SDND104 - Computation of the power of wear of a mass rubbing under seismic excitation harmonic

Abstract:

One considers a mass in contact rubbing with a rigid plane on which one imposes a vibratory motion of harmonic type. Friction is modelled by the model of Coulomb. The computation of the response of the mass is of transitory type nonlinear. One calculates the power of wear resulting from the phases of sliding between the mass and the rigid plane. The computation of the power of wear being developed in *Aster* only for modal computations, the analysis is carried out on modal base (commonplace) system. In order to avoid the numerical problems resulting from the nullity of the single mode of rigid body of the mass, a spring far from stiff is introduced, flexible the mass at a point interdependent of the vibrating rigid plane.

The reference solution is a quasi analytical computation of the transient response, whose numerical estimates are programmed with Maple.

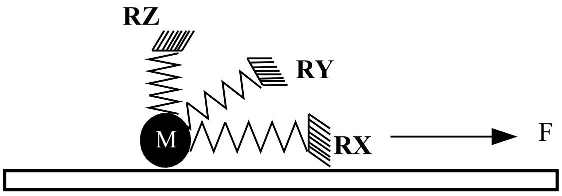
The single modelization *Aster* retained tests the explicit algorithms of integration to constant step of Eulerian (order 1), Devogeleare (order 4) and the algorithms with variable step `ADAPT_ORDRE2` (order 2) and `RUNGE_KUTTA` (orders 54 and 32) developed in the operator dedicated to the vibratory dynamics, for various amplitudes of the harmonic acceleration of seismic energization of the plane of rigid support. According to this amplitude, the mode of the response of the mass is of the adherent type for any time (stick), successively member and slipping (stick-slip), or always slipping with inversion of the meaning of sliding (slipway-slipway).

An account is given owing to the fact that in the case of a sufficiently low amplitude of excitation (first mode, permanent dependancy), the power of wear is strictly null.

1 Problem of reference

1.1 Geometry

the system considered consists of a simple heavy mass posed on a rigid support subjected to an imposed vibration of type seismic, sinusoidal. The contact, as well as solid friction are modelled by penalization. The system thus has two degrees of freedom of translation (horizontal and vertical).



A very weak come out from stiffness connects the mass to the support in the three directions. This spring is an artifice of computation, intended to avoid the nullity of the frequency associated with the rigid mode with horizontal adjustment with the mass. The Aster *results* taking into account the presence of this spring are not very different from the results which one would get without spring.

1.2 Properties of the model

Stiffness of spring (according to the three directions):	$k=3.10^{-5} N/m$
mass:	$m=1\,kg$
gravity:	$g=10\,m/s^2$
coefficient of Coulomb:	$\mu=0{,}1$

1.3 Boundary conditions, initial conditions and loadings

the mass rests on the rigid level with the dimension $z=0$.

The harmonic acceleration imposed on the base has as an equation $a=a_0 \sin(\omega t)$. In particular, it is null at initial time. The displacement of the support satisfies the equation $X(t) = -(a_0 / \omega^2)\sin(\omega t)$, and thus begins its motion towards the left, with the non-zero initial velocity $\dot{X}(0) = -a_0 / \omega$.

The initial displacement (with $t = 0$) of the mass is taken null. The mass is regarded as in a state of dependancy at initial time. It thus has the same non-zero velocity as the support with $t = 0$.

Computations are carried out for various values of maximum acceleration:

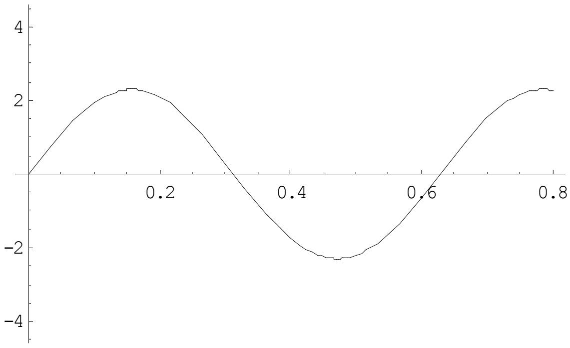
$a_0 = 15$ m/s ^{2} , $a_0 = 1{,}5$ m/s ^{2} , $a_0 = 1{,}01$ m/s ^{2} and $a_0 = 0{,}99$ m/s ^{2}

and a value of pulsation: $\omega=2\pi$.

