SSNV124 - Regularized limit analysis. Model of Norton - Hoff

Summarized

This test makes it possible to validate the operators used in regularized limit analysis. One calculates the Yield-point load by a kinematical approach regularized by the method of Norton-Hoff-Friaâ.

One considers a rectangular plate (modelization A) or a cube (modelization B) or an axisymmetric cylinder (modelization C). The constitutive material checks the strength criterion of von Mises and the structure is subjected to loadings on edges. The computation allows to obtain the parameter of the Yield-point load in the direction of the loading.

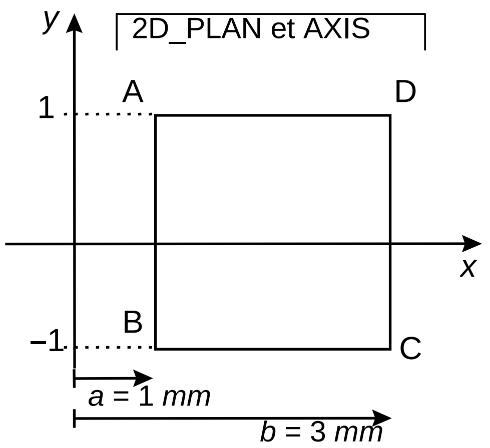
The structure is modelled by incompressible elements and the loading which one seeks the limiting load parameter is standardized (unit power).

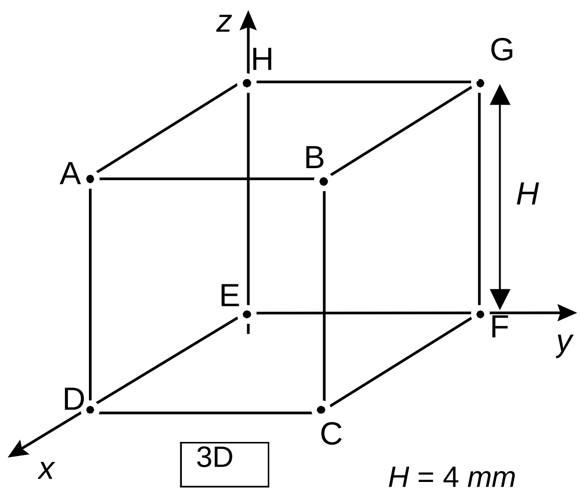
The resolution by the regularized method of Norton-Hoff-Friaâ is carried out by control in command `STAT_NON_LINE`. A postprocessing in command `POST_ELEM` makes it possible to obtain the value of a higher limit of the Yield-point load, as well as an estimate by lower value, when there is no constant loading applied.

The reference solution is analytical and the results are in perfect agreement with the values of reference.

1 Problem of reference

1.1 Geometry





1.2 Material properties

Elastic limit: $\sigma_y = 10\,MPa$.

1.3 Limiting boundary conditions and

loadings Conditions in 2D:

on AB : $DX = 0.$
on BC : $DY = 0.$

Limiting conditions in 3D:

$EFGH$ ($FACEXINF$): $DX = 0.$
$ADEH$ ($FACEYINF$): $DY = 0.$
$DCFE$ ($FACEZINF$): $DZ = 0.$

Limiting conditions in AXIS:

on BC and AD : $DY = 0.$

The loading parameterized by λ is:

in 2D:

$FY = -1.$ on AD

in 3D:

$FX = -0.2$ on $ABCD$ ($FXSUP$)
$FY = -0.8$ on $BCFG$ ($FYSUP$)

in AXIS:

$FX = 1.$ on AB .

2 Reference solution

2.1 Method of calculating used for the reference solution

the constitutive material checks the criterion of von Mises, with for threshold σ_y. The structure is subjected to pressures on edges horizontal $-\alpha f$ and vertical $-(1-\alpha) f$ with $\alpha \geq 0.5$ ($\alpha = 1.$ in 2D, $\alpha = 0.8$ in 3D). In 2D plane, one considers two ways of making: on the one hand by amplifying the two pressures together, on the other hand by amplifying only the horizontal pressure, and by leaving the constant vertical pressure. Into axisymmetric, the solid is subjected to the pressure only interns $-\alpha f$. One obtains the exact Yield-point load and that by the method of regularization [R7.07.01] in this direction of loading, for the criterion of von Mises, with the threshold σ_y.

