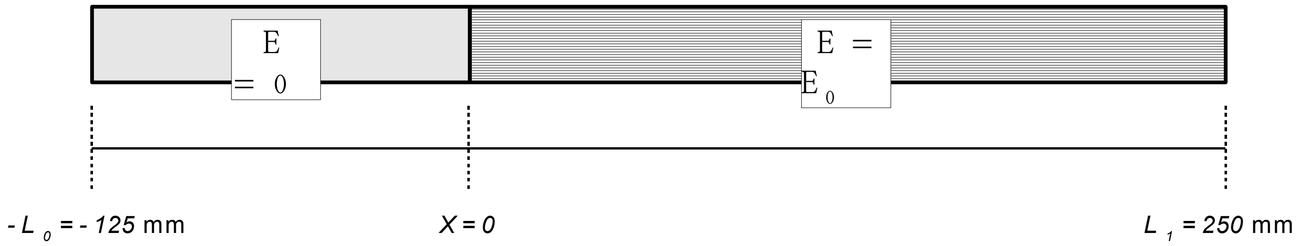
SSNL125 - Tension of a brittle bar: damage with Summarized

gradient:

This test allows the validation of the brittle damage model gradient ENDO_SCALAIRE in a nonhomogeneous unidimensional situation. From its character 1D, this problem admits an analytical solution which exhibe two modes of boundary layers: one finite length (existence of a free border between the damaged zone and the healthy zone) and the other infinite length (it extends to the border from the part).

1 Problem of reference

1.1 Geometry



the studied structure is a bar $375\,mm$ length. The problem being purely $1\mathrm{D}$, its section is without influence.

1.2 Properties of the material

the material obeys a brittle elastic constitutive law (`ENDO_SCALAIRE`) with gradient of damage (modelization `*_GRAD_VARI`).

ELAS	ENDO_SCALAIRE	NON_LOCAL
E = $30\,000\,MPa$ NU = 0	SY = $3\,MPa$ GAMMA = 4	C_GRAD_VARI = $1.875\,N$ PENA_LAGR = 1.5

1.3 Conditions of loading

One forces the left part of the bar ($125\,mm$ length) to remain rigid (blocking of the degrees of freedom of displacement). As for the right part of the bar, it is subjected to a uniform axial strain ε_0, i.e. with an imposed displacement whose spatial is linear. Only one parameter thus controls the intensity of the loading: the level of imposed strain ε_0.
In the directions perpendicular to the axis of the bar, displacements are blocked: the problem is purely $1\mathrm{D}$. Moreover, as the Poisson's ratio is null, no stress of fastening does not develop in these directions.

2 Reference solution

We consider one of the constitutive laws with gradient of damage [R5.04.01] of which the density of free energy arises in the following form:
$\Phi(\varepsilon, a)=A(a)w(\varepsilon)+\omega(a)+c/2(\nabla a)^2$, where $0\leqslant a\leqslant 1$ indicates the variable of damage and $w(\varepsilon)$ elastic strain energy. For model ENDO_SCALAIRE, $\omega(a)=ka$ and the prefactor of damage is written like $A(a)=\left(\frac{1-a}{1+\gamma a}\right)^2$. The solution depends then on three material parameters γ c , and k (see [R5.03.18]).

In the general case, two partial derivative equations must be solved: the balance equation ($\delta\Phi(\varepsilon, a)/\delta\varepsilon=0$) and the equation of behavior $\delta\Phi(\varepsilon, a)/\delta a=0$. To obtain an analytical solution proves generally delicate, even for unidimensional structures. To validate nevertheless this model, one sticks to a simpler problem for which the balance equation does not require to be solved, i.e. the field of displacement is built-in everywhere. The equation of behavior is then controlled by known elastic w strain energy in any point of space. In this simplified case, the semi-analytical solution is obtained by a numerical integration of a differential equation with the initial conditions of Cauchy.

