right)$$

and:

$$C_p = \xi \frac{(3\mu)^2}{(\sigma_{eq})^2} \frac{1}{3\mu + R'}$$

$$\xi = \begin{cases} 1 & \text{si } \sigma_{eq}^- - R(p^-) = 0 \\ 0 & \text{sinon} \end{cases}$$

In the case of the first increment of loading, therefore if the state at the previous moment corresponds in a nonconstrained initial state, the tangent operator is identical to the operator of elasticity.

3.3 Calculation of the constraints and the internal variables

The decomposition of the deformations makes it possible to write:

$$\Delta \boldsymbol{\varepsilon} = \Delta \boldsymbol{\varepsilon}^p + \Delta \left(\mathbf{A}^{-1} \boldsymbol{\sigma} \right) + \Delta \boldsymbol{\varepsilon}^{th}$$

Maybe, by taking the spherical and deviatoric parts:

$$\Delta \tilde{\boldsymbol{\varepsilon}} = \Delta \boldsymbol{\varepsilon}^p + \Delta \left(\frac{\tilde{\boldsymbol{\sigma}}}{2\mu} \right) \text{ because } \Delta \tilde{\boldsymbol{\varepsilon}}^{th} = 0 .$$

$$tr\,\Delta\,\boldsymbol{\varepsilon}=\Delta\left(\frac{\mathrm{tr}\,\boldsymbol{\sigma}}{3\mathrm{K}}\right)+tr\,\Delta\,\boldsymbol{\varepsilon}^{th} \text{ because } \mathrm{tr}\,\Delta\,\boldsymbol{\varepsilon}^{p}=0\,.$$

By direct implicit discretization of the relations of behaviour for isotropic work hardening, one obtains then:

$$2\mu\,\Delta\,\tilde{\boldsymbol{\varepsilon}}-(\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}})=\frac{3}{2}2\mu\,\Delta\,p\frac{\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}}{(\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}})_{eq}}-2\frac{\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}$$

$$tr\,\boldsymbol{\sigma}=\frac{3\mathrm{K}}{3\mathrm{K}^{-}}tr\,\boldsymbol{\sigma}^{-}+3K\,tr\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}-3K\,tr\boldsymbol{\Delta}\,\boldsymbol{\varepsilon}^{th}$$

$$\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}-R\left(p^{-}+\Delta\,p\right)\leq 0$$

$$\Delta\,p=0\,\mathrm{si}\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}<R\left(p^{-}+\Delta\,p\right)$$

$$\Delta\,p\geq 0\,\mathrm{si}\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}=R\left(p^{-}+\Delta\,p\right)$$

One defines, to simplify the notations, the tensor σ^{e} such as:

$$\tilde{\boldsymbol{\sigma}}^{e}=\frac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}+2\mu\,\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}} \text{ and } \mathrm{tr}\,\sigma^{e}=\mathrm{tr}\,\sigma\,.$$

Two cases arise:

- $\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}<R\,(p^{-}+\Delta\,p)$

  in this case: $\Delta\,p=0\,\mathrm{soit}\,\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{e}$, thus: $\left(\tilde{\boldsymbol{\sigma}}^{e}\right)_{eq}<R\left(p^{-}\right)$

- $\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}=R\left(p^{-}+\Delta\,p\right)$

  in this case: $\Delta\,p\geq 0$ thus: $\left(\tilde{\boldsymbol{\sigma}}^{e}\right)_{eq}\geq R\left(p^{-}\right)$

One from of deduced the algorithm from resolution:

- if $\tilde{\sigma}^{e}_{eq}\leq R\left(p^{-}\right)$ then $\Delta\,p=0\mathrm{soit}\,\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}=\tilde{\boldsymbol{\sigma}}^{e}$
- if $\tilde{\sigma}^{e}_{eq}>R\left(p^{-}\right)$

  then it is necessary to solve: $\tilde{\boldsymbol{\sigma}}^{e}=\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}+\frac{3}{2}2\mu\,\Delta\,p\frac{\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}}{(\boldsymbol{\sigma}^{-}+\Delta\,\boldsymbol{\sigma})_{eq}}$

thus while factorizing $\tilde{\boldsymbol{\sigma}}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}$ and by taking the equivalent value of Von Mises:

$$\sigma^{e}_{eq}=\left(1+\frac{3}{2}\frac{2\mu\,\Delta\,p}{\left(\boldsymbol{\sigma}^{-}+\Delta\,\boldsymbol{\sigma}\right)_{\mathrm{eq}}}\right)\left(\boldsymbol{\sigma}^{-}+\Delta\,\boldsymbol{\sigma}\right)_{\mathrm{eq}}$$

that is to say:

$$\sigma^{e}_{eq}=R\left(p^{-}+\Delta\,p\right)+3\mu\,\Delta\,p$$

because: $\sigma_{eq}=\left(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta}\,\boldsymbol{\sigma}\right)_{eq}=R\left(\overline{p}+\Delta\,p\right)$

It is a scalar equation in $\Delta\,p$, linear or not according to $R(p)$. $\Delta\,p$ is calculated in the following way:

- if work hardening is linear (relation VMIS_ISOT_LINE), one obtains directly:

$$\Delta p = \frac{\sigma_{eq}^{e} - \sigma_y - R' p^-}{R' + 3\mu} \quad R' = \frac{E E_T}{E - E_T} \text{ with}$$