2.2 Plane case

the structure is subjected to pressures on edges horizontal: $-\alpha f$ and vertical: $-(1-\alpha) f$, with: $\alpha \geq 1/2$, and one exerts a blocking in z. One considers two ways of controlling the loading:

case 1: the two pressures horizontal and vertical are parameterized by λ,
case 2: the horizontal pressure is parameterized by λ, while the vertical pressure is constant $-(1-\alpha) f_0$, with $f_0 = \lambda_0 f$.

2.2.1 Solution in limit analysis

the solution is homogeneous (biaxées stresses σ : $\sigma_{xx} = \alpha f, \sigma_{yy} = (1-\alpha) f \quad \sigma_{xy} = 0$, plane strains ε). One obtains [bib2] the Yield-point load in these directions of loading, for the criterion of von Mises, in plane strains, with the threshold σ_y :

$$\text{case 1: } \lambda_{\lim}.f = \frac{2\sigma_y}{\sqrt{3}.|2\alpha - 1|} \qquad \text{éq 2.2.1-1}$$

$$\text{cases 2: } \lambda_{\lim}.f = \frac{2\sqrt{3}\sigma_y}{3.|\alpha|} + \frac{1-\alpha}{|\alpha|}\lambda_0.f \qquad \text{éq 2.2.1-2}$$

One checks that if one takes $\lambda_0 = \lambda_{\lim}$ in the cas2, one finds the cas1 then.

2.2.2 Solution in regularized limit analysis

the solution is homogeneous. The plane strains are necessarily of the form:

$$\varepsilon(\mathbf{u}) = \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \sqrt{\varepsilon(\mathbf{u}).\varepsilon(\mathbf{u})} = |\gamma|\sqrt{2} \qquad \text{éq 2.2.2-1}$$

By the model of Norton-Hoff, the coefficient $m \in [1, 2]$ being given, one obtains the deviatoric stresses:

$$\sigma^D = A(m)\sqrt{2}^{m-2}|\gamma|^{m-2}\gamma. \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \|\sigma^D\|_{VM} = A(m)\sqrt{2}^{m-2}|\gamma|^{m-1}\sqrt{3} \qquad \text{éq 2.2.2-2}$$

the standardization (unit power) of the loading which one seeks the limiting load parameter, cf [R7.07.01] [§1.2], led to:

$$\text{case 1: } \gamma f = \frac{1}{H(b-a)(2\alpha - 1)} \qquad \text{éq 2.2.2-3}$$

cases 2: $\gamma f=\frac{1}{H(b-a)\alpha}$ éq the 2.2.2-4

terms of the continuation $\hat{\lambda}_m$ of approximations per excess of the Yield-point load in these two parameter settings of the loading are then:

case 1: $\hat{\lambda}_m . f=\frac{2\sqrt{3}\sigma_y}{3.|2\alpha-1|} \qquad \forall m$ éq 2.2.2-5

cases 2: $\hat{\lambda}_m . f=\frac{2\sqrt{3}\sigma_y}{3.|\alpha|}+\frac{1-\alpha}{|\alpha|}\lambda_0 . f \quad \forall m$ éq 2.2.2-6

invariance according to m observed here (what is a typical case) results owing to the fact that one is in a statically determinate situation. In case 1, one can also exploit the continuation of the approximations by default of the Yield-point load $\underline{\lambda}_m$:

case 1: $\underline{\lambda}_m . f=\frac{2\sqrt{3}\sigma_y}{3m.|2\alpha-1|}$ éq 2.2.2-7

One thus obtains the exact $\lambda_{\lim}$ Yield-point load when $m\rightarrow 1^+$.