2.1 Characterization of the solution

more precisely, one considers a bar of which a part is obligation to remain without strain while the other is subjected to a homogeneous strain. One then studies the boundary layer of damage which develops with the interface of these two zones. The differential equation of behavior is the following one in the zones where the criterion is reached, i.e. where the damage evolves:

$$c\frac{\partial^2 a}{\partial x^2}-\frac{\partial A}{\partial a}w-k=0 \text{ , where } \frac{\partial A}{\partial a}(a)=-2(1+\gamma)\frac{(1-a)}{(1+\gamma a)^3} , \qquad \text{éq 2.1-1}$$

where x the variable of space indicates.

Elastic strain energy w is known in each of the two parts of the bar: $w=0$ on the left and $w(\varepsilon_0)=const$ on the right. The solution is given by a differentiable continuous function $a\in C^1(-L_{0,}L_1)$.

One has three following boundary conditions:

$$\frac{da}{dx}(-L_0)=\frac{da}{dx}(L_1)=0 \text{ and } a(-L_0)=0 \qquad \text{éq 2.1-2}$$

like two conditions of interface:

$$[|a|]_{x=0}=0 \qquad \left[\left|\frac{da}{dx}\right|\right]_{x=0}=0 \qquad \text{éq 2.1-3}$$

On the whole, we thus have five boundary conditions for five unknowns. In spite of this fact, the problem “is badly posed” to be able to be solved numerically, because the boundary conditions are defined in the various points of space. We proceed in following the identification of the unknowns using the boundary conditions finally obtaining an ordinary differential equation of the Cauchy-Lipschitz type.

2.2 Resolution of the problem in the discharged zone

One supposes that the left end of the bar $x=-L_0$ is sufficiently far from the interface $x=0$ so that the assumption of a boundary layer finite length in the discharged zone remains valid. Then let us note $b_0>0$ the length (*a priori* dependant on the level of strain ε_0 in the zone charged) on which develops this boundary layer.

In the part $x\in[L_0,-b_0]$ the damage does not evolve and remains null:

$$a(x)=0 \text{ on } [-L_0,-b_0] \qquad \text{éq 2.2-1}$$

In the part $x\in[-b_0,0]$, the criterion is reached and the field of damage evolves according to the equation [éq 2.1-1], which is simplified considerably in this part of the bar since elastic strain energy is there null:

$$c\frac{d^2a}{dx^2}=k \text{ on } [-b_0,0] \qquad \text{éq 2.2-2}$$

Per definition of b_0, one has the following conditions:

$$a(-b_0)=0 \text{ and } \frac{da}{dx}(-b_0)=0 \qquad \text{éq 2.2-4}$$

One can then analytically express the variable of damage while integrating [éq 2.2-2]:

$$a(x)=\frac{k}{2c}(x+b_0)^2 \text{ on } [-b_0,0] \qquad \text{éq 2.2-5}$$

One then knows the values of the damage and his derivative to the interface $x=0$, according to the unknown b_0 :

$$a(0^-)=\frac{kb_0^2}{2c} \text{ and } a'(0^-)=\frac{kb_0}{c} \qquad \text{éq 2.2-6}$$

and the damage with the interface a_0 being strictly understood enters 0 and 1, one from of deduced the following framing the length b_0 :

$$0<a_0<1 \quad \Rightarrow \quad 0<b_0<\sqrt{\frac{2c}{k}} \qquad \text{éq 2.2-7}$$

It is thus enough to take the left part of the bar longer than $\sqrt{2c/k}$ to be certain to have the profile of damage confined on the left.

2.3 Resolution of the problem in the zone charged

This part of the bar sees a homogeneous strain $\varepsilon_0>0$ associated with an elastic strain energy w_0. In this zone, the boundary layer is not any more limited and asymptotically extends towards the homogeneous response. It will thus be supposed that the right end of the bar $x=L_1$ is sufficiently far from the interface $x=0$ to be able null to make the approximation of a derivative for the field of damage "ad infinitum". In the vicinity of the right end of the bar, the equation of behavior [éq 2.1-1] is reduced then to:

$$\frac{-\partial A}{\partial a}(a_\infty)w_0=k \quad \text{éq 2.3-1}$$

where a_∞ the asymptotic value of the field of damage indicates. Taking into account the statement of $\partial A/\partial a$ (*cf* [éq 2.1-1]), one can then parameterize the level of loading by the only value a_∞ :

$$\text{for } a_\infty\in]0,1[,\quad w_0=\frac{k}{2(1+\gamma)}\frac{(1+\gamma a_\infty)^3}{(1-a_\infty)} \quad \text{éq 2.3-2}$$