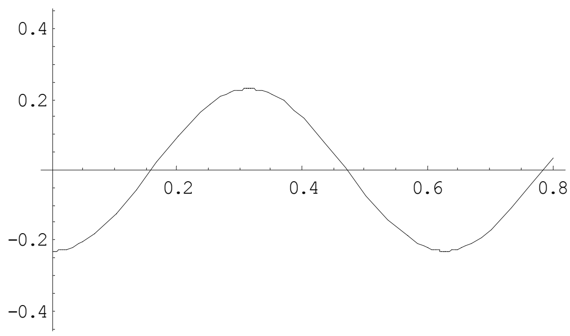
2 Reference solution

the reference solution, which is analytical, is calculated in the following way.
That is to say $x(t)$ the X-coordinate of the mass in the fixed reference and $X(t)$ the X-coordinate of the vibrating support in this same reference.

Initially, it is supposed that the mass is adherent on its support. It remains to it a certain time then after initial time $t=0$. It undergoes of this fact the acceleration imposed by the rigid support, that is to say $\ddot{x}(t)=\ddot{X}(t)=a_0\sin\omega t$. The tangential force exerted by the mass on the support is then $F_T=-m\ddot{x}(t)=-ma_0\sin\omega t$ (null at time initial, which justifies the starting assumption that initially, the mass is adherent on its support). The mass remains adherent as long as $\left|F_T\right|=ma_0\left|\sin\omega t\right|\leq\mu F_N=\mu mg$. If $a_0\leq\mu g$, the mass thus remains indefinitely adherent on its support, and its motion is exactly the same one as this one. By introducing the adimensional coefficient $\eta=\dfrac{\mu g}{a_0}$, the condition of permanent dependancy is written $\eta\geq 1$. The curve of acceleration of the mass, like support, then takes the following form according to time:

