SSNP152 – Inclusion of two contours

Summarized:

This test is used to the contact evaluate the performances of Aster with regard to the processing between two structures with geometrical compatibility between surfaces Master and slave at initial time and various positions of geometrical incompatibility in the course of time. It makes it possible of the contact to validate the processing with the continuous method in taking into account of large rotations.

One considers a structure made up of two concentric contours. External contour is subjected to a uniform pressure on its free edge whereas one imposes a rotation finished on interior contour. Their stiffness, represented by their Young's moduli plays an important role in the evaluating of the value of the strains and the fluctuations of the contact pressure. One also seeks to know which are the effects of the use of a mesh of a higher nature.

An analytical solution was developed for this problem in order to validate the calculated numerical results. The validation of this test relates to the values of the contact pressure.

1 Problem of reference

1.1 Geometry

the structure is made up of two concentric circular rings. The internal radius R_2 of external contour is equal to the external radius of interior contour (Figure 1.1-1).
Dimensions structural feature are:

$R_1=1{,}0\,m\,;\,R_2=0{,}6\,m\,;\,R_3=0{,}2\,m$

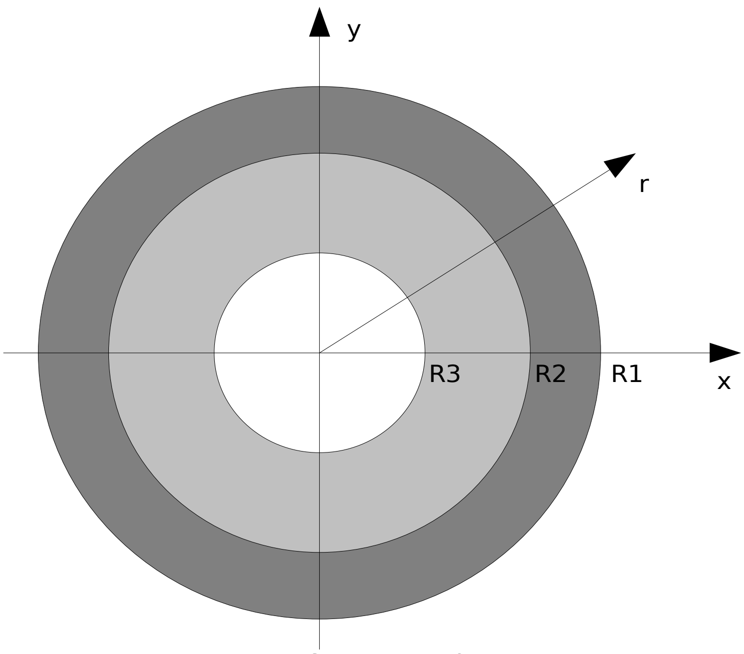

Figure 1.1-1: Geometry of the structure

1.2 Properties of the materials

the Young's modulus and the Poisson's ratio of the material of external contour are given by E_1 and ν_1 (respectively E_2 and ν_2 for interior contour).

1.3 Boundary conditions and loadings

1.3.1. Boundary conditions in the case without rotation

external contour is subjected to a displacement are equivalent to the application of a pressure p on its edge ($r=R_1$) whereas the edge of interior contour ($r=R_3$) is left free of stress.

$$\xi_X(r)=f(r)\cos(\arctan(\frac{Y}{X}))$$
$$\xi_Y(r)=f(r)\sin(\arctan(\frac{Y}{X}))$$

The function $f(r)$ of radial displacement is given according to the properties of the materials and the pressure p . In the case of plane strains (`MODELISATION = "D_PLAN"`) one a:

$$f(R_1)=\frac{1+\nu_1}{E_1}(A_1(1-2\nu_1)R_1+\frac{B_1}{R_1})$$
$$f(R_3)=\frac{1+\nu_2}{E_2}(A_2(1-2\nu_2)R_3+\frac{B_2}{R_3})$$ éq 1.1

with:

$$A_1=\frac{-pR_1^2+\lambda R_2^2}{R_1^2-R_2^2};\ B_1=(-p+\lambda)\frac{R_1^2R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda\frac{R_2^2}{R_2^2-R_3^2};\ B_2=-\lambda\frac{R_2^2R_3^2}{R_2^2-R_3^2}$$ éq 1.2

Where λ is the contact pressure whose analytical statement is:

$$\lambda=2p(1-\nu_1)\frac{\frac{R_1^2}{R_1^2-R_2^2}}{\frac{R_1^2+R_2^2(1-2\nu_1)}{R_1^2-R_2^2}+\frac{E_1}{E_2}\frac{(1+\nu_1)}{(1+\nu_2)}\frac{R_2^2(1-2\nu_2)+R_3^2}{R_2^2-R_3^2}}$$ éq. 1.3

It is noticed that in the case of plane strains the Poisson's ratios does not intervene in the equations, whereas in the case of plane stresses (MODELISATION = "C_PLAN") one a:

$$f(R_1)=\frac{1}{E_1}(A_1(1-\nu_1)R_1+\frac{B_1(1+\nu_1)}{R_1})$$
$$f(R_3)=\frac{1}{E_2}(A_2(1-\nu_2)R_3+\frac{B_2(1+\nu_2)}{R_3})$$ éq the 1.4

statements of A_1 B_1, A_2 and B_2 are the same ones as those of equations 1.2 while the contact pressure is given by:

$$\lambda=2p\frac{\frac{R_1^2}{R_1^2-R_2^2}}{\frac{R_1^2(1+\nu_1)+R_2^2(1-\nu_1)}{R_1^2-R_2^2}+\frac{E_1}{E_2}\frac{R_2^2(1-\nu_2)+R_3^2(1+\nu_2)}{R_2^2-R_3^2}}$$ éq 1.5

1.3.2. Boundary conditions in the case with rotation

In addition to radial displacement, one imposes on the internal surface of interior contour ($r=R_3$) a displacement involving the rotation of this one. The statement of displacement must take into account the contraction of interior contour due to the application of the pressure, its new internal radius being $R_3+f(R_3)$:

$$\xi_X(X,Y,i)=(R_3+f(R_3))\cos(\arctan(\frac{Y}{X})+\omega(i))-X$$
$$\xi_Y(X,Y,i)=(R_3+f(R_3))\sin(\arctan(\frac{Y}{X})+\omega(i))-Y$$ éq 1.6
$$\omega(i)=\frac{2\pi}{N_tN_e}i,\ i\in\{n\in\mathbb{Z},0\leq n\leq N_t\}$$

where N_t is the number of time step and N_e the number of elements along contact surface. As a rotation of angle is imposed $\frac{2\pi}{N_e}$, at the last moment there is a configuration similar to the initial configuration but where interior contour was shifted of an element. In this last configuration the master meshes and slave of contact surfaces are again in opposite.