In addition, the nonlinear differential equation [éq 2.1-1] can be written in the following form:

$$\frac{c}{2}2a'a''=a'(A'(a)w-k) \quad \text{éq 2.3-3}$$

it thus admits an integral first:

$$\forall x\in[0,L_1],\quad \left[\frac{c}{2}\left(\frac{da}{dx}(s)\right)^2\right]_0^x=[A(a(s))+k\,a(s)]_0^x \quad \text{éq 2.3-4}$$

One recalls that the condition of interface [éq 2.1-3] imposes a connection C^1 in $x=0$, by taking account of the statements flexible $a(0^+)$ and $a'(0^+)$ with the length b_0 of the boundary layer one can then write:

$$a_0=a(0^+)=\frac{kb_0^2}{2c} \text{ and } a'_0=a'(0^+)=\sqrt{\frac{2k}{c}a_0} \quad \text{éq 2.3-5}$$

Moreover, the boundary conditions [éq 2.1-2] ensure the nullity of derivative of the damage in $x=L_1$, by making the approximation $a(L_1)=a_\infty$, the integral first [éq 2.3-4] evaluated in $x=L_1$ cost with:

$$A(a_0)w_0+k\,a_0-\frac{c}{2}a'^2_0=A(a_\infty)w_0+k\,a_\infty \quad \text{éq 2.3-6}$$

the equation [éq 2.3-6] is simplified according to [éq 2.3-5], it is reduced then to trinomial quadratic according to unknown a_0 :

$$(f(a_\infty)\gamma^2-1)a_0^2+2(f(a_\infty)\gamma+1)a_0+(f(a_\infty)-1)=0 \text{ where } f(a_\infty)=A(a_\infty)+\frac{k\,a_\infty}{w_0} \quad \text{éq 2.3-7}$$

One thus has a simple relation between the loading parameter a_∞ and the value a_0 of the damage as well as its derivative a'_0 with the interface.

The resolution of the nonlinear EDO [éq 2.1-1] on $[0,L_1]$ was carried out while bringing back it to a differential connection of a nature 1, integrated numerically with a diagram of Runge-Kutta into order 4 (in space). The implementation of this diagram requires only the knowledge of the initial conditions (a_0,a'_0) associated with each loading parameter a_∞, which is ensured by the relation [éq 2.3-7]

2.4 Resolution of the problem in the whole bar

In order to calculate the reference solution on all the bar, one adopts the following diagram:

- choice of a loading parameter $a_{\infty} \in]0,1[$
- Resolution in the part charged:
    - determination with [éq 2.3-2] of elastic strain energy w_0 associated
    - determination with [éq 2.3-7] with the set of boundary conditions $(a_{0,} a'_0)$
    - numerical integration with a diagram of Runge-Kutta to order 4
- Resolution in the discharged part:
    - determination with [éq 2.3-7] and [éq 2.3-5] of the size b_0 of the boundary layer in the zone discharged
    - obtaining with [éq 2.2-5] from the analytical field $a(x)$ in this zone

2.5 numerical Application

For elasticity, hardening and the nonlocal parameter, one adopts the following characteristics:

$$E=3.10^4 MPa \quad \sigma^y=3. MPa \quad c=1.875 N \qquad \nu=0. \quad \gamma=4.$$ éq 2.5-1

These choices lead to an elastic strain energy threshold:

$$w_y=1.5\,10^{-4} MPa$$ éq 2.5-2

2.6 Results of reference

the reference solution is obtained by taking a bar length $-L_0=-125\,mm$ and $L_1=250\,mm$. One examines the value of the field of damage a for three levels of loading and in two places, one in the discharged zone, the other in the zone charged.