- if work hardening is given by a traction diagram closely connected per pieces, (relation VMIS_ISOT_TRAC), one benefits from the linearity per pieces to determine exactly Δp (see §Annexe 1);
- in the case of a work hardening defined by a law in power (relation VMIS_ISOT_PUIS), Δp is solution of the nonlinear equation: $R\left(p^- + \Delta p\right) + 3\,\mu\,\Delta p - \sigma_{eq}^{e} = 0$. This equation is solved by an iterative method (algorithm of the secant type). In the vicinity of the origin, one linearizes $R(p)$, because the derivative $R' = \frac{E}{an}\left(\frac{E}{a\,\sigma_y} p\right)^{\frac{1}{n}-1}$ is infinite in $p = 0$. Thus if $p < p_0$, one replaces $R(p)$ by $R^{lin}(p) = \sigma_y + \frac{p}{p_0}\left(R(p_0) - \sigma_y\right)$, which avoids the search for a solution numerically almost worthless. In practice, one chooses $p_0 = 10^{-10}$.

Once Δp determined, one calculates σ by:

$$\tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}} = \frac{\sigma_{eq}^{e} - 3\mu\Delta p}{\sigma_{eq}^{e}} . \tilde{\boldsymbol{\sigma}}^{e}$$

and

$$\operatorname{tr}\left(\boldsymbol{\sigma}^- + \Delta\boldsymbol{\sigma}\right) = \operatorname{tr}\boldsymbol{\sigma}^{\mathbf{e}} .$$

Options RAPH_MECA and FULL_MECA both carry out the preceding calculation, which clarifies the calculation of $\mathbf{R}\left(\mathbf{u}_i^n\right)$. It is noticed that actually, $\mathbf{R}(\mathbf{u}_i^n) = \mathbf{Q}^T \boldsymbol{\sigma}_i^n$ where $\boldsymbol{\sigma}_i^n$ is calculated not according to $\mathbf{u}_i^n$, but of $\boldsymbol{\sigma}_{i-1}$ et $\Delta\,\mathbf{u}_i^n$.

Note:

Typical case of the plane constraints.

The model of Von Mises with isotropic work hardening (VMIS_ISOT_LINE, VMIS_ISOT_PUIS or VMIS_ISOT_TRAC) is also available in plane constraints, i.e. for modelings C_PLAN, DKT, COQUE_3D, COQUE_AXIS, COQUE_D_PLAN, COQUE_C_PLAN, PIPE, TUYAU_6M.

In this case, the system to be solved comprises an additional equation. This calculation is detailed in appendix 2.

3.4 Tangent operator. Option FULL_MECA

The option FULL_MECA allows to calculate the tangent matrix K_i^n with each iteration. The tangent operator who is used for building it is calculated directly on the preceding discretized system (one notes to simplify: $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}}, p = p^- + \Delta p$) and one writes the expressions only in the isothermal case.

- If the tensor of the constraints is on the border of the field, $f = 0$ then one has, by differentiating the expression of the law of normality in $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^- + \Delta\tilde{\boldsymbol{\sigma}}$:

$$2\mu\,\delta\boldsymbol{\epsilon}^{p}=2\mu\,\delta\,\tilde{\boldsymbol{\varepsilon}}-\delta\,\tilde{\boldsymbol{\sigma}}=\frac{3}{2}2\mu\left(\delta\,p\frac{\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}+\Delta\,p\frac{\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}-\frac{3}{2}\Delta\,p\frac{\tilde{\sigma}:d\,\tilde{\sigma}}{\sigma_{eq^3}}.\tilde{\boldsymbol{\sigma}}\right)$$

where $\boldsymbol{\delta\,\varepsilon}^{p},\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}},\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}$ represent infinitesimal increases around the solution in the incremental elastoplastic problem obtained previously.

Like:

$$\frac{3}{2}\frac{\tilde{\boldsymbol{\sigma}}:d\,\tilde{\boldsymbol{\sigma}}}{\sigma_{eq}}=R^{'}(p)\,dp$$

by carrying out the tensorial product of the first equation by $\tilde{\boldsymbol{\sigma}}$ one a:

$$2\mu\,\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}}-\tilde{\boldsymbol{\sigma}}:\delta\,\tilde{\boldsymbol{\sigma}}=2\mu\,\delta\,p.\sigma_{eq}$$

while eliminating $\delta\,p$ of the two last equations:

$$\tilde{\boldsymbol{\sigma}}:\delta\tilde{\boldsymbol{\sigma}}=\frac{2\mu\,\tilde{\boldsymbol{\sigma}}:\delta\,\tilde{\boldsymbol{\varepsilon}}}{1+\dfrac{3\mu}{R^{'}(p)}}\;.$$

- So on the other hand if the tensor of the constraints is inside the field, $f<0$, then the tangent operator is the operator of elasticity.