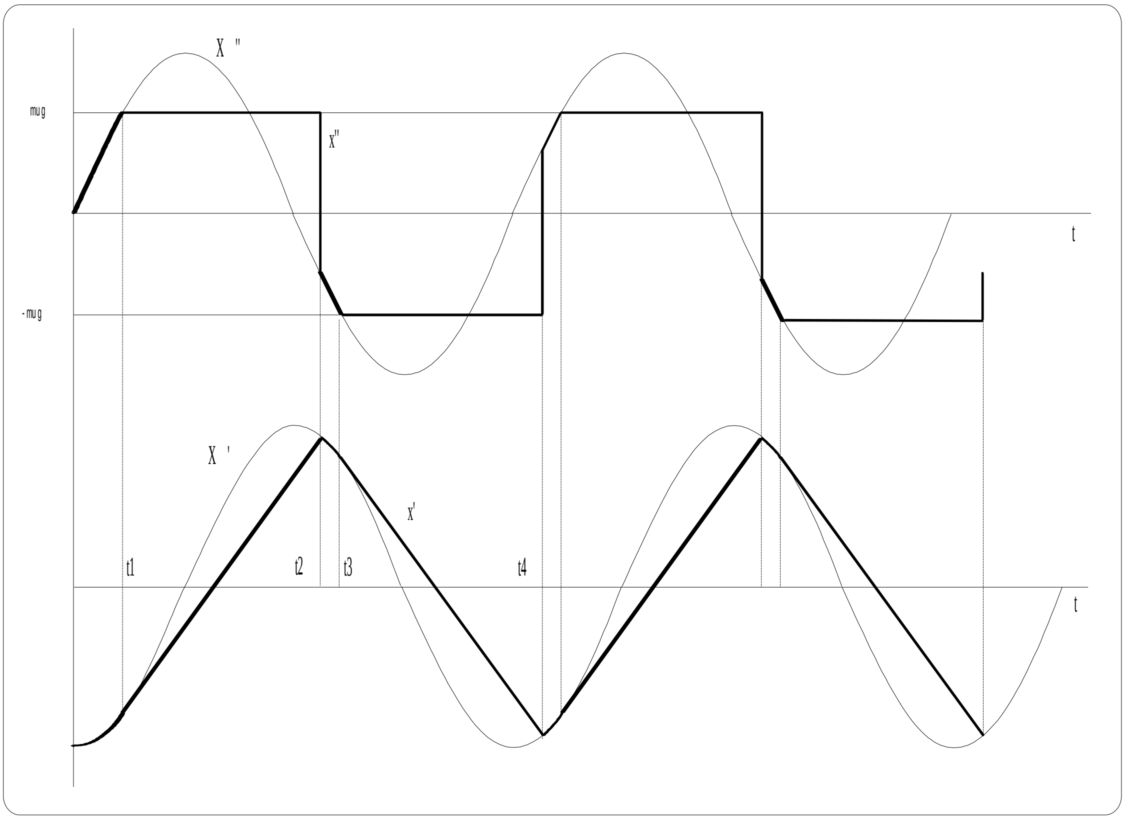


As for the velocity, it takes the following form (single primitive of average null):



If $a_0 > \mu g$, there exists a smaller time $t = t_1$ such as $\left|F_T\right| = ma_0\left|\sin\omega t_1\right| = \mu mg$. This smaller time is necessarily such as $\sin\omega t_0 > 0$, which makes it possible to remove the absolute value in the preceding statement, and to obtain explicit statement $t_1 = \frac{1}{\omega}\arcsin\frac{\mu g}{a_0} = \frac{1}{\omega}\arcsin\eta$. In particular $t_1 \le \frac{T}{4} = \frac{2\pi}{4\omega} = \frac{\pi}{2\omega}$.

After this time, the mass slips towards the left compared to the support, therefore it checks the dynamic equation $\ddot{x}(t) = \mu g$, that is to say $\dot{x}(t) = \mu g(t - t_1) + \dot{x}(t_1)$. Its velocity thus increases linearly with time, while leaving to t_1 negative value $\dot{x}(t_1) = -\frac{a_0}{\omega}\cos\omega t_1 = -\frac{a_0}{\omega}\sqrt{1-\eta^2}$ (indeed, $\sin\omega t_1 = \eta$).



Motion for $\eta > \eta^*$, mode of "stick-slip", succession of dependancy and sliding

Necessarily, for a certain value of satisfactory t_2 time $\pi / 2\omega \le t_2 \le 2\pi / \omega$, the velocity of the mass becomes again equal at the speed of the support. A this time, motion becomes again adherent *if and only if* the acceleration which the mass at the beginning of the dependancy undergoes is lower in absolute value than μg. One examines the translation of this condition in the continuation. One expresses to begin the value of t_2.

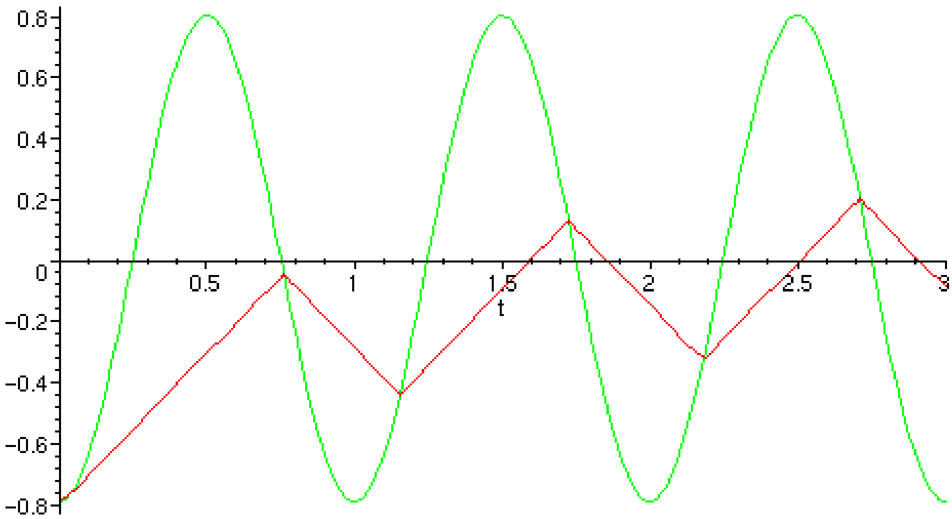
Time t_2 satisfies the equation $\dot{x}(t_2) = \dot{X}(t_2)$, that is to say $\mu g(t_2 - t_1) - \frac{a_0}{\omega}\cos\omega t_1 = -\frac{a_0}{\omega}\cos\omega t_2$, or $\eta\omega(t_2 - t_1) - \cos\omega t_1 + \cos\omega t_2 = 0$.