ε_0	w_0 (MPa)	$a(x=-7.5\,mm)$	$a(x=+7.5\,mm)$
2,70000000000000E-04	1,09350000000000E-003	1,93274688119012E-02	1,41846675324338E-01
7,34846922834953E-04	8,10053999999999E-003	1,39107889370765E-01	3,80008828951670E-01
1,10464444958548E-02	1,83060308787201E+000	6,14240950943351E-01	9,77312427816067E-01

Array 2.6-1 - Results of reference

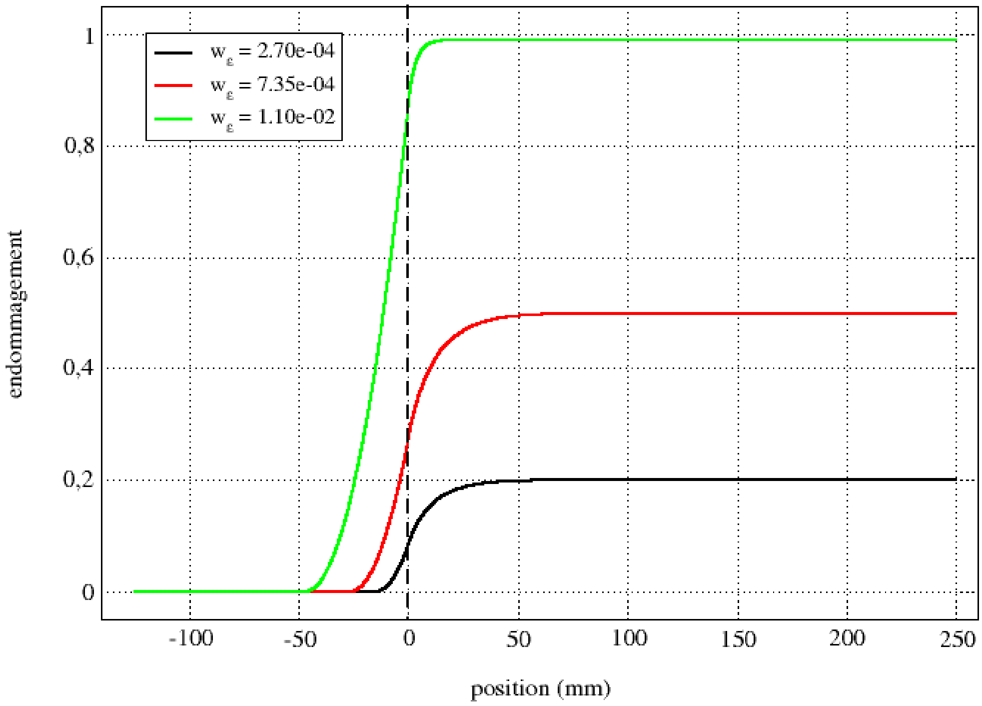


Figure 2.6-1: Profile of damage for the reference solution

3 Modelization A

3.1 Characteristic of the modelization and the mesh

It acts of an axisymmetric modelization (AXIS_GRAD_VARI). The corresponding geometry is a rectangle, i.e. the bar is laid out in a vertical way and its section (without influence) is circular.

The mesh consists of only one element according to the radius. According to the axis, the elements have a size of $2.5\,mm$. The mesh thus generated finally consists of 150 quadrangular elements with 8 nodes.

3.2 Quantities tested and results

One validates the modelization and the algorithm of integration of nonlocal models by examining the level of damage (local variable $V1$) on the various levels of loading and the various loci listed in [Array 2.6-1]. The results are joined together in the extract of results file Ci - below.

Identification	Time	Reference	*Aster*	Difference (%)
$V1(x=-7.5)$	2,70000000000000E-04	1,93274688119012E-02	1,93347814045151E-02	0,038
$V1(x=+7.5)$	2,70000000000000E-04	1,41846675324338E-01	1,41975480236050E-01	0,091
$V1(x=-7.5)$	7,34846922834953E-04	1,39107889370765E-01	1,39227327099270E-01	0,086
$V1(x=+7.5)$	7,34846922834953E-04	3,80008828951670E-01	3,80279953561891E-01	0,071
$V1(x=-7.5)$	1,10464444958548E-02	6,14240950943351E-01	6,15183197007750E-01	0,153
$V1(x=+7.5)$	1,10464444958548E-02	9,77312427816067E-01	9,77966469784146E-01	0,067

4 Summary of the results

One notes very a good agreement between the modelization and the analytical solution.