In case 2, where the vertical pressure is constant, the power of this “permanent” loading in solution displacement is:

case 2: $L_0(\mathbf{u})=\frac{1-\alpha}{|\alpha|}.\lambda_0 f$ éq 2.2.2-8

2.3 Cases axisymmetric

In 2D axisymmetric one considers the same geometry, but the solid, on which one imposes a complete axial blocking, is only subjected to a pressure on the internal wall: αf parameterized par. λ

2.3.1 Solution in limit analysis

One obtains [bib2] the Yield-point load in this direction of loading, for the criterion of von Mises, into axisymmetric and null axial strains, with the threshold σ_y :

$$\lambda_{\lim}.\alpha f=\frac{2\sqrt{3}}{3}\sigma_y \ln\frac{b}{a}$$ éq 2.3.1-1

2.3.2 Solution in regularized limit analysis

the solution is homogeneous. Displacement being only radial, the isochoric strains are necessarily of the form:

$$u_r(r)=\frac{\gamma}{r}; \qquad \varepsilon(\mathbf{u})=\frac{\gamma}{r^2}\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix} \quad ; \qquad \sqrt{\varepsilon(\mathbf{u}).\varepsilon(\mathbf{u})}=\frac{|\gamma|}{r^2}\sqrt{2}$$ éq 2.3.2-1

By the model of Norton-Hoff, the coefficient $m\in[1,2]$ being given, one obtains the deviatoric stresses:

$$\sigma^D=A(m)\sqrt{2}^{m-2}|\gamma|^{m-2}\gamma . r^{-2m+2}.\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix};\left\|\sigma^D\right\|_{VM}=A(m)\sqrt{2}^{m-2}|\gamma|^{m-1}.r^{-2m+2}.\sqrt{3}$$ éq the 2.3.2-2

balance equations axial and radial result in determining the average constraint:

$$\mathrm{tr}\,\sigma(r)=3A(m)\sqrt{2}^{m-2}.\gamma.|\gamma|^{m-2}.r^{-2m+2}.\frac{2-m}{1-m}+3\tau$$ éq 2.3.2-3

where τ is a constant, which is calculated from the boundary condition of pressure null out of external wall. The components of the stresses then are obtained:

$$\begin{cases} \sigma_{rr}(r) = \beta\left(b^{-2m+2} - r^{-2m+2}\right) \\ \sigma_{zz}(r) = \beta\left(b^{-2m+2} - (2-m)\, r^{-2m+2}\right) \\ \sigma_{\theta\theta}(r) = \beta\left(b^{-2m+2} - (3-2m)\, r^{-2m+2}\right) \end{cases} \quad avec:\ \beta = \frac{A(m)\sqrt{2}^{m-2}}{(m-1)(\alpha f H)^{m-1}} \qquad \text{éq 2.3.2-4}$$

the standardization (unit power) of the loading which one seeks the limiting load parameter, cf [R7.07.01] [§1.2], led to: $\alpha f \gamma = \frac{1}{H}$.

The terms of the continuation $\hat{\lambda}_m$ of approximations per excess of the Yield-point load for this loading are then:

$$\hat{\lambda}_m.\alpha f = \frac{2\sqrt{3}}{3}\sigma_y H \int_a^b \frac{|\gamma|}{r^2} r dr = \frac{2\sqrt{3}}{3}\sigma_y \ln\frac{b}{a} \quad \forall m \qquad \text{éq the 2.3.2-5}$$

terms of the continuation $\underline{\lambda}_m$ of the approximations by default of the Yield-point load for this loading are:

$$\underline{\lambda}_m.\alpha f = \frac{2\sqrt{3}}{3m}\sigma_y \int_a^b r^{-2m}.\left(\underset{]a,b[}{Max}\left(r^{-2m+2}\right)\right)^{-1} r dr = \frac{\sigma_y\sqrt{3}}{3m(1-m)} \frac{b^{-2m+2} - a^{-2m+2}}{a^{-2m+2}} \qquad \text{éq 2.3.2-6}$$

In $m \to 1^+$, one finds: $\underline{\lambda}_{1^+}.\alpha f = \frac{2\sqrt{3}}{3}\sigma_y \ln\frac{b}{a}$, i.e. the same value as $\hat{\lambda}_m$ and $\lambda_{\lim}$.