This equation, transcendent, allows the determination of t_2 according to t_1 and η, that is to say finally, taking into account the statement of t_1, the determination of t_2 according to the physical parameters of the system η and ω. If the acceleration of the support in t_2 is lower in absolute value than μg, motion remains adherent then up to one time t_3 for which the acceleration of the support and the mass reach the value $-\mu g$, time which for reasons of clear symmetries on the graphs above, exactly satisfied $t_3 = t_1 + \pi / \omega$. The mass then starts a phase of sliding up to one time t_4, after which motion reproduces periodically.

It is understood that for sufficiently small values of η, motion will not be able to become adherent as from time t_2, because the acceleration of the mass would exceed the threshold μg. There thus exists a breaking value η^* such as for $\eta > \eta^*$, the motion of the mass passes without phase of dependancy of a sliding to a shift in opposite meaning. A reflection on the continuity of the function response of velocity of the mass compared to the parameter η shows that for $\eta \le \eta^*$, later motion is always slipping (mode of "slipway-slipway", of alternate meanings). For $\eta < \eta^*$, motion periodically alternates phases of dependancy and sliding.

The breaking value η^* admits a simple analytical statement. Indeed, for $\eta = \eta^*$, times t_2 and t_3 coincide. Thus $t_2 - t_1 = t_3 - t_1 = \pi / \omega$ and the equation $\eta\omega(t_2 - t_1) - \cos\omega t_1 + \cos\omega t_2 = 0$ becomes $\pi\eta^* = 2\cos\omega t_1 = 2\sqrt{1 - \eta^{*2}}$. While passing squared, one obtains $\pi^2\eta^{*2} = 4 - 4\eta^{*2}$, that is to say $\eta^* = \frac{2}{\sqrt{\pi^2 + 4}} \approx 0{,}537$.

For $\eta \le \eta^*$, motion is **only asymptotically** periodic. The continuation (t_n) of times of change of meaning of sliding checks $t_{n+1} - t_n \to \pi / \omega$ when n tends towards the infinite one. Figure Ci - below watch typical pace (broken line) velocity of the mass in the situation of slipway-slipway.



Motion for $\eta \le \eta^*$: mode of "slipway-slipway", no dependancy

Let us summarize the conclusions:

There is the adimensional coefficient $\eta = \frac{\mu g}{a_0}$ and its value criticizes * such as $\eta^* = \frac{2}{\sqrt{\pi^2 + 4}} \approx 0{,}537$.

If $\eta^* < \eta < 1$	the established mode is of standard "stick-slip": alternation of phases of dependancy and sliding;
If $\eta < \eta^*$,	the established mode is of standard "slipway-slipway": alternate permanent sliding;
If $\eta > 1$,	the established mode is of standard "stick": permanent dependancy with the base.

In the results of analytical comparison computation/*Aster* which follow, the choices of the amplitude a_0 are such as these three situations are visited. One takes indeed $m = 1\,kg$ $g = 10\,m/s^2$ $\mu = 0.1$

$a_0 = 15\,m/s^2$ $a_0 = 1.5\,m/s^2$, $a_0 = 1.01\,m/s^2$ and $a_0 = 0.99\,m/s^2$.

