SSLL14 - Plane gantry articulated in foot

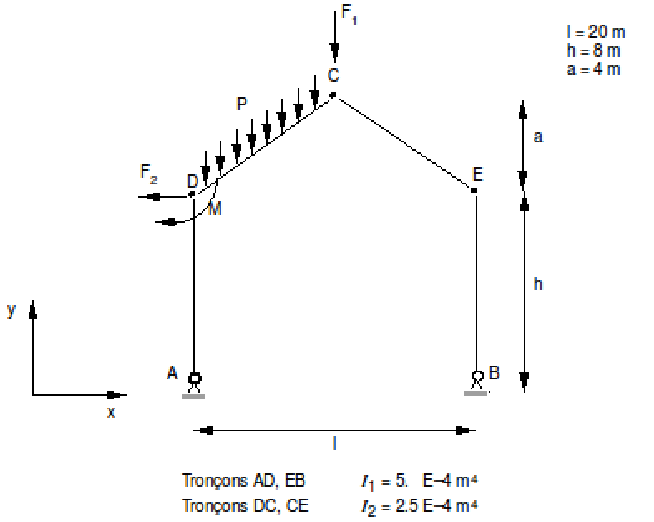
Summarized

This test relates to the study of a gantry made up of hurled beams, articulated in foot, in linear static analysis.

The gantry is modelled with elements linear `SEG2` and subjected to four loadings (distributed or specific).

1 Problem of reference

1.1 Geometry



Geometry of the gantry (m) :

- $l=20$
- $h=8$
- $a=4$

Quadratic moments of the beams (m^4) :

- Sections AD EB : $I_1=5.0\,E^{-4}$
- Sections DC CE : $I_1=2.5\,E^{-4}$

The gantry consists of symmetric beams of sections, so that $IY=IZ$.
One takes account only of the energy of bending, because the beams are very slender. This is why the other characteristics of section of beam do not intervene.

1.2 Material properties

isotropic linear elastic Material: $E=2.1\,E11\,Pa$

1.3 Boundary conditions and loadings

Feet of columns A and B pin-jointed.

Loadings

Forces nodal in C :	$F\,y=-2000\,N=F\,1$
Nodal force in D :	$F\,x=-10\,000\,N=F\,2$
Moment in D :	$Mx=-100\,000\,N.m=M$
Distributed force on the section DC :	$P\,z=-3000\,N/m$

2 Reference solution

2.1 Méthode de calcul used for the reference solution

the méthode de calcul and the solution were determined by F. Voldoire (EDF R & D/AMA) and are exposed in the appendix.

2.2 Results of reference

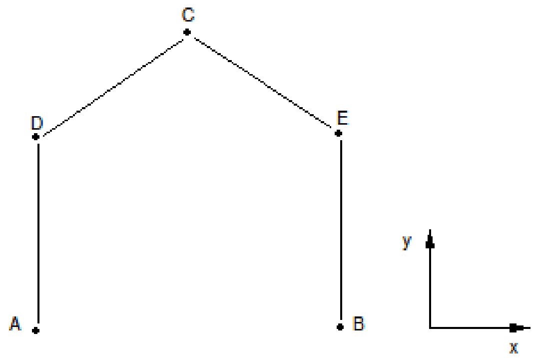
horizontal and Fx vertical Reactions Fy to the point A.
Bending moment Mz in C.
Displacements horizontal Dx and vertical Dy of the point C.

2.3 Uncertainty on the analytical

solution Solution.

3 Modelization A

3.1 Characteristic of the modelization



- Modelization POU_D_E
- 10 elements section by section, is 40 elements SEG2
- Displacement in the plane: $DZ=0$ on all the mesh
- Feet of columns A and B pin-jointed: $DX=DY=0$

3.2 Quantities tested and Values

3.2.1 results tested

Loading	Node	Value tested	Reference
p	C	$Dx(m)$	0.0110476
		$Dy(m)$	– 0.012422374
		$Mz(N.m)$	18672.994
	A	$Fx(N)$	5175.37
		$Fy(N)$	24233.24
F_1	C	$Dx(m)$	0.00000
		$Dy(m)$	– 0.01497330
		$Mz(N.m)$	41422.161
	A	$Fx(N)$	4881.487
		$Fy(N)$	10000.00
F_2	C	$Dx(m)$	– 0.03000956
		$Dy(m)$	– 0.00299466
		$Mz(N.m)$	8284.432
	A	$Fx(N)$	5976.297
		$Fy(N)$	4000.00
C	C	$Dx(m)$	0.0273532
		$Dy(m)$	– 0.001215646
		$Mz(N.m)$	4916.724
	A	$Fx(N)$	4576.394
		$Fy(N)$	5000.00