2 Reference solution

We develop here an analytical solution to the problem presented above. This solution will be developed on the assumption of small strains by considering that the materials of contours isotropic, are governed by a linear elastic model without temperature variation.
Because of symmetries of the problem, the solution in displacement of the problem has the following generic form:

$$u=u_r(r,z).\underline{e_r}+u_z(r,z).\underline{e_z}$$

2.1.1. Case 1: plane strains

By means of symmetries of the problem and the assumption of invariance according to Z of the plane stresses, the solution of the problem takes the following shape:

$$\begin{aligned} u_r&=u_r(r)\\ u_\theta&=0\\ u_z&=0 \end{aligned} \qquad \text{éq 2.1}$$

By means of the equation of Lamé-Navier:

$$(\lambda+\mu)\underline{grad}(div(\underline{u}))+\mu\Delta\underline{u}+\underline{fd}=\underline{0} \qquad \text{éq 2.2}$$

where $\underline{fd}=\underline{0}$ are the null voluminal forces here, and the F ormule of the Laplacian:

$$\Delta\underline{u}=\underline{grad}(div(\underline{u}))+\underline{rot}\,\underline{rot}(\underline{u}) \qquad \text{éq 2.3}$$

One can write éq 2.2 pennies the form:

$$(\lambda+2\mu)\underline{grad}(div(\underline{u}))+\mu\,\underline{rot}\,\underline{rot}(\underline{u})+\underline{fd}=\underline{0} \qquad \text{éq 2.4}$$

is still by means of $\underline{rot}(\underline{u})=\vec{0}$ et $\underline{fd}=\vec{0}$ et $u=u_r(r).\underline{e_r}$:

$$\begin{aligned} div(\underline{u})&=\frac{d}{dr}u_r(r)+\frac{1}{r}u_r(r)\\ \underline{grad}(div\,\underline{u})&=\frac{d}{d_r}[\frac{1}{r}\frac{d}{d_r}(r\,u_r(r))].\underline{e}_r\\ \text{soit encore }(\lambda+2\mu)&\frac{d}{d_r}[\frac{1}{r}\frac{d}{d_r}(r\,u_r(r))]=0 \end{aligned} \qquad \text{éq 2.5}$$

by integrating the equation, one obtains for contours (external and interior) the following shape of the field of displacement:

$$u_r=C_i r+\frac{D_i}{r} \quad u_\theta=0 \quad u_z=0 \qquad \text{éq 2.6}$$

to determine C_i and D_i, it remains us to impose the limiting conditions in pressure and displacement. For that, the strains should initially be calculated then the stresses associated with the field with displacement.

The strains are the symmetric part of the gradient of displacements. One obtains:

$$\epsilon_{rr}=C_i-\frac{D_i}{r^2}$$
$$\epsilon_{\theta\theta}=C_i+\frac{D_i}{r^2} \quad \text{éq 2.7}$$
$$\epsilon_{zz}=\epsilon_{r\theta}=\epsilon_{rz}=\epsilon_{\theta z}=0$$

By applying the Hooke's law:

$$\underline{\underline{\sigma}}=\lambda\, tr(\underline{\underline{\epsilon}})\underline{\underline{1}}+2\mu\underline{\underline{\epsilon}} \quad \text{éq 2.8}$$

one obtains the following general form for the stresses:

$$\sigma_{rr}=\frac{E}{1+\nu}\left(\frac{C_i}{1-2\nu}-\frac{D_i}{r^2}\right)$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\frac{C_i}{1-2\nu}+\frac{D_i}{r^2}\right) \quad \text{éq 2.9}$$
$$\sigma_{zz}=\frac{2\nu E C_i}{(1+\nu)(1-2\nu)}$$
$$\sigma_{r\theta}=\sigma_{rz}=\sigma_{\theta z}=\underline{0}$$

While posing:

$$A_i=\frac{E_i}{(1+\nu_i)(1-2\nu_i)}C_i \qquad B_i=\frac{E_i}{1+\nu_i}D_i \quad \text{éq the 2.10}$$

non-zero stresses become:

$$\sigma_{rr}=A_i-\frac{B_i}{r^2}$$
$$\sigma_{\theta\theta}=A_i+\frac{B_i}{r^2} \quad \text{éq 2.11}$$
$$\sigma_{zz}=2\nu A_i$$

It any more but does not remain us to calculate the values of A_i and B_i for each contour. One will note λ_n the contact pressure between two contours such as:

$$\underline{\underline{\sigma}}_{1\mathrm{rr}}(R_2).(-\underline{e}_r)=\lambda_n\underline{e}_r$$
$$\underline{\underline{\sigma}}_{2\mathrm{rr}}(R_2).\underline{e}_r=-\lambda_n\underline{e}_r \quad \text{éq 2.12}$$

with the boundary conditions:

$$\underline{\underline{\sigma}}_{1\mathrm{rr}}(R_1).\underline{e}_r=-p.\underline{e}_r$$
$$\underline{\underline{\sigma}}_{2\mathrm{rr}}(R_3).(-\underline{e}_r)=\underline{0} \quad \text{éq 2.13}$$

the condition of continuity on displacement with the interface between the two groups of contact gives moreover:

$$u_{r;1}(R2)=u_{r;2}(R2)$$ éq 2.14

We thus have 5 equations for the 5 unknowns $A_1, B_1, A_2, B_2, \lambda_n$.

The system of the first 4 equations enables us to obtain:

$$A_1=\frac{-p\,R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2};\ B_1=(-p+\lambda_n)\frac{R_1^2 R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda_n\frac{R_2^2}{R_2^2-R_3^2};\ B_2=-\lambda_n\frac{R_2^2 R_3^2}{R_2^2-R_3^2}$$ éq 2.15

and the continuity equation on displacement finally makes it possible to have the contact pressure:

$$\lambda_n=\frac{2\,p\,R_1^2(1-\nu_1)}{R_1^2+R_2^2(1-2\nu_1)+\frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-2\nu_2)+R_3^2)}$$ éq 2.16

2.1.2. Cases 2: plane stresses

We suppose initially that there are no stresses according to the direction perpendicular to the plan of contours ($\underline{\underline{\sigma}}_i\,.\,\underline{e}_z=\underline{0}$). Symmetries of the problem bring us to a stress field which can be written in the form:

$$\underline{\underline{\sigma_i}}(r,z)=\sigma_{rr;i}\,\underline{e}_r\otimes\underline{e}_r+\sigma_{\theta\theta;i}\,\underline{e}_\theta\otimes\underline{e}_\theta+\sigma_{r\theta;i}\,\underline{e}_r\otimes\underline{e}_\theta$$

where the index i is worth 1 for external contour and 2 for interior contour. In the absence of volume forces and by considering the quasi-static problem (one neglects the effects of acceleration), one a:

$$div\,\underline{\underline{\sigma}}=\underline{0}$$

By means of the solution in displacement of the generic problem:

$$u=u_r(r,z).\underline{e_r}+u_z(r,z).\underline{e_z}$$

The strain field is written then:

$$\begin{gathered}\epsilon_{rr}=\frac{\partial u_r(r,z)}{\partial r}\\ \epsilon_{\theta\theta}=\frac{u_r}{r}\\ \epsilon_{zz}=\frac{\partial u_z(r,z)}{\partial z}\\ \epsilon_{\theta z}=\epsilon_{r\theta}=0\\ \epsilon_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\end{gathered}$$ éq 2.17