While expressing $\delta\,p$ and $\tilde{\boldsymbol{\sigma}}:\delta\,\tilde{\boldsymbol{\sigma}}$ in the first equation, one obtains:

$$2\mu\,\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}}-\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}=\frac{3\mu\,\Delta\,p}{\sigma_{eq}}\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}+C_{p}.(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}})_{+}\tilde{\boldsymbol{\sigma}},$$

with:

$$C_{p}=\frac{9\mu^{2}}{\sigma_{eq}^{2}}\left(1-\frac{R^{'}(p)\Delta\,p}{\sigma_{eq}}\right)\frac{1}{R^{'}(p)+3\mu}$$

The positive part $(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}})_{+}$ allows to gather in only one equation the two conditions:

- that is to say $f<0$, which implies $\Delta\,p=0$
- that is to say $f=0$

One obtains then:

$$\boldsymbol{\delta}\,\tilde{\boldsymbol{\sigma}}=\frac{2\mu}{a}\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}}-\frac{C_{p}}{a}(\tilde{\boldsymbol{\sigma}}:\boldsymbol{\delta}\,\tilde{\boldsymbol{\varepsilon}})_{+}\tilde{\boldsymbol{\sigma}}$$

while posing:

$$a=1+\frac{3\mu\,\Delta\,p}{R(p+\Delta\,p)}$$

The tangent operator binds the vector of virtual deformations $\boldsymbol{\varepsilon}^{*}$ with a vector of virtual constraints $\boldsymbol{\sigma}^{*}$.

The matrix of tangent rigidity is written for an elastic behavior:

$$\boldsymbol{\sigma}^* = \left(K\, \vec{\mathbf{1}} \otimes \vec{\mathbf{1}} + 2\mu\, \mathbf{P} \right) \boldsymbol{\varepsilon}^*$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^* = \left(K\, \vec{\mathbf{1}} \otimes \vec{\mathbf{1}} + \frac{2\mu}{a} \mathbf{P} - \frac{C_p}{a} \mathbf{s} \otimes \mathbf{s} \right) \boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of the deviatoric constraints associated with $\boldsymbol{\sigma}^-$ defined by:

$$\mathbf{s}^{\mathbf{T}} = \left(\tilde{\sigma}_{11}^-, \tilde{\sigma}_{22}^-, \tilde{\sigma}_{33}^-, \sqrt{2}\, \tilde{\sigma}_{12}^-, \sqrt{2}\, \tilde{\sigma}_{23}^-, \sqrt{2}\, \tilde{\sigma}_{31}^- \right)$$

and:

$$\xi = \begin{cases} 1 \text{ si } \Delta\varepsilon \text{ conduit à une plastification} \quad \text{et } \tilde{\boldsymbol{\sigma}} . \Delta \tilde{\boldsymbol{\varepsilon}} \geq 0 \\ 0 \text{ sinon} \end{cases}$$

It is noted that the tangent operator with the system resulting from the implicit discretization differs from the tangent operator to the problem of speed (`RIGI_MECA_TANG`). One finds it while making: $\Delta p = 0$ in the expressions of C_p and a.

3.5 Internal variables of the behaviors VMIS_ISOT_LINE, VMIS_ISOT_PUIS, VMIS_ISOT_TRAC and VMIS_JOHN_COOK

Relations of behavior `VMIS_ISOT_LINE`, `VMIS_ISOT_PUIS` and `VMIS_ISOT_TRAC` two internal variables produce:

- p cumulated equivalent plastic deformation,
- and χ indicator of plasticity at the moment considered (useful for the calculation of the tangent operator).

`VMIS_JOHN_COOK` use two internal variables besides the two preceding ones:

- $\dot{p}^-$ speed of plastic deformation equivalent cumulated to the moment less,
- and Δt^- the increment of step of time at the moment less.

4 Relation of Von Mises with linear kinematic work hardening

4.1 Expression of the relation of behavior, case general

This relation is obtained by the keyword VMIS_CINE_LINE keyword factor BEHAVIOR.

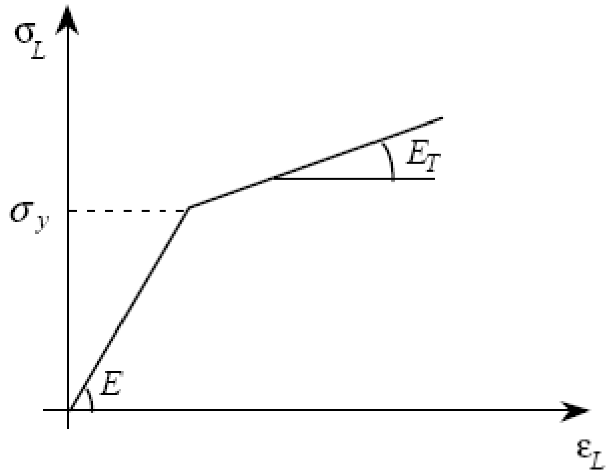
She is written (always in small deformations):

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \frac{3}{2}\dot{p}\frac{\tilde{\boldsymbol{\sigma}}-\tilde{\mathbf{X}}}{(\sigma-X)_{eq}} = \frac{3}{2}\dot{p}\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{(\sigma-X)_{eq}} = \dot{\boldsymbol{\varepsilon}} - \overbrace{\mathbf{A}^{-1}\boldsymbol{\sigma}}^{\cdot} - \dot{\boldsymbol{\varepsilon}}^{th} \\ \mathbf{X} = C\,\boldsymbol{\varepsilon}^p \quad , \quad \boldsymbol{\varepsilon}^{th} = \alpha(T - T_{ref})\,\mathbf{Id} \\ (\boldsymbol{\sigma}-\mathbf{X})_{eq} - \sigma_y \le 0 \\ \begin{cases} \dot{p} = 0 \text{ si } (\boldsymbol{\sigma}-\mathbf{X})_{eq} - \sigma_y \le 0 \\ \dot{p} \ge 0 \text{ si } (\mathbf{s}-\mathbf{X})_{eq} - \sigma_y = 0 \end{cases} \end{cases} \qquad \text{éq 4.1-1}$$

σ_y is the elastic limit (the choice of σ_y fall to the user: it can correspond at the end of linearity of the real traction diagram, either to a lawful or conventional elastic limit… At all events, one uses here the single value defined under ECRO_LINE).
C is the coefficient of work hardening deduced from the data by a simple tensile test.