The power of wear is physically null during the phases of dependancy.
In *Code_Aster*, with the operator `DYNA_VIBRA` used here, the dependancy is not detected because the integration of motion is made by regularization of the friction law. The respect of result no one of the power of wear during phases of dependancy required the introduction of a criterion on the velocity of sliding, so that below a certain value, it must regarded as null, and adherent motion. One can consult documentation of reference Operator of computation of wear/Models of Archard [R7.04.10].

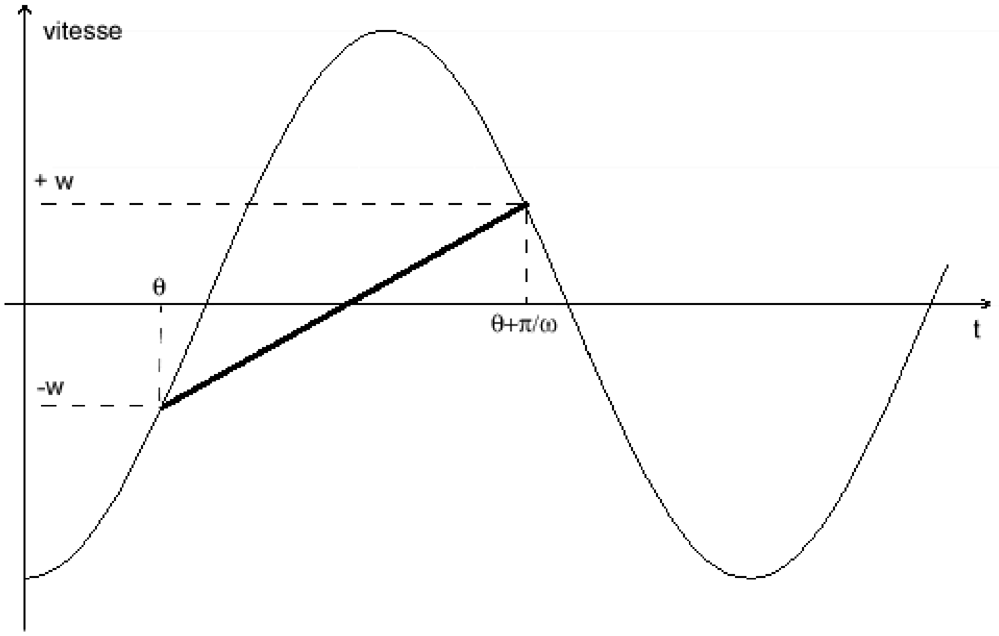
During the phases of sliding, the power of wear follows the model $P_u(t) = \mu mg |V_R(t)|$, where $V_R(t) = \dot{x}(t) - \dot{X}(t)$ is the relative velocity of sliding of the mass on the support. In the situation of the mode of stick-slip, for which motion becomes strictly periodic at the end of a finished time, the energy of wear during a half-period is exactly

$$E_u = \int_{t_1}^{t_2} mg |V_R(t)| dt = mg \int_{t_1}^{t_2} |\dot{X}(t) - \dot{x}(t)| dt = mg \int_{t_1}^{t_2} (-\frac{a_0}{\omega} \cos\omega t - (\mu g(t - t_1) - \frac{a_0}{\omega} \cos\omega t_1)) dt$$

$$= mg \left[\frac{a_0}{\omega} ((t_2 - t_1)\cos\omega t_1 - \frac{1}{\omega}(\sin\omega t_2 - \eta)) - \frac{\mu g}{2}(t_2 - t_1)^2 \right].$$

The transcendent formulation of t_2 apparently does not make it possible to simplify the statement of this energy of wear. The power of average wear $\overline{P}_u$ is simply the energy of wear E_u divided above by the half-period of the response $T/2 = \pi/\omega$.