Like:

$$\begin{gathered}div\,\underline{\underline{\sigma}}=\underline{0}\text{ devient }\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0\\ \text{avec}\\ \sigma_{rr}=\frac{E}{1+\nu}\left(\epsilon_{rr}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{\theta\theta}=\frac{E}{1+\nu}\left(\epsilon_{\theta\theta}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{zz}=\frac{E}{1+\nu}\left(\epsilon_{zz}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)\\ \sigma_{zz}=\sigma_{z\theta}=\sigma_{zr}=0\\ \text{impliquant}\\ \epsilon_{zr}=0=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\end{gathered}$$ éq 2.18

the fact that $\sigma_{zz}=0$ we gives:

$$\begin{gathered}\frac{E}{1+\nu}\left(\epsilon_{zz}+\frac{\nu}{1-2\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}+\epsilon_{zz})\right)=0\\ \epsilon_{zz}(1-\nu)=-\nu(\epsilon_{rr}+\epsilon_{\theta\theta})\\ \epsilon_{zz}=\frac{-\nu}{(1-\nu)}(\epsilon_{rr}+\epsilon_{\theta\theta})\end{gathered}$$ éq 2.19

what implies:

$$\begin{gathered}\sigma_{rr}-\sigma_{\theta\theta}=\frac{E}{1+\nu}(\epsilon_{rr}-\epsilon_{\theta\theta})\\ tr\,\underline{\underline{\epsilon}}=\frac{\nu-1}{\nu}\epsilon_{zz}+\epsilon_{zz}\\ tr\,\underline{\underline{\epsilon}}=\frac{2\nu-1}{\nu}\epsilon_{zz}\end{gathered}$$ éq 2.20

and makes it possible to write:

$$tr\,\underline{\underline{\epsilon}}=\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})$$ éq 2.21

One replaces then $tr\underline{\underline{\epsilon}}$ by his value in σ_{rr} and $\sigma_{\theta\theta}$ to obtain:

$$\sigma_{rr}=\frac{E}{1+\nu}(\epsilon_{rr}+\frac{\nu}{1-2\nu}\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}(\epsilon_{\theta\theta}+\frac{\nu}{1-2\nu}\frac{1-2\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$

éq 2.22

$$\sigma_{rr}=\frac{E}{1+\nu}(\epsilon_{rr}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$
$$\sigma_{\theta\theta}=\frac{E}{1+\nu}(\epsilon_{\theta\theta}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$

the balance equation becomes as follows:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \quad\rightarrow\quad \frac{\partial\epsilon_{rr}}{\partial r}+\frac{\nu}{1-\nu}(\frac{\partial}{\partial r}(\epsilon_{rr}+\epsilon_{\theta\theta}))+\frac{\epsilon_{rr}-\epsilon_{\theta\theta}}{r}=0$$ éq 2.23

As we have:

$$\epsilon_{rr}=\frac{\partial u_r(r,z)}{\partial r}$$
$$\epsilon_{\theta\theta}=\frac{u_r}{r}$$ éq 2.24

the substitution of the stresses by the strains in the balance equation makes it possible to write finally:

$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{\nu}{1-\nu}(\frac{\partial^2 u_r(r,z)}{\partial^2 r}+\frac{\partial u_r(r,z)}{\partial r})+\frac{1}{r}(\frac{\partial u_r(r,z)}{\partial r}-\frac{u_r(r,z)}{r})=0$$
$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{\nu}{1-\nu}(\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial u_r(r,z)}{\partial r}-\frac{1}{r^2}u_r(r,z))+\frac{1}{r}(\frac{\partial u_r(r,z)}{\partial r}-\frac{u_r(r,z)}{r})$$
$$\frac{\partial^2 u_r(r,z)}{\partial r^2}+\frac{1}{r}\frac{\partial u_r(r,z)}{\partial r}-\frac{1}{r^2}u_r(r,z)=0$$
$$\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(r.u_r(r,z))]=0$$ éq 2.25

By successive integrations of éq 2.25 we obtain the following shape of the field $u_r(r,z)$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r.u_r)=f(z)$$
$$\frac{\partial}{\partial r}(r.u_r)=r.f(z)$$
$$r.u_r=\frac{r^2}{2}f(z)+g(z)$$
$$u_r=\frac{r}{2}f(z)+\frac{g(z)}{r}$$ éq 2.26

Is:

$$u_r(r,z)=C(z)r+\frac{D(z)}{r} \quad \text{éq 2.27}$$

As we have:

$$\begin{aligned}
&\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}=0\\
&\epsilon_{rr}=\frac{u_r(r,z)}{\partial r}=C(z)-\frac{D(z)}{r^2}\\
&\epsilon_{\theta\theta}=\frac{u_r}{r}=C(z)+\frac{D(z)}{r^2}\\
&\epsilon_{zz}=\frac{\partial u_z}{\partial z}=\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta})\\
&\epsilon_{zz}=\frac{-2\nu}{1-\nu}C(z)
\end{aligned} \quad \text{éq 2.28}$$

One leads thus to:

$$\frac{\partial u_z}{\partial z}=\frac{-2\nu}{1-\nu}C_1 z+C_2 \quad \text{éq 2.29}$$

One can thus write while integrating ϵ_{zz} that $u_z(r,z)=f(z)+g(r)$. By means of the first relation of éq 2.28 one obtains then:

$$\begin{aligned}
&\frac{\partial u_z}{\partial r}=g'(r)=\frac{-\partial u_r}{\partial z}\\
&g'(r)=-C'(z)r-\frac{D'(z)}{r}\\
&C'(z)=cte\rightarrow C(z)=C_1 z+C_2\\
&D'(z)=cte\rightarrow D(z)=D_1 z+D_2\\
&g(r)=-C_1\frac{r^2}{2}-D_1\ln(r)+C_0
\end{aligned} \quad \text{éq 2.30}$$