In this case (tensor of constraints uniaxial, tensor of plastic deformations isochoric and orthotropic):

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} X_L & 0 & 0 \\ 0 & -\frac{X_L}{2} & 0 \\ 0 & 0 & -\frac{X_L}{2} \end{pmatrix}$$



$$(\boldsymbol{\sigma}-\mathbf{X})_{eq} = \sigma_L - \frac{3}{2}X_L \text{ and } X_L = C\,\varepsilon_L^p = C\left(\varepsilon_L - \frac{\sigma_L}{E}\right)$$

The data material are those provided under the keyword factor `ECRO_LINE` or `ECRO_LINE_FO` of the operator `DEFI_MATERIAU` :

```
/ ECRO_LINE =_F (D_SIGM_EPSI =  E_T  ,  SY =  σ_y  )
/ ECRO_LINE_FO =_F (D_SIGM_EPSI =  E_T  , SY =  σ_y  )
```

`ECRO_LINE_FO` corresponds if E_T and σ_y depend on the temperature and are then calculated for the temperature of the point of current Gauss.

The Young modulus E and the Poisson's ratio are those provided under the keywords factors `ELAS` or `ELAS_FO`.

For $\varepsilon_L > \frac{\sigma_y}{E}$ $\sigma_L = \sigma_y + E_T \left(\varepsilon_L - \frac{\sigma_y}{E} \right)$,

but one also has:

$$\begin{cases} \sigma_L - \frac{3}{2} X_L = \sigma_y \\ X_L = C \left(\varepsilon_L - \frac{\sigma_L}{E} \right) \end{cases}$$

from where, while eliminating X_L and while identifying:

$$C = \frac{2}{3} \frac{E E_T}{E - E_T} .$$

4.2 Expression of the relation of behavior in 1D

For reasons of performances the relation is also written in 1D for a use with finite elements of standard beam multifibre. The preceding equations are identical, the sizes σ_L , X_L and ε_L are scalars.
The data material are those provided under the keyword factor `ECRO_LINE` of the operator `DEFI_MATERIAU [U4.43.01]` :

```
/ECRO_LINE = _F (
        ♦ D_SIGM_EPSI   =  E_T                              [ R éel]
        ♦ SY            =  σ_y                              [ R éel]
       ◊  SIGM_ELS      = sigmels                           [R éel]
       ◊  EPSI_ELU      =  epsielu                          [ R éel]
)
```

Operands `SIGM_ELS` and `ESPI_ELU` allow to define the terminals which correspond with the states limiting of service and ultimate, classically used at the time of study in civil engineer.

◊ SIGM_ELS = sgels
Definition of the ultimate stress of service.

◊ EPSI_ELU = epelu
Definition of the ultimate limiting deformation.

These terminals are obligatory when the behavior is used `ECRO_CINE _1D` (Cf [U4. 5 1. 11] non-linear Behaviors, [U4.42.07] DEFI_MATER_GC). In the other cases they are not taken into account.

Supported modeling is `1D`, numbers of internal variables is of 6.

- $V1$: Criterion ELS: `CRITELS`. This variable gives information compared to the absolute limit of service. This variable represents the absolute value of the constraint divided by the ultimate stress with the ELS material. If this variable is in $[0,1]$ the material respects the ELS.
- $V2$: ELECTED criterion: `CRITELU`. This variable gives information compared to the ultimate absolute limit. This variable represents the absolute value of the total deflection divided by the deformation limits to the ELU material. If this variable is in $[0,1]$ the material respects the ELECTED OFFICIAL.
- $V3$: Kinematic work hardening: `XCINXX`. In 1D only a scalar is necessary.
- $V4$: Plastic indicator: `INDIPLAS`. Indicate if the material exceeded the elastic criterion.
- $V5$: nonrecoverable dissipation: `DISSIP`. During seismic calculations it can be useful for the user to know nonrecoverable dissipated energy. The variable `DISSIP` represent the nonrecoverable office plurality of energy. The nonrecoverable increment of energy is written in the form:

$$\Delta Eg=\frac{1}{2}\left(E^{+}\Delta\varepsilon-\left(\sigma^{+}-\sigma^{-}\right)\right)\Delta\varepsilon$$

- $V6$: thermodynamic dissipation: `DISSTHER` . The thermodynamic increment of dissipation is written in the form: $\Delta Eg=\sigma_y\dot{p}$.

4.3 Tangent operator. Option RIGI_MECA_TANG

The goal of this paragraph is to calculate the tangent operator K_{i-1} (option of calculation `RIGI_MECA_TANG` called with the first iteration of a new increment of load) starting from the results known at the previous moment t_{i-1} .

For that, if the tensor of the constraints with t_{i-1} is on the border of the field of elasticity, one writes the condition:

$$\dot{f}=0$$

who must be checked (for the continuous problem in time) jointly in the condition:

$$f=0$$

with

$$f=f\left(\boldsymbol{\sigma}^{-},\mathbf{X}^{-}\right)=\left(\boldsymbol{\sigma}^{-}-\mathbf{X}^{-}\right)_{eq}-\sigma_y$$

So on the other hand the tensor of the constraints with t_{i-1} is inside the field, $f<0$, then the tangent operator is the operator of elasticity.