In the case of a motion always slipping ($\eta \leq \eta^*$), the interval of integration to be taken is form $[t_n, t_{n+1}]$ with n sufficiently large, so that $t_{n+1} - t_n$ is sufficiently close to the limiting value π/ω. One can avoid numerical computation by recurrence of this continuation, knowing that the average asymptotic velocity is null. Indeed, the continuation $t_n - n\pi/\omega$ has a finished limit ϑ. The satisfied properties by ϑ are illustrated on the following figure:



The line segment has as an equation

$$v = \mu g(t-\vartheta) - w = \mu g(t-\vartheta) - \frac{a_0}{\omega}\cos(\omega\vartheta)\ ,$$

and for $t = \vartheta + \pi/\omega$, the velocity v is to take the opposed value $w = \frac{a_0}{\omega}\cos(\omega\vartheta)$, which gives the equation

$$\mu g\pi/\omega - \frac{a_0}{\omega}\cos(\omega\vartheta) = \frac{a_0}{\omega}\cos(\omega\vartheta)\ ,$$

that is to say

$$\mu g\pi = 2a_0\cos(\omega\vartheta)\ ;$$

whose solution is

$$\vartheta = \frac{1}{\omega}\arccos\left(\frac{\mu g\pi}{2a_0}\right) = \frac{1}{\omega}\arccos\left(\frac{\eta\pi}{2}\right).$$

Let us note that one finds although for $\eta = \eta^*$, the acceleration of the support calculated at time $t = \vartheta$ gives the limiting value μg. Indeed

$$a_0\sin(\omega\vartheta) = a_0\sin(\arccos(\eta^*\pi/2)) = a_0\sqrt{1-\eta^{*2}\pi^2/4} = a_0\sqrt{1-(1-\eta^{*2})} = a_0\eta^* = \mu g\ .$$

In the case of motion always slipping, the energy of wear during one asymptotic period is given exactly by the formula

$$E_u = \int_{\vartheta}^{\vartheta+\pi/\omega} mg\left|V_R(t)\right|dt$$

which one can clarify according to preceding computation, while taking $t_1 = \vartheta$ and $t_2 = \vartheta + \pi/\omega$, which gives

$$E_u = mg\left[\frac{a_0}{\omega}(t\cos\omega\vartheta - \frac{1}{\omega}\sin\omega t) - \frac{\mu g}{2}(t-\vartheta)^2\right]_{\vartheta}^{\vartheta+\pi/\omega} = mg\left[\frac{a_0}{\omega}(\frac{\pi}{\omega}\frac{\eta\pi}{2} + \frac{2}{\omega}\sqrt{1-\frac{\eta^2\pi^2}{4}}) - \frac{\mu g}{2}\frac{\pi^2}{\omega^2}\right],$$

that is to say

$$E_u = \frac{mga_0}{\omega^2}\sqrt{4-\pi^2\eta^2} = \frac{mg}{\omega^2}\sqrt{4a_0^2-\pi^2\mu^2 g^2} \; .$$

The power of average wear (over one period) asymptotic is then

$$\overline{P}_u = \frac{E_u}{\pi/\omega} = \frac{mga_0}{\pi\omega}\sqrt{4-\pi^2\eta^2} = \frac{mga_0}{\omega}\sqrt{\frac{4}{\pi^2}-\eta^2} \; .$$