One leads thus to:

$$\frac{\partial u_z}{\partial r}=g'(r) \quad \text{éq 2.31}$$

By integrating the two partial derivative equations éq 2.29 and éq 2.31 one obtains for each of two contours i=1 external and i=2 interior:

$$\begin{aligned}
&u_z=\frac{-2\nu}{1-\nu}(C_1\frac{z^2}{2}+C_2 z)-C_1\frac{r^2}{2}-D_1\ln(r)+C_0\\
&u_z(r,0)=g(r)+f(0)=0\rightarrow C_0=C_1=D_1=0\\
&u_z(r,z)=\frac{-2\nu}{1-\nu}C_2 z
\end{aligned} \quad \text{éq 2.32}$$

from where one obtains:

$$\sigma_{rr}=\frac{E}{1+\nu}(\epsilon_{rr}+\frac{\nu}{1-\nu}(\epsilon_{rr}+\epsilon_{\theta\theta}))$$
$$\sigma_{rr}=\frac{E}{1+\nu}(\frac{\partial u_r(r,z)}{\partial r}+\frac{\nu}{1-\nu}(\frac{\partial u_r(r,z)}{\partial r}+\frac{u_r}{r})) \quad \text{éq 2.33}$$
$$\sigma_{rr}=\frac{E}{1+\nu}(\frac{1+\nu}{1-\nu}C_i+\frac{D_i}{r^2})$$

While posing:

$$A_i=\frac{E}{1+\nu}\frac{1+\nu}{1-\nu}C_i \qquad B_i=\frac{E}{1+\nu}D_i \quad \text{éq 2.34}$$

the field of the stresses is written:

$$\sigma_{rr;i}(r)=A_i-\frac{B_i}{r^2}$$
$$\sigma_{\theta\theta;i}(r)=A_i+\frac{B_i}{r^2} \quad \text{éq 2.35}$$
$$\underline{\underline{\sigma}}_i(r)=A_i\underline{\underline{1}}-\frac{B_i}{r^2}(\underline{e}_r\otimes\underline{e}_r-\underline{e}_\theta\otimes\underline{e}_\theta)$$

It any more but does not remain us to calculate the values of A_i and B_i for each contour. One will note λ_n the contact pressure between two contours such as:

$$\underline{\underline{\sigma}}_1(R_2).(-\underline{e}_r)=\lambda_n\underline{e}_r$$
$$\underline{\underline{\sigma}}_2(R_2).\underline{e}_r=-\lambda_n\underline{e}_r \quad \text{éq 2.36}$$

with the boundary conditions:

$$\underline{\underline{\sigma}}_1(R_1).\underline{e}_r=-p.\underline{e}_r$$
$$\underline{\underline{\sigma}}_2(R_3).(-\underline{e}_r)=\underline{0} \quad \text{éq 2.37}$$

one a:

$$A_1=\frac{-p R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2};\, B_1=(-p+\lambda_n)\frac{R_1^2R_2^2}{R_1^2-R_2^2}$$
$$A_2=-\lambda_n\frac{R_2^2}{R_2^2-R_3^2};\, B_2=-\lambda_n\frac{R_2^2R_3^2}{R_2^2-R_3^2} \quad \text{éq 2.38}$$

in addition the constitutive law of the materials makes it possible to write:

$$\underline{\underline{\epsilon}}_i=\frac{1+\nu_i}{E_i}\underline{\underline{\sigma}}_i-\frac{\nu_i}{E_i}Tr(\underline{\underline{\sigma}}_i)\underline{\underline{1}}=\frac{1}{E_i}[A_i(1-\nu_i)\underline{\underline{1}}-B_i(1+\nu_i)\frac{1}{r^2}(\underline{e}_r\otimes\underline{e}_r-\underline{e}_\theta\otimes\underline{e}_\theta)]$$
$$=\frac{\partial u_{r;i}}{\partial r}\underline{e}_r\otimes\underline{e}_r+\frac{1}{2}(\frac{1}{r}\frac{\partial u_{r;i}}{\partial\theta}+\frac{\partial u_{\theta;i}}{\partial r}-\frac{u_{\theta;i}}{r})(\underline{e}_r\otimes\underline{e}_\theta+\underline{e}_\theta\otimes\underline{e}_r)+(\frac{1}{r}\frac{\partial u_{\theta;i}}{\partial\theta}+\frac{u_{r;i}}{r})\underline{e}_\theta\otimes\underline{e}_\theta \quad \text{éq 2.39}$$

what makes it possible to obtain:

$$\underline{u}_i=\frac{1}{E_i}[A_i(1-\nu_i)r+B_i(1+\nu_i)\frac{1}{r}]\underline{e}_r=f_{CP}(r)\underline{e}_r \quad \text{éq 2.40}$$

the function $f(r)$ of radial displacement is given according to the properties of the materials and the pressure p . In the case of plane stresses (MODELISATION = "C_PLAN") one a:

$$f(R_1)=\frac{1}{E_1}(A_1(1-\nu_1)R_1+\frac{B_1(1+\nu_1)}{R_1})$$
$$f(R_3)=\frac{1}{E_2}(A_2(1-\nu_2)R_3+\frac{B_2(1+\nu_2)}{R_3}) \quad \text{éq 2.41}$$

to obtain the value of λ_n , one imposes the continuity of the vector displacement in $r=R_2$:

$$\underline{u}_1(R_2)=\underline{u}_2(R_2)$$
$$\frac{1}{E_1}[A_1(1-\nu_1)R_2+B_1(1+\nu_1)\frac{1}{R_2}]\underline{e}_r=\frac{1}{E_2}[A_2(1-\nu_2)R_2+B_2(1+\nu_2)\frac{1}{R_2}]\underline{e}_r$$
$$\frac{1}{E_1}[\frac{-p R_1^2+\lambda_n R_2^2}{R_1^2-R_2^2}(1-\nu_1)R_2+(-p+\lambda_n)\frac{R_1^2 R_2^2}{R_1^2-R_2^2}(1+\nu_1)\frac{1}{R_2}]\underline{e}_r \quad \text{éq 2.42}$$
$$=\frac{1}{E_2}[-\lambda_n\frac{R_2^2}{R_2^2-R_3^2}(1-\nu_2)R_2-\lambda_n\frac{R_2^2 R_3^2}{R_2^2-R_3^2}(1+\nu_2)\frac{1}{R_2}]\underline{e}_r$$

is still:

$$\lambda_n=\frac{2pR_1^2}{R_1^2(1+\nu_1)+R_2^2(1-\nu_1)+\frac{E_1}{E_2}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-\nu_2)+R_3^2(1+\nu_2))} \quad \text{éq 2.43}$$

2.1.3. Remark

Once one calculated the value of displacements and the contact pressure in the case of plane stresses, one can easily calculate these values in plane strains by replacing the values of the Young modulus and the Poisson's ratio:

$$E_{DP}=\frac{E_{CP}}{1-\nu_{CP}^2};\nu_{DP}=\frac{\nu_{CP}}{1-\nu_{CP}} \quad \text{éq 2.44}$$

Where E_{CP} and ν_{CP} takes the values E_i and ν_i of the §2.1.2.