One poses:

$$\boldsymbol{\sigma}^{dev}=\tilde{\boldsymbol{\sigma}}^{-}-\mathbf{X}^{-}\ \text{et}\ \gamma=\begin{cases}1 & \text{si}\ \left(\boldsymbol{\sigma}^{-}-\mathbf{X}^{-}\right)_{eq}-\sigma_y=0\quad(\text{variable interne}\ \chi)\\ 0 & \text{sinon}\end{cases}$$

The problem of speed is written in this case:

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \begin{cases} \dfrac{1}{2\mu}\dfrac{3}{2}\left(\dfrac{2\mu}{\sigma_y}\right)^2 \dfrac{\left(\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right).\dot{\tilde{\boldsymbol{\varepsilon}}}\right)\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right)}{C+2\mu} \text{ si } \left(\boldsymbol{\sigma}-\mathbf{X}\right)-\sigma_y=0 \\ 0 \text{ si } \left(\mathbf{s}-\mathbf{X}\right)_{eq}-\sigma_y<0 \end{cases} \\ \dot{\sigma}_{ij}=K\,\dot{\varepsilon}_{kk}\,\delta_{ij}+2\mu\left(\dot{\tilde{\varepsilon}}_{ij}-\dot{\varepsilon}^p_{ij}\right) \end{cases}$$

The tangent operator binds the vector of virtual deformations $\boldsymbol{\varepsilon}^*$ with a vector of virtual constraints $\boldsymbol{\sigma}^*$.
The matrix of tangent rigidity is written for an elastic behavior:

$$\boldsymbol{\sigma}^*=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\,\mathbf{P}\right)\boldsymbol{\varepsilon}^*$$

and for a plastic behavior:

$$\boldsymbol{\sigma}^*=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\mu\,\mathbf{P}-C_p\,\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of the deviatoric constraints associated with $\boldsymbol{\sigma}^{dev}$ defined by:

$$\mathbf{s}^{\mathbf{T}}=\left(\sigma^{dev}_{11},\sigma^{dev}_{22},\sigma^{dev}_{33},\sqrt{2}\,\sigma^{dev}_{12},\sqrt{2}\,\sigma^{dev}_{23},\sqrt{2}\,\sigma^{dev}_{31}\right)$$

and:

$$C_p=\gamma\frac{3}{2}\left(\frac{2\mu}{\sigma_y}\right)^2\frac{1}{2\mu+C}$$

In the case of the first increment of loading, therefore if the state at the previous moment corresponds in a nonconstrained initial state, the tangent operator is identical to the operator of elasticity.

4.4 Calculation of the constraints and internal variables

The direct implicit discretization of the continuous relations results in solving:

$$\begin{cases} 2\mu\,\Delta\,\boldsymbol{\varepsilon}^p=2\mu\left(\Delta\,\tilde{\boldsymbol{\varepsilon}}+\dfrac{\tilde{\boldsymbol{\sigma}}^-}{2\mu^-}-\dfrac{\tilde{\boldsymbol{\sigma}}}{2\mu}\right)=\dfrac{3}{2}2\mu\,\Delta\,p\,\dfrac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y} \\ \mathbf{X}=\dfrac{C}{C^-}\mathbf{X}^-+C\,\Delta\,\boldsymbol{\varepsilon}^p \\ \left(\boldsymbol{\sigma}-\mathbf{X}\right)_{eq}\leq\sigma_y \\ \Delta\,p=0 \text{ si } \left(\boldsymbol{\sigma}-\mathbf{X}\right)_{eq}<\sigma_y \\ \Delta\,p\geq 0 \text{ sinon} \\ \operatorname{tr}\left(\boldsymbol{\sigma}^-+\Delta\,\boldsymbol{\sigma}\right)=\dfrac{3K}{3K^-}\operatorname{tr}\boldsymbol{\sigma}^-+3K\operatorname{tr}\Delta\,\boldsymbol{\varepsilon}-3K\operatorname{tr}\Delta\,\boldsymbol{\varepsilon}^{th} \end{cases}$$

One still poses:

$$\tilde{\boldsymbol{\sigma}}^e=\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-+2\mu\,\Delta\,\tilde{\boldsymbol{\varepsilon}}-\frac{C}{C^-}\mathbf{X}^-\ .$$

The first equation is also written:

$$\left(2\mu\,\mathbf{\Delta}\,\tilde{\boldsymbol{\varepsilon}}+\frac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}\right)=\tilde{\boldsymbol{\sigma}}+\frac{3}{2}2\mu\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}$$

while cutting off $\mathbf{X}=\frac{C}{C^{-}}\mathbf{X}^{-}+C\,\mathbf{\Delta}\,\boldsymbol{\varepsilon}^{p}$ has each term, one obtains:

$$2\mu\,\mathbf{\Delta}\,\tilde{\boldsymbol{\varepsilon}}+\frac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}-\frac{C}{C^{-}}\mathbf{X}^{-}=\tilde{\boldsymbol{\sigma}}-\mathbf{X}+\frac{3}{2}2\mu\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}+C\,\mathbf{\Delta}\,\boldsymbol{\varepsilon}^{\mathbf{P}}$$

or, by using the law of flow:

$$\tilde{\boldsymbol{\sigma}}^{e}=\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right)\left(1+\frac{3}{2}\left(2\mu+C\right)\frac{\Delta p}{\sigma_y}\right)$$