Following the Maple program allows the computation of the power of exact wear in a specified time interval, as well as the layout of the graph showing the convergence of the function velocity of the mass towards a periodic function limits, for any value of the physical parameters and of excitation such as the mode is of standard slipway-slipway ($\eta \le \eta^*$), and the exact value of the average power of wear over one period (the only useful one for what interests us) in the case of the stick-slip.

```
# This program calculates, on transitory part
# of the beginning of the signal, of the power of exact wear,
# until a time specifies at the beginning of program.
Digits: = 20:
pi: = evalf (pi):
T: = 1:                    # period of the motion of the Omega
support: = 2*pi/T:
tmin: = 4:
tmax: = 12:                # duration of the transient considers
ncycle: = floor (tmax/T) +2:  # iteration count of Ti computation [I] and tf
[I]
Nmax: = 100*ncycle:      # to replace the function sin by one line brisee
m: = 1:
G: = 10:
driven: = 0.1:
a0: = 1.5:
eta: = mu*g/a0:
Omega: = 2*pi/T:
etaetoile: = 2/sqrt (pi^2+4):
Ti [1]: = 1/omega*arcsin (eta):
dX: = T - > - a0/omega*cos (omega*t):
dxmoins [0]: = dX (T):
lignedx: = [Ti [1], dX (Ti [1])] :
Eusure: = 0:   # wear is null on the phase of dependancy [0, Ti [1]]
#
# To note that Ti [i+1] is necessarily in the interval [i*T-T/4, i*T+T/2]
# and    that tf [I]   is necessarily in the interval [i*T-3*T/4, i*T].
# These two intervals are recovered, but there is always tf [I] <ti [i+1].
#
yew eta<etaetoile then            # mode of slipway-slipway
  for I from 1 to ncycle C
    dxplus [I]: = mu*g* (T-Ti [I]) + subs (t=ti [I], dxmoins [i-1]):
    tf [I]: = fsolve (dX (T) =dxplus [I], t= (i*T-3*T/4). (i*T)) :
    lignedx: = lignedx, [tf [I], dX (tf [I])] :
    tinf: = max (Ti [I], tmin):
    tsup: = min (tf [I], tmax):
    yew tinf<tsup then
      Eusure: = Eusure + int (m*g* (dX (T) - dxplus [I]), t=tinf. .tsup):
    fi:
```

```
      dxmoins [I]: = - mu*g* (t-tf [I]) + subs (t=tf [I], dxplus [I]):
      Ti [i+1]: = fsolve (dX (T) =dxmoins [I], t= (i*T-T/4). (T/2+i*T)) :
      lignedx: = lignedx, [Ti [i+1], dX (Ti [i+1])] :
      tinf: = max (tf [I], tmin):
      tsup: = min (Ti [i+1], tmax):
      yew tinf<tsup then
        Eusure: = Eusure + int (m*g* (dxmoins [I] - dX (T)), t=tinf. .tsup):
      fi:
    od:
  # courbedX: = stud ([seq ([j*tmax/Nmax, dX (j*tmax/Nmax)], j=0. Nmax)]):
  # courbedx: = stud ([lignedx]):
  # with (studs):
  # display ([courbedX, courbedx]);
    theta: = arccos (pi*eta/2) /omega:
    dxinfini: = T - > mu*g* (T-theta) +dX (theta):
    Vginfini: = dxinfini - dX:
    Eumoyana: = - int (m*g*Vginfini (T), t=theta. (theta+pi/omega)) :
    Eumoyanaana: = m*g*a0/omega^2*sqrt (4-eta^2*pi^2):
    Pumoyana: = 2*Eumoyana/T:
    Pumoyanaana: = 2*Eumoyanaana/T:
    Pusure: = Eusure/(tmax-tmin);
  elif (eta>etaetoile and eta<1) then    # mode of stick-slip
    lignedx: = [Ti [1], dX (Ti [1])] :
    dxplus [1]: = mu*g* (T-Ti [1]) + subs (t=ti [1], dxmoins [0]):
    tf [1]: = fsolve (dX (T) =dxplus [1], t= (T-3*T/4). T):
    dxplus: = unapply (dxplus [1], T):
    Vg: = dxplus - dX:
    Have: = - int (m*g*Vg (T), t=ti [1]. .tf [1]):
    Pusuremoy: = 2*Eu/T;
  else                                   # mode of permanent dependancy
    Have: = 0;
  fi:
```

The solution *Aster* considered is the computation of the power of average wear during a transitional stage going from 4 to $11{,}99\,secondes$ (of $8\pi/\omega$ with $24\pi/\omega$). The energy of wear for this transitory length of time differs somewhat from the energy of average wear (asymptotic) over this period (as well in situation stick-slip as slipway-slipway). It is thus appropriate, to precisely compare it with the Aster *results*, to do an exact calculation of this energy in the time interval $[4\mathrm{s}, 11{,}99\,s]$.

For $a_0 = 15\mathrm{m}/s^2$, the power of average wear asymptotic is of $15{,}1146144886\,Watt$ whereas the power of average wear on the temporal interval $[4\mathrm{s}, 11{,}99\,s]$ is of $15{,}257521794\,Watt$. It is this last value which constitutes result reference.