The values of displacements thus are obtained:

$$\underline{u}_i=\frac{1+\nu_i}{E_i}[A_i(1-2\nu_i)r+\frac{B_i}{r}]\underline{e}_r=f_{DP}(r)\underline{e}_r \quad \text{éq 2.45}$$

and of the contact pressure:

$$\lambda_n=\frac{2\,p\,R_1^2(1-\nu_1)}{R_1^2+R_2^2(1-2\nu_1)+\frac{E_1}{E_2}\frac{1+\nu_2}{1+\nu_1}\frac{R_1^2-R_2^2}{R_2^2-R_3^2}(R_2^2(1-2\nu_2)+R_3^2)}$$ éq 2.46

One notices that the values of A_i and B_i remain unchanged.

2.1.4. Values tested

One tests the contact pressure to the interface between 2 contours. For a node of interface, the analytical solution is obtained by equations 1.3 and 1.5 in plane strains and plane stresses respectively.

The value of the pressure applied to edge to each time step is given by the formula:

$$p(t)=p_0 10^{\frac{t}{10}-1,1}, t\in\{n\in\mathbb{Z}, 1\leqslant n\leqslant 21\}, p_0=1,0\,MPa$$ éq 2.47

for $t=1$, the pressure is worth $0,1\,MPa$ and for $t=21$ it is worth $10\mathrm{MPa}$. It is noticed that simulations were made by neglecting the effects of acceleration (command STAT_NON_LINE) what implies that time step have arbitrary units.

The analytical solution is calculated in small strains. The calculations will be done with behavior GROT_GDEP, to compare the solutions with and without rotation. The analytical variation compared to the solution will thus increase with the increase in the value of the external pressure applied.

3 Modelization A

3.1 Characteristic of the modelization

It acts of a modelization in plane stresses (C_PLAN), without rotation of interior contour and where one imposes a loading increasing exponentially with time, the goal being measuring the difference between the computed values and those analytically obtained in order to better know the field of validity of the solution.

The Young's moduli $E_1=E_2$ and the Poisson's ratios $\nu_1=\nu_2$ are respectively $1.0E9\,Pa$ and 0.2. The pressure applied to edge of external contour is worth $1.0E6\,Pa$ and it varies from 10% to 1000% of its value in the course of time.

External contour defines main surface.

3.2 Characteristics of the mesh

The mesh (Figure 3.2-a) comprises:
- 160 meshes of type SEG2;
- 240 meshes of type QUAD4.

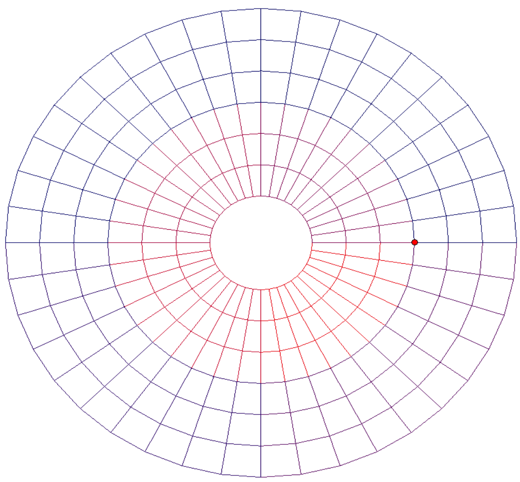

Figure 3.2-1: The mesh of the modelization A

3.3 Quantities tested and results

One calculates the contact pressure (LAGS_C) for the node A of coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. For each step of load, one 1.5 compares the computed value with that given by the equation. The tolerance is fixed at 2% compared to the analytical value.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$p(t)$	node	2.10-2

3.4 Comments

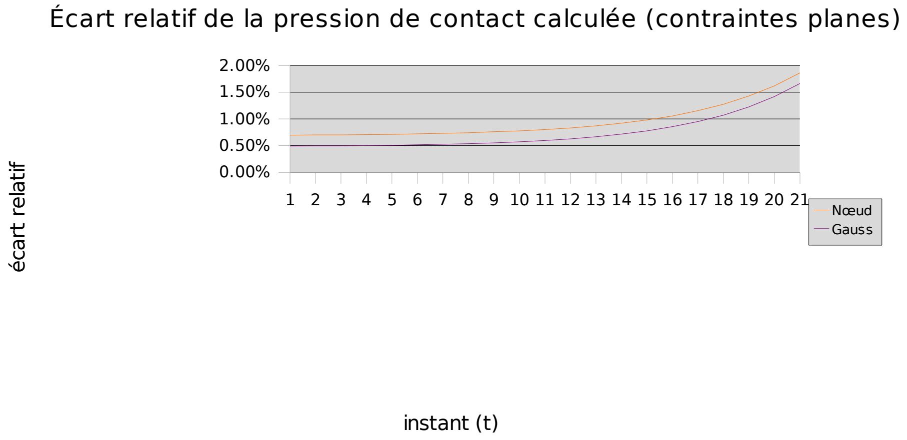

Figure 3.4-1: evolution of the variation enters the analytical solution and the solution given by Code_Aster for an integration to the nodes and Gauss points.

Since simulation is made on the assumption of an elastic behavior, the configuration at one time t depends by no means on previous times: all occurs as if several independent simulations were carried out, each one with a value of different loading. The gap with the analytical solution widens because one uses the constitutive law GROT_GDEP which utilizes large displacements.

4 Modelization B

4.1 Characteristic of the modelization

The modelization is identical to the modelization A, but in this case one will work in plane strains (D_PLAN) and, like already mentioned above, the Poisson's ratios do not intervene any more in the solution.
External contour defines main surface.

4.2 Characteristics of the mesh

Idem modelization A.

4.3 Grandeurs tested and results

One calculate the contact pressure (LAGS_C) for the node A of coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. For each step of load, one compares the computed value with that given by equation 1.3. The tolerance is fixed at 2% compared to the analytical value.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$p(t)$	node	2.10-2

4.4 Comments

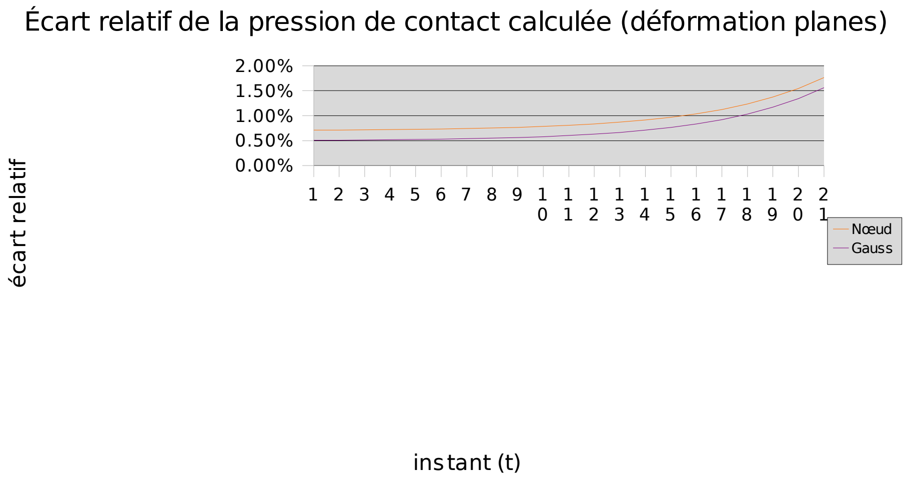

Figure 4.4-1: evolution of the variation enters the analytical solution and the solution given by Code_Aster for an integration to the nodes and Gauss points.