One still obtains a scalar equation in Δp by taking the equivalent values of Von Mises:

$$\sigma_{eq}^{e}=\sigma_y+\frac{3}{2}\left(2\mu+C\right)\Delta p$$

what gives directly:

$$\Delta p=\frac{\sigma_{eq}^{e}-\sigma_y}{\frac{3}{2}\left(2\mu+C\right)}$$

And σ is obtained by: $\tilde{\boldsymbol{\sigma}}=\frac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}+2\mu\,\mathbf{\Delta}\,\tilde{\boldsymbol{\varepsilon}}+2\mu\,\mathbf{\Delta}\,\boldsymbol{\epsilon}^{p}$

By noticing that: $\mathbf{\Delta}\,\boldsymbol{\varepsilon}^{p}=\frac{3}{2}\Delta p\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}=\frac{3}{2}\Delta p\frac{\tilde{\boldsymbol{\sigma}}^{e}}{\sigma_{eq}^{e}}$ because: $\frac{\tilde{\boldsymbol{\sigma}}-\mathbf{X}}{\sigma_y}=\frac{\tilde{\boldsymbol{\sigma}}^{e}}{\sigma_{eq}^{e}}$

one thus has:

$$\tilde{\boldsymbol{\sigma}}=\frac{2\mu}{2\mu^{-}}\tilde{\boldsymbol{\sigma}}^{-}+2\mu\,\mathbf{\Delta}\,\tilde{\boldsymbol{\varepsilon}}-\frac{2\mu}{2\mu+C}\frac{\left(s_{eq}^{e}-\sigma_y\right)_{+}}{\sigma_{eq}^{e}}.\tilde{\boldsymbol{\sigma}}^{e}$$

Internal variables $\mathbf{X}$ are calculated by:

$$\mathbf{X}=\frac{C}{C^{-}}\mathbf{X}^{-}+C\,\mathbf{\Delta}\,\boldsymbol{\varepsilon}^{p}=\frac{C}{C^{-}}\mathbf{X}^{-}+\frac{3}{2}C\,\Delta p\frac{\tilde{\boldsymbol{\sigma}}^{e}}{\sigma_{eq}^{e}}$$

Note: *Typical case of the plane constraints.*

The direct taking into account of the assumption of the plane constraints in the integration of the model of Von Mises with linear kinematic work hardening was not made in *Code_Aster*. To take into account this assumption, i.e. to use an elastoplastic behavior of Von Mises with a linear kinematic work hardening (law of Prager) with modelings `C_PLAN`, `DKT`, `COQUE_3D`, `COQUE_AXIS`, `COQUE_D_PLAN`, `COQUE_C_PLAN`, `PIPE`, `TUYAU_6M`, one can:

- that is to say to use the method of condensation static (due to R. of Borst [R5.03.03]) which makes it possible to obtain a state plan of constraints with convergence of the total iterations of the algorithm of Newton;
- that is to say to use the behavior `VMIS_ECMI_LINE` (cf [R5.03.16]).

4.5 Tangent operator. Option `FULL_MECA`

The option `FULL_MECA` allows to calculate the tangent matrix $\mathbf{K}_i^n$ with each iteration. The tangent operator who is used for building it is calculated directly on the preceding discretized system (one notes to simplify: $\tilde{\boldsymbol{\sigma}}=\boldsymbol{\sigma}^{-}+\Delta\,\tilde{\boldsymbol{\sigma}}, p=p^{-}+\Delta\, p$) and one writes the expressions only in the isothermal case.

One poses $\boldsymbol{\sigma}^{dev}=\tilde{\boldsymbol{\sigma}}-\mathbf{X}$ and $\gamma=\begin{cases}1 & \text{si} \quad \Delta\, p>0\,\text{et}\left(\tilde{\boldsymbol{\sigma}}-\mathbf{X}\right).\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}\geq 0\\ 0 & \text{sinon}\end{cases}$

The tangent operator binds the vector of virtual deformations $\boldsymbol{\varepsilon}^*$ with the vector of virtual constraints $\boldsymbol{\sigma}^*$. Then the matrix of tangent rigidity is written:

$$\boldsymbol{\sigma}^*=\left(K\,\vec{\mathbf{1}}\otimes\vec{\mathbf{1}}+2\,\mu\, a_2 P-C_p\,\mathbf{s}\otimes\mathbf{s}\right)\boldsymbol{\varepsilon}^*$$

with $\mathbf{s}$ the vector of constraints associated with $\boldsymbol{\sigma}^{dev}$ by:

$$\mathbf{s}^{\mathbf{T}}=\left(\sigma_{11}^{dev},\sigma_{22}^{dev},\sigma_{33}^{dev},\sqrt{2}\,\sigma_{12}^{dev},\sqrt{2}\,\sigma_{23}^{dev},\sqrt{2}\,\sigma_{31}^{dev}\right)$$

and:

$$C_p=\gamma\frac{3}{2}\left(\frac{2\,\mu}{\sigma_y}\right)^2\frac{1}{2\,\mu+C}a_1 \quad \text{with} \quad a_1=\frac{1}{1}+\frac{3}{2}\frac{(2\,\mu+C)\Delta\, p}{\sigma_y} \quad \text{and} \quad a_2=a_1\left(1+\frac{3}{2}C\frac{\Delta\, p}{\sigma_y}\right)$$

4.6 Internal variables of the model `VMIS_CINE_LINE`

The internal variables are 7:

- the tensor $\mathbf{X}$ stored on 6 components,
- the scalar variable χ.