Note:

As a computation of average power, the power of wear calculated on an interval is not obligatorily increasing with the period of the interval. If one adds to the interval a period over which there is dependancy, the power of average wear will be lower.

2.1 Results of reference

Value of acceleration $max.\,a0$ **(** ms^{-2} **)**	**Value of the average power of wear On the interval** $[4\mathrm{s}, 11{,}99\,s]$**, in Watt**
15 (slipway-slipway)	15,26709959
1,5 (stick-slip)	0,40906245
1,01 (stick-slip)	2,261641E-4
0,99 (stick)	0

2.2 Uncertainty on the quasi-analytical

solution Solution (presence of transcendent equations solved numerically with an arbitrary accuracy).

2.3 Bibliographical references

1 B. WESTERMO, F. UDWADIA: Periodic Response of has sliding oscillator system to harmonic excitation. Earthquake Engineeering and structural dynamics Flight 14.135-146 (1983)

2 Documentation of *the Code_Aster* [R7.04.10]

3 Modelization A

3.1 Characteristic of the modelization

element of type a DIS_T on a mesh POI1 is used to model the system.
The computation is done on modal base. One blocks displacements in Y and in Z, modal base thus contains only one mode.
One uses the functionality of dynamic computation on the basis of operator DYNA_VIBRA modal base , with key key CHOC to model to it not local linearity.
An obstacle of the type PLAN_Z (two parallel planes separated by a clearance) is used to simulate the slip surface. One chooses to take for generator of this plane Oy is NORM_OBST : $(0.,1.,0.)$. The origin of the obstacle is ORIG_OBST : $(0.,0.,1.)$, its clearance which gives the half-spacing between the planes is of 0.5.
One places oneself in the relative reference (loading mono-bearing) and one applies a loading in acceleration with CALC_CHAR_SEISME.
One time step uses one $3.10^{-5} s$ for temporal integration to limit the computing time. This time step is quite lower than $\min(2/\sqrt{K/M}, 2/\sqrt{K_N/M}) = 7.10^{-4} s$.
The tangential stiffness of friction is taken as large as possible to ensure the stability of the diagram, that is to say $K_T = 900000\, N/m$. The value $K_T = 1000000\, N/m$ leads to a numerical instability.
The normal stiffness K_N must be taken equal to $20\, N/m$ compensate for the weight of the mass exactly. (the value of clearance is of $0{,}50\, m$). Any other value leads to aberrant results.

3.2 Characteristics of the mesh

Many nodes: 1
Number of meshes and types: 1 POI1

4 Results of the modelization A

4.1 Values tested

Identification	Reference	*Aster* ADAPT ORDRE2	*Aster* DEVOGE	*Aster* EULER	*Aster R-K 54*	*Aster R-K 32%*	difference max
$a0 = 15$	15,2671	15,2661	15,2665	15,2668	15,2655	15,2661	0,0065%
$a0 = 1{,}5$	0,409062	0,409067	0,409067	0,409067	0,409071	0,409068	0,0078%
$a0 = 1{,}01$	2,26164E-4	2,2715E-4	2,26108E-4	2,26112E-4	2,26105E-04	2,31715E-04	2,45%
$a0 = 0{,}99$	0	0	0	0	0	0	0%

5 Summary of the results

the benchmark validates the computation of the power of wear with POST_DYNA_MODA_T after a transient computation on modal base, as well on a diagram with variable steps (ADAPT_ORDRE2, RUNGE_KUTTA54 and RUNGE_KUTTA32) as on diagrams with steps constant (Eulerian and Devogeleare). In particular the tangential microphone-velocities induced by the model of contact by penalization, during the phases of dependancy, are correctly cancelled.

The influence of added spring remains in on this side precise details obtained.

The tangential stiffness of the contact is the element limiting for a higher accuracy. The convergence of the results towards the reference solution was checked. The tangential stiffness was taken as large as possible to ensure the stability of the diagram with $dt=10^{-4} s$.

The tolerances in the tests-resu are taken just above found differences.