Idem modelization A.

5 Modélisation C

5.1 Characteristic of the modelization

Thereafter, one will work only with the modelization in plane stresses.

The Young's moduli and the Poisson's ratios remain the same ones. One fixes the value of the pressure on edge of contour external with ($p=1.0\mathrm{E}7$) and one imposes a rotation of interior contour in accordance with equation 1.6 for a number $N=100$ of time step.
External contour defines main surface.

5.2 Characteristics of the mesh

Idem modelization A.

5.3 Grandeurs tested and results

One calculate the contact pressure (LAGS_C) for the node A of coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0\mathrm{E}7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the meshes are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$\lambda=9.26\mathrm{E}6$	node	4.10-2

5.4 Comments

In addition to the computation of the contact pressure, one is also interested by his variation in the course of time. The incompatibility of meshes main and slaves induced of the fluctuations on the value this pressure. Figure 5.4-1 shows (in a a little coarse way) this effect. A way of attenuating it consists in using either a finer mesh, or of the elements of a higher nature.

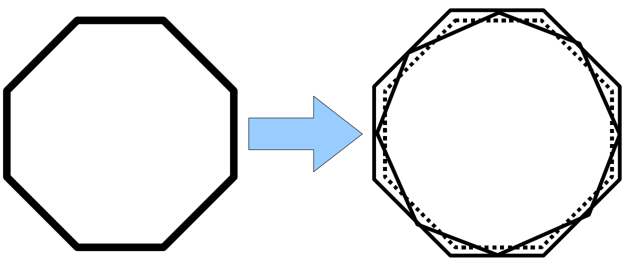

Figure 5.4-1: Fluctuation of the contact pressure due to the incompatibility of meshes

One gives Figure Figure 5.4-2 Figure Figure 5.4-3 evolutions of the contact pressure calculated for the 100 time step corresponding to the rotation of an element along circumference for $R=R_2$. The results presented here are got for $p=1.0\mathrm{E}4$. To consider most correctly possible the contact pressure, during rotation, when contact surfaces Master and slave are not compatible any more, it is necessary compared to to use the diagrams of integration of nature the highest possible with a substantial refinement of the mesh situation with compatible meshes of the modelizations A and B (in this case it is necessary to refine 10 times more on the circumference and radially and to use a diagram of Gauss of order 10 or Simpson to order 4 to bring back the error in pressure to 15% for semi path). Result is very clearly improved by the recourse to the quadratic elements (cf modelization G with an error of about 4%).

The rigidification of interior contour (see Figure 5.4-4) does not seem to have of effect on quality of the solution, contrary to what can occur for a quadratic mesh (see Figure 9.3-22).

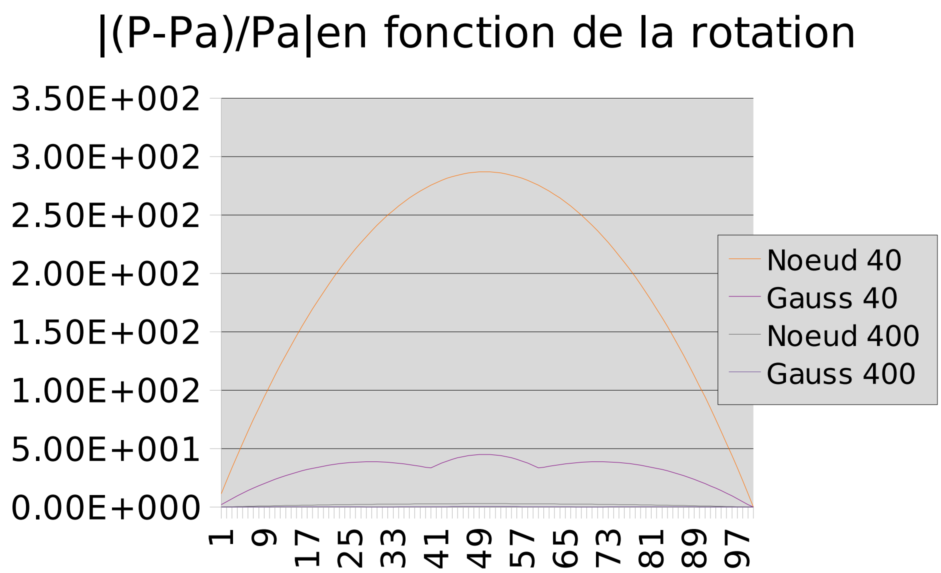

Figure 5.4-2: Evolution of the contact pressure during rotation for 160 and 1600 elements on the circumference $R=R2$. With the mesh with 160 elements one reaches an error of 30000% with integration to the nodes and 5000% with integration with 2 Gauss points for an error lower than 4% without rotation with compatibility of the meshes of surfaces Master and slave.