5 Bibliography

1) P. MIALON, Elements of analysis and digital resolution of the relations of elastoplasticity. EDF - Bulletin of the Management of the Studies and Research - Series C - N° 3 1986, p. 57 - 89.

2) E.LORENTZ, J.M.PROIX, I.VAUTIER, F.VOLDOIRE, F.WAECKEL "Initiation with the thermo - plasticity in *Code_Aster*", EDF/DER/HI - 74/96/013 Notes

Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	J.M.Proix, E.Lorentz, P.Mialon EDF-R&D	Initial text
8.5	J.M.Proix EDF-R&D/AMA	Correction page 10, cf drives REX 11079
10.2	J.M.Proix EDF-R&D/AMA	Modification page 7 of the drafting on the way of calculation R (p) (cf card-indexes 15001).
11.1	S. Fayolle EDF-R&D/AMA	Addition of VMIS_JOHN_COOK

Annexe 1 Relation VMIS_ISOT_TRAC: complements on integration

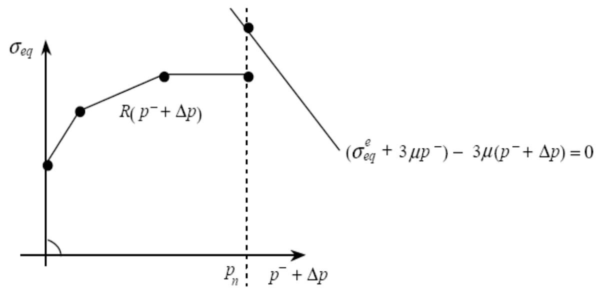
Implicit discretization of the relation of behavior led to solve an equation in Δp (see §3.3):

$$\sigma_{eq}^{e}-3\mu\Delta p-R\left(p^{-}+\Delta p\right)=0$$

One solves the equation exactly while drawing left the linearity per pieces.

One examines initially if the solution could be apart from the terminals of the points of discretization of the curve $R(p)$, i.e., if $p\geq p_n$ is a possible solution. For that:

- if $\sigma_{eq}^{e}+3\mu\left(p^{-}-p_n\right)-\sigma_n\geq 0$, then one is in the following situation:



  - if the prolongation on the right is linear then: $\Delta p=\dfrac{\sigma_{eq}^{e}-H_{n-1}}{\alpha_{n-1}+3\mu}$

    with: $\alpha_{n-1}=\dfrac{\sigma_n-\sigma_{n-1}}{p_n-p_{n-1}}$ $\qquad H_{n-1}=\sigma_{n-1}+\alpha_{n-1}\left(p^{-}-p_{n-1}\right)$

  - if the prolongation is constant: $\Delta p=\dfrac{\sigma_{eq}^{e}-\sigma_n}{3\mu}$

- if not, the solution p is to be sought in the interval $\left[p_i,p_{i+1}\right]$ such as:

$$\sigma_{i+1}>\sigma_{eq}^{e}+3\mu\left(p^{-}-p_{i+1}\right) \text{ and } \sigma_i\leq\sigma_{eq}^{e}+3\mu\left(p^{-}-p_i\right)$$

then the solution is: $\Delta p=\dfrac{\sigma_{eq}^{e}-H_i}{\alpha_i+3\mu}$ et $p^{-}+\Delta p\in\left[p_i,p_{i+1}\right]$

with:

$$\alpha_i=\frac{\sigma_{i+1}-\sigma_i}{p_{i+1}-p_i} \quad ; \quad H_i=\sigma_i+\alpha_i\left(p^{-}-p_i\right) \text{ pour } i=1 \text{ à } n-1$$

Annexe 2 Isotropic work hardening in plane constraints

In this case, the system to be solved comprises an equation moreover: $\Delta\sigma_{33}=0$. The following system then is obtained:

$$\begin{cases} 2\mu\,\Delta\tilde{\boldsymbol{\varepsilon}}-\Delta\tilde{\boldsymbol{\sigma}}=\frac{3}{2}2\mu\,\Delta p\,\frac{\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}}}{(\boldsymbol{\sigma}^-+\Delta\boldsymbol{\sigma})_{eq}} \\ \mathrm{tr}\,\Delta\boldsymbol{\sigma}=3K\,\mathrm{tr}\,\Delta\boldsymbol{\varepsilon} \\ (\boldsymbol{\sigma}^-+\Delta\boldsymbol{\sigma})_{eq}-R(p^-+\Delta p)\le 0 \\ \Delta p=0 \text{ si } (\boldsymbol{\sigma}^-+\Delta\boldsymbol{\sigma})_{eq}<R(p^-+\Delta p) \\ \Delta p\ge 0 \text{ si } (\boldsymbol{\sigma}^-+\Delta\boldsymbol{\sigma})_{eq}=R(p^-+\Delta p) \\ \Delta\sigma_{33}=0 \end{cases}$$

With this assumption, $\Delta\boldsymbol{\varepsilon}$ is not entirely known: $\Delta\varepsilon_{33}$ cannot be only calculated from $\Delta\mathbf{u}_i^n$.