2.4 Three-dimensional case

In 3D one considers the same geometry, but the solid, of unit thickness, is free in the direction antiplane z. The solid is subjected to pressures on the walls horizontal: $-\alpha f$ and vertical: $-(1-\alpha) f$, with: $\alpha \geq 1/2$. The two pressures horizontal and vertical are parameterized par. λ

2.4.1 Solution in limit analysis

the solution is homogeneous (biaxées stresses σ : $\sigma_{xx} = \alpha f, \sigma_{yy} = (1-\alpha) f \quad \sigma_{xy} = 0 \quad \sigma_{zz} = 0$, strains ε). One obtains the Yield-point load in this direction of loading [bib2], for the criterion of von Mises, with the threshold σ_y :

$$\lambda_{\lim}.f = \frac{\sigma_y}{\sqrt{3\alpha^2 - 3\alpha + 1}} \qquad \text{éq 2.4.1-1}$$

2.4.2 Solution in regularized limit analysis

the solution is homogeneous. The isochoric strains are necessarily of the form:

$$\varepsilon(\mathbf{u})=\gamma\begin{pmatrix}1&0&0\\0&\delta&0\\0&0&-1-\delta\end{pmatrix} \quad ; \quad \sqrt{\varepsilon(\mathbf{u}).\varepsilon(\mathbf{u})}=|\gamma|\sqrt{2\left(1+\delta+\delta^2\right)}$$ éq 2.4.2-1

By the model of Norton-Hoff, the coefficient $m\in[1,2]$ being given, one obtains the deviatoric stresses:

$$\sigma^D=\beta.\begin{pmatrix}1&0&0\\0&\delta&0\\0&0&-1-\delta\end{pmatrix} \quad ; \quad \left\|\sigma^D\right\|_{VM}=|\beta|\sqrt{3\left(1+\delta+\delta^2\right)} \ avec : \ \beta=A(m)\sqrt{2\left(1+\delta+\delta^2\right)}^{m-2}|\gamma|^{m-2}\gamma$$

éq 2.4.2-2

One deduces from $\sigma_{zz}=0$: $\mathrm{tr}\,\sigma=3\beta(1+\delta)$. From where stresses: $\sigma=\beta.\begin{pmatrix}2+\delta&0&0\\0&1+2\delta&0\\0&0&0\end{pmatrix}$.

The equilibrium of solid imposes that $\sigma_{xx}.(1-\alpha)=\sigma_{yy}.\alpha$. One from of deduced the parameter $\delta=\dfrac{3\alpha-2}{1-3\alpha}$.

The standardization (unit power) of the loading which one seeks the limiting load parameter, cf [R7.07.01] [§1.2], led to:

$$\gamma f=\frac{1}{H(b-a)(\alpha+\delta(1-\alpha))}$$ éq the 2.4.2-3

terms of the continuation $\hat{\lambda}_m$ of hight delimiters of the Yield-point load in this case of loading are thus identical to:

$$\hat{\lambda}_m.f=\frac{2\sqrt{3}\sigma_y}{3}.\frac{\sqrt{2\left(1+\delta+\delta^2\right)}}{\alpha+\delta(1-\alpha)}=\frac{\sigma_y}{\sqrt{3\alpha^2-3\alpha+1}}$$ éq 2.4.2-4

2.5 Results of reference

Modelization	case	$\lambda_{\lim}$	$\lambda_{\lim}^{\sup}=\hat{\lambda}_m$	$\lambda_{\lim}^{estimée}=\bar{\lambda}_m$	power $L_0(\mathbf{u})$
A (case 1)	2D plane	$\lambda_{\lim}=\dfrac{2}{\sqrt{3}\lvert 2\alpha-1\rvert}.\dfrac{\sigma_y}{f}$	$\dfrac{2}{\sqrt{3}\lvert 2\alpha-1\rvert}.\dfrac{\sigma_y}{f}$	$\dfrac{2}{\sqrt{3}\lvert 2\alpha-1\rvert}.\dfrac{\sigma_y}{mf}$	0
Abis (cas2)	2D plane	$\dfrac{2\sqrt{3}}{3\lvert\alpha\rvert}\dfrac{\sigma_y}{f}+\dfrac{1-\alpha}{\lvert\alpha\rvert}\lambda_0 f$	$\dfrac{2\sqrt{3}}{3\lvert\alpha\rvert}\dfrac{\sigma_y}{f}+\dfrac{1-\alpha}{\lvert\alpha\rvert}\lambda_0 f$	nothing	$\dfrac{1-\alpha}{\lvert\alpha\rvert}.\lambda_0 f$
B	3D	$\lambda_{\lim}=\dfrac{1}{\sqrt{3\alpha^2-3\alpha+1}}.\dfrac{\sigma_y}{f}$	$\dfrac{1}{\sqrt{3\alpha^2-3\alpha+1}}.\dfrac{\sigma_y}{f}$	$\dfrac{1}{\sqrt{3\alpha^2-3\alpha+1}}.\dfrac{\sigma_y}{mf}$	0
C $\alpha=1$	2D AXIS	$\lambda_{\lim}=\dfrac{2\sqrt{3}}{3}.\dfrac{\sigma_y}{f}.\ln\dfrac{b}{a}$	$\dfrac{2\sqrt{3}}{3}.\dfrac{\sigma_y}{f}.\ln\dfrac{b}{a}$	$\dfrac{\sigma_y\sqrt{3}\left((b/a)^{-2m+2}-1\right)}{3m(1-m)}$	0