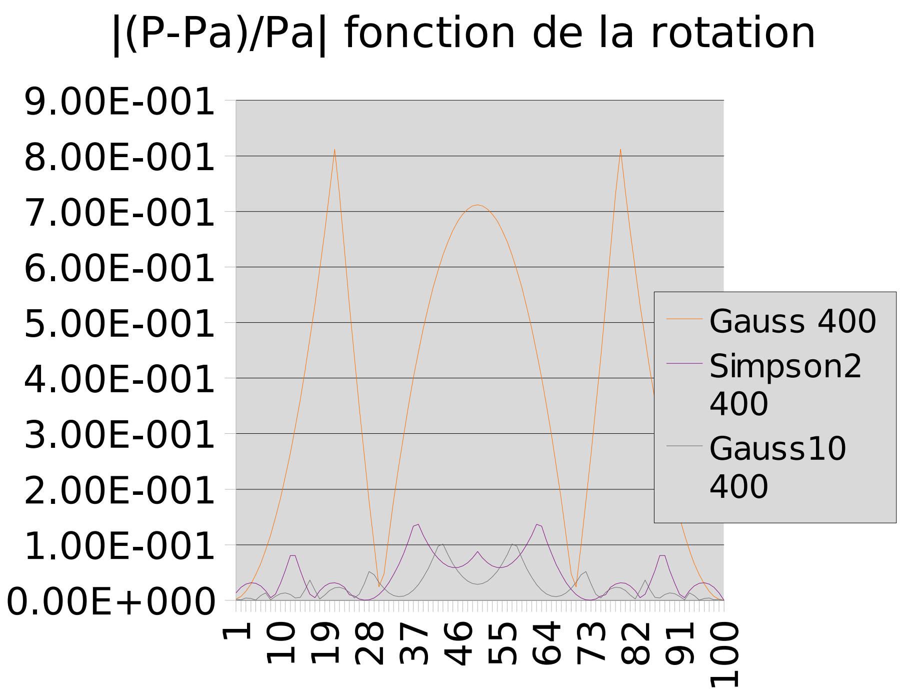

Figure 5.4-3: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$ for a rigid interior contour. An integration with 2 Gauss points led to a maximum error of 80%. Only the Gauss quadratures with 10 and of Simpson to order 4 make it possible to bring back the maximum error to the neighborhoods of 15% for incompatible meshes.

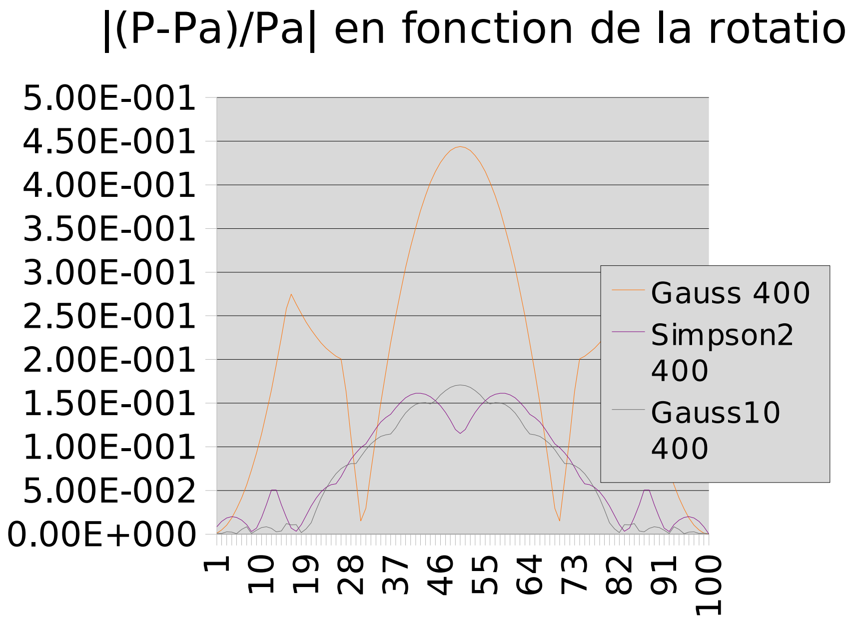

Figure 5.4-4: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$. An integration with 2 Gauss points led to a maximum error of 45%. Only the Gauss quadratures with 10 and of Simpson to order 4 make it possible to bring back the maximum error to the neighborhoods of 15% for incompatible meshes.

6 Modelization D

6.1 Characteristic of the modelization

Idem modelization C but one plays with the Young's moduli Poisson's ratios and the:

External contour: $E_1=1.0E9$, $\nu_1=0.3$

Interior Contour: $E_2=1.0E8$, $\nu_2=0.2$

External contour defines main surface.

6.2 Characteristics of the mesh

Idem modelization A.

6.3 Grandeurs tested and results

One calculate the contact pressure (LAGS_C) for the node A of coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0E7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the meshes are again in opposite.

_identification	Reference	Aster	tolerance
LAGS_C with node A	$\lambda=2.418E6$	Analytical	4.10-2

6.4 Comment

When the displacement become too important due with a weak stiffness of contour, the value calculate himself draw aside of that calculate analytically, once que this solution have be develop on the assumption of small strain and that the simulation have be make in large displacement.

Other with dimensions, when the stiffness of contours is too important, the fluctuation of the contact pressure increases appreciably. This comes owing to the fact that when one imposes a displacement on a structure, the stresses to which this one is subjected can be too high (see infinite) in order to be compatible with the models of the mechanics. For a very rigid structure, a small displacement is possible only with considerable stresses.

7 Modelization E

7.1 Characteristic of the modelization

Idem modelization C but interior contour defines main surface in order to satisfy condition LBB ($P1$ in contact and $P2$ displacement).

7.2 Characteristics of the mesh

The mesh comprises:

- 80 meshes of type SEG2;
- 80 meshes of type SEG3;
- 120 meshes of type QUAD4 on external contour;
- 120 meshes of type QUAD8 on interior contour.

7.3 Quantities tested and results

One calculates the contact pressure (LAGS_C) for the node A of coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0E7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the meshes have a maximum shift of a surface half-element.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$\lambda=9.26E6$	node	4.10-2

8 Modelization F

8.1 Characteristic of the modelization

Idem modelization C.

external contour defines main surface in order to satisfy condition LBB ($P1$ in contact and $P2$ displacement).

8.2 Characteristics of the mesh

We will work with elements of order 2 in order to observe the effects of their uses during computations. For the modelization E, interior contour will comprise elements of order 2 and the interior one of the elements of order 1. For the modelization F, one reverses. For the modelization G, all the mesh will have elements of order 2.

The mesh comprises:
- 80 meshes of type SEG2 ;
- 80 meshes of type SEG3 ;
- 120 meshes of type QUAD4 on interior contour;
- 120 meshes of type QUAD8 on external contour.

8.3 Quantities tested and results

One calculates the contact pressure (LAGS_C) for the node A of coordinates $(0.6, 0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0E7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the meshes have a maximum shift of a surface half-element.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$\lambda=9.26E6$	node	4.10-2

9 Modelization G

9.1 Characteristic of the modelization

Idem modelization C.
external contour defines main surface.

9.2 Characteristics of the mesh

The mesh quadratic comprises:

- 160 meshes of type SEG3;
- 240 meshes of type QUAD8.