Notice :

> *In the case as of modelings other than `C_PLAN` , therefore for example for modelings of hulls (`DKT` , `COQUE_3D`), assumptions on the transverse terms of shearing $\Delta\sigma_{13}$ and $\Delta\sigma_{23}$ are defined by these modelings (in general, the behavior related to transverse shearing linear, elastic and is uncoupled from the equations above). These terms thus do not enter on account here.*

One poses $\Delta\boldsymbol{\varepsilon}=\Delta\boldsymbol{\varepsilon}^q+\Delta\boldsymbol{\varepsilon}^y$ with $\Delta\boldsymbol{\varepsilon}^q$ entirely known from $\Delta\mathbf{u}_i^n$ and of elasticity, therefore $\Delta\varepsilon_{33}^q=-\frac{\nu}{1-\nu}\left(\Delta\varepsilon_{11}^q+\Delta\varepsilon_{22}^q\right)$ et $\Delta\boldsymbol{\varepsilon}^y=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\Delta y\end{pmatrix}$ is unknown.

Compared to the preceding system, there is an additional unknown factor, Δy .

- If $(\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}})_{eq}<R(p^-+\Delta p)$ alors $\Delta p=0$ thus $2\mu\,\Delta\tilde{\boldsymbol{\varepsilon}}=\Delta\tilde{\boldsymbol{\sigma}}$, i.e. $\Delta y=0$.
- If not, the technique of resolution consists in expressing Δy according to Δp . One then obtains a nonlinear scalar equation in Δp .

One poses: ' $\tilde{\boldsymbol{\sigma}}^e=\frac{2\mu}{2\mu^-}\tilde{\boldsymbol{\sigma}}^-+2\mu\,\Delta\tilde{\boldsymbol{\varepsilon}}^q$. In the same way that for integration except plane constraints, one obtains:

$$\tilde{\boldsymbol{\sigma}}_e+2\mu\,\Delta\tilde{\boldsymbol{\varepsilon}}^y=\left(\tilde{\boldsymbol{\sigma}}^-+\Delta\tilde{\boldsymbol{\sigma}}\right)\left(1+\frac{3\mu\,\Delta p}{R(p+\Delta p)}\right) .$$

But this expression utilizes an additional unknown factor Δy : In particular:

$$\tilde{\sigma}_{33}+2\mu\,\Delta\,\tilde{\varepsilon}^{y}_{33}=\left(\tilde{\sigma}^{-}_{33}+\Delta\,\tilde{\sigma}_{33}\right)\left(1+\frac{3\mu\,\Delta\,p}{R(p+\Delta\,p)}\right)$$

however

$$\Delta\,\tilde{\varepsilon}^{y}_{33}=\frac{2}{3}\Delta\,y$$

and $\operatorname{tr}(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})=3K\operatorname{tr}\boldsymbol{\Delta\varepsilon}^{q}+3K^{+}\Delta\,y+\frac{3K^{+}}{3K^{-}}\operatorname{tr}\boldsymbol{\sigma}^{-}-3K^{+}\Delta\,\varepsilon^{th}$

Like:

$$\tilde{\sigma}^{e}_{33}+\Delta\,\tilde{\sigma}_{33}=\sigma^{e}_{33}+\Delta\,\sigma_{33}=\frac{\operatorname{tr}(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})}{3}=0-\frac{\operatorname{tr}(\boldsymbol{\sigma}^{-}+\boldsymbol{\Delta\sigma})}{3}$$

One obtains an equation flexible Δy and Δp :

$$\tilde{\sigma}_{33}+2\frac{2}{3}\Delta\,y=\left(1+\frac{3\mu\,\Delta\,p}{R(p+\Delta\,p)}\right)\left(\frac{-\operatorname{tr}\sigma_{e}-3K\,\Delta\,y}{3}\right)$$

with:

$$\operatorname{tr}\boldsymbol{\sigma}_{e}=\frac{3K}{3K^{-}}\operatorname{tr}\boldsymbol{\sigma}^{-}+3K\operatorname{tr}\boldsymbol{\Delta\varepsilon}^{q}-3K\,\Delta\,\varepsilon^{th}$$

That is to say:

$$\Delta\,y\left(\frac{4\mu}{3}+K\left(1+\frac{3\mu\,\Delta\,p}{R\left(p^{-}+\Delta\,p\right)}\right)\right)=\tilde{\sigma}^{e}_{33}-\frac{\operatorname{tr}\sigma_{e}}{3}\left(1+\frac{3\mu\,\Delta\,p}{R\left(p^{-}+\Delta\,p\right)}\right)$$

by noticing that:

$$\tilde{\sigma}^{e}_{33}=\sigma^{e}_{33}-\frac{\operatorname{tr}\sigma_{e}}{3}=0-\frac{\operatorname{tr}\sigma_{e}}{3}$$

and while clarifying μ, K , one obtains:

$$\Delta\,y=\frac{3\left(1-2\nu\right)\Delta\,p}{E\,\Delta\,p+2\left(1-\nu\right)R\left(p+\Delta\,p\right)}\tilde{\sigma}^{e}_{33}$$

to defer in the equation in Δp (identical to the preceding cases):

$$(\tilde{\boldsymbol{\sigma}}^{e}+2\mu\,\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}^{y})_{eq}-3\mu\,\Delta\,p-R(p^{-}+\Delta\,p)=0$$

where Δy express yourself according to Δp since: $\boldsymbol{\Delta}\,\tilde{\boldsymbol{\varepsilon}}^{y}=\frac{\Delta\,y}{3}\begin{pmatrix}-1 & & \\ & -1 & \\ & & 2\end{pmatrix}$

The scalar equation in Δp thus obtained is always nonlinear. This equation is solved by a research method of zeros of functions, based on an algorithm of secant. Once the solution Δp known one calculates Δy then σ .