Modelization	case	λ_{lim}	$\lambda_{\text{lim}}^{\text{sup}}$	$\lambda_{\text{lim}}^{estimée}$ ($m=1{,}2$)	$\lambda_{\text{lim}}^{estimée}$ ($m=1{,}0001$)	power $L_0(\mathbf{u})$
A	2D plane	11.547	11.547	9.6225	11.5458	0
Abis	2D plane	14.6837	14.6837	nothing	nothing	0.25
B ($\alpha=0{,}8$)	3D	13.8675	13.8675	11.5562	13.8661	0
C	2D AXIS	12.6857	12.6857	8.5545	12.6830	0

Note::

Code_Aster calculates the opposite value of the power of the “permanent” loading in solution displacement $L_0(\mathbf{u})$.

2.6 Bibliographical references

1) F.VOLDOIRE, E.LORENTZ, J.M.PROIX, E.VISSE: Computation of Yield-point load by the method of Norton-Hoff-Friaâ. [R7.07.01]. F.VOLDOIRE

2) , Yield design and limit analysis of structures, note EDF HI-74/93/082. Modelization

3 A Characteristic

3.1 of the modelization One considers

a rectangular plate modelled by element QUAD8 of an incompressible type: miplqu8. The two cases are studied: the first with the two amplified loads, the second with the amplified horizontal pressure and the constant vertical. Characteristics

3.2 of the mesh Many

nodes: 8 Number of meshes

and types: 1 mesh of the type QUAD8, incompressible finite element. Values

3.3 tested Identification

Case Reference		higher
Yield-point load A Abis 11.547	14.683 7	estimated
Yield-point load () A Abis $m=1,2$ 9.6225	nothin g	Power
permanent loading A Abis 0	– 0.25	NOEUD=' N
3 ' EPSI_ELNO: EPXX A - 0.3125		NOEUD=' N
3 ' EPEQ_ELNO: INVA_2 A 0.360844		Modelization

4 B Characteristic

4.1 of the modelization One consider

a cube modelled by element HEXA20 of an incompressible type: minc_hexa 20. Characteristics

4.2 of the mesh Many

nodes: 20. Number of meshes

and types: 1 mesh of the incompressible type HEXA20 finite element. Values

4.3 tested Identification

Reference	higher
Yield-point load 13.867505	estimated
Yield-point load () 11.5562 $m=1,2$	NOEUD=' N1
' EPSI_ELNO: EPXX 0.0407692	Modelization

5 C Characteristic

5.1 of the modelization One consider

a cylinder modelled by axisymmetric elements QUAD8 of `the incompressible` type: miaxqu8, according to a structured mesh. Characteristics

5.2 of the mesh Many

nodes: 96 Number of meshes

and types: 25 meshes of incompressible type QUAD8 finite element. Values

5.3 tested Identification

Reference	higher
Yield-point load 12.6857	estimated
Yield-point load () 8.5545	
$m = 1{,}2$ Summary	

6 of the results the numerical

results are in perfect agreement with the values of reference. In the axisymmetric case, the light differences are explained by the fact why displacement is in $1/r$ the analytical solution, which is not understood in the base of the finite elements selected.