9.3 Quantities tested and results

One calculates the contact pressure (LAGS_C) for the node A of coordinates $(0.6, 0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0E7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 4% compared to the analytical value. One tests when the meshes are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$\lambda=9.26E6$	node	4.10-2

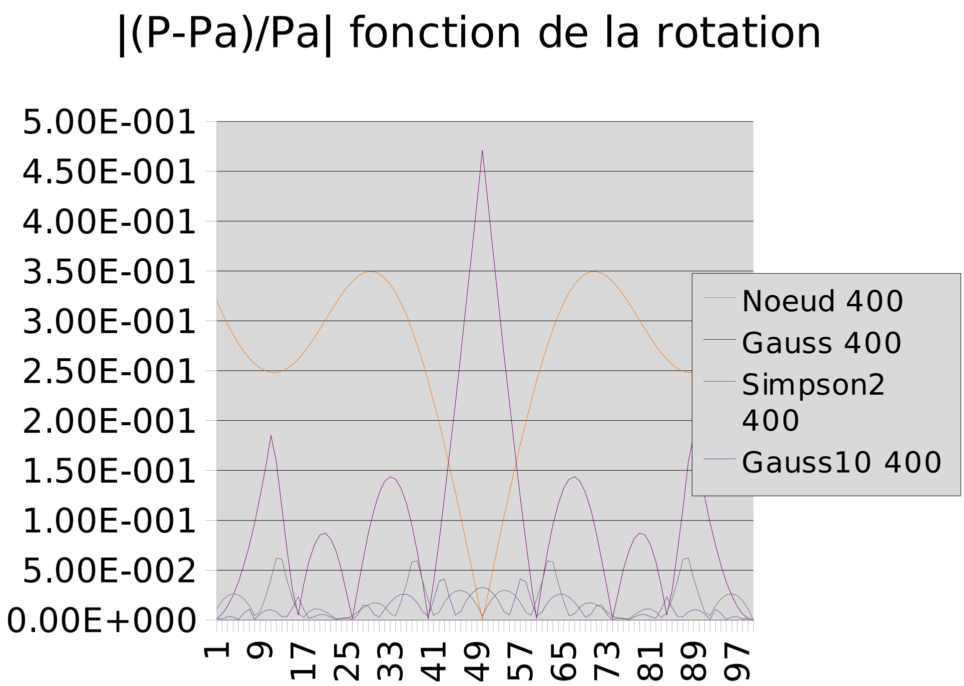

Figure 9.3-11: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$. An integration with 3 Gauss points led to a maximum error of 45%. A nodal integration does not give correct results when the master meshes and slave are compatible. Only the Gauss quadratures with order 10 and of Simpson to order 4 make it possible to bring back the maximum error to the neighborhoods of 5% for incompatible meshes.

One gives above the evolutions of the contact pressure calculated for the 100 time step corresponding to the rotation of an element along circumference for $R=R_2$. The results presented here are got for $p=1.0\text{E}4$. To consider most correctly possible the contact pressure, during rotation, when contact surfaces Master and slave are not compatible any more, it is necessary compared to to use the diagrams of integration of nature the highest possible with a substantial refinement of the mesh situation with compatible meshes of the modelizations A and B (in this case it is necessary to refine 10 times more and to use diagrams of Gauss of order 10 or Simpson to order 4 to bring back the error in pressure to 5%). Result is very clearly improved compared to that obtained with linear elements (on the same refined mesh, the diagrams of order 10 of Gauss and order 4 of Simpson lead to errors in pressure of 15%).

So finally one rigidifies interior contour for the same pressure $p=1.0\text{E}4$, one obtains result according to. All the diagrams of integration behave correctly with an error lower than 1% except the diagram with the node which gives a constant error to 33,4%.

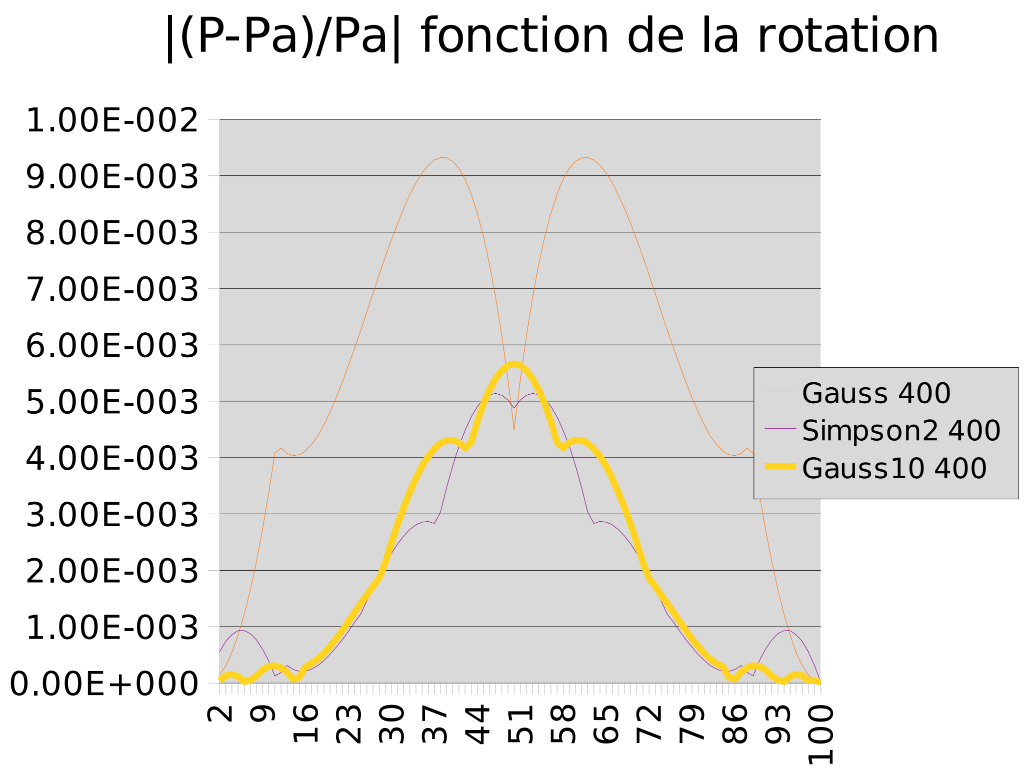

Figure 9.3-22: Evolution of the contact pressure during rotation for 1600 elements on the circumference $R=R2$ for a rigid interior disc. All the diagrams of Gauss to order 3, Gauss with order 10 and Simpson with order 4 give an error lower than 1%. A diagram with the node gives a constant error on the group of the rotation of 33,4% not represented here.

10 Modelization H

10.1 Characteristic of the modelization

Idem modelization G except that one uses an under-integrated modelization, "C_PLAN_SI".
External contour defines main surface.

10.2 Characteristics of the mesh

The mesh quadratic comprises:

- 160 meshes of type SEG3;
- 240 meshes of type QUAD8.

10.3 Quantities tested and results

One calculates the contact pressure (LAGS_C) for the node A coordinates $(0.6,0.0)$, that which at initial time is more on the right of the interface between two contours. The computed values are compared with the value obtained according to the equation 1.5 for an external pressure of $p=1.0E7$. The rotation of interior contour is applied. One looks at the variations of the external pressure during this rotation. The tolerance is fixed at 2% compared to the analytical value. One tests when the meshes are again in opposite.

Identification	Reference	Aster	tolerance
LAGS_C to the Analytical A	$\lambda=9.26E6$	node	2.10-2