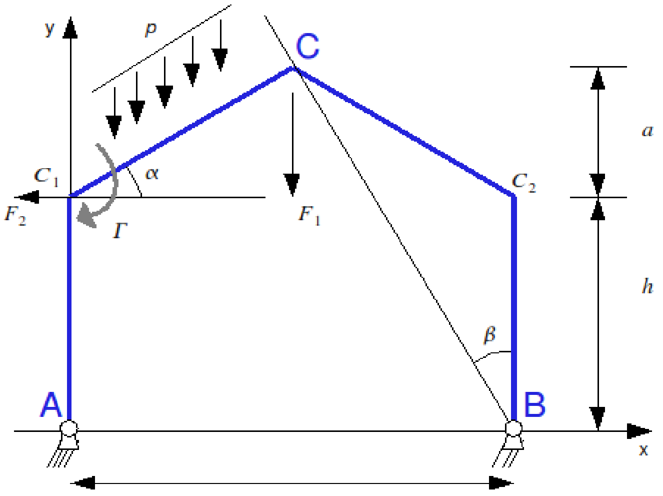
4 Summary of the results

the results got with modelization POU_D_E are in very good agreement with the analytical solution and thus validate the computation of truss of beams subjected to forces specific or distributed.

5 Appendix

5.1 Presentation

One considers the gantry opposite, subjected to various loads.



One considers the gantry opposite, subjected to various loads.

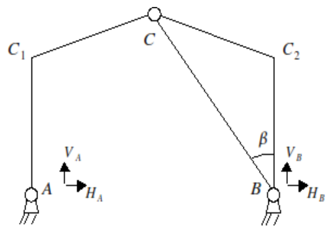
Hyperstaticity of degree 1.
Redundant unknown: X:
moment in C.

Top-load distributed p on C_1C.

Two forces F_1, F_2 and a couple in C_1.

Hyperstaticity of degree 1.
Redundant unknown: X
Loads applied:

- moment in C,
- top-load distributed p on C_1C_2,
- force $F1$, $F2$ applied in C_1,
- couple Γ applied in C_1



$$\tan(\alpha)=\frac{2a}{l}=0.4(\Rightarrow(\cos(\alpha))^{-1}=\sqrt{1.16}=1.077033)$$

$$\tan(\beta)=\frac{l}{2(a+h)}=\frac{1}{1.2}$$

$$b=\frac{l}{2\cos(\alpha)} \quad ; \quad \sin(\alpha)=\frac{a}{b}$$

5.2 statically determinate Requests under real load distributed p to C_1C

5.2.1 statically determinate Reactions of bearings

$$H_A+H_B=0 \qquad V_A+V_B=\frac{pl}{2\cos(\alpha)} \qquad lV_B=\frac{pl^2}{8\cos(\alpha)}$$

the partieest CB pin-jointed and only charged at its ends

$$\begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC = 0 \Leftrightarrow H_B = -V_B \tan(\beta)$$

From where statically determinate reactions

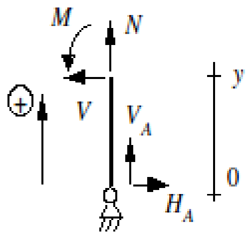
$$H_A = \frac{pl}{8\cos(\alpha)}\tan(\beta) \quad ; \; V_A = \frac{3\,pl}{8\cos(\alpha)} \; ; \; H_B = \frac{-pl}{8\cos(\alpha)}\tan(\beta) \; ; \; V_B = \frac{pl}{8\cos(\alpha)}$$

Note:

$$\frac{l\tan(\beta)}{8\cos(\alpha)} = \frac{bl}{8(a+h)}$$

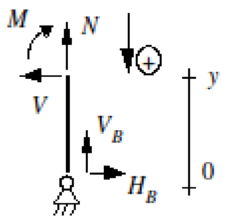
5.2.2 Requests

Beam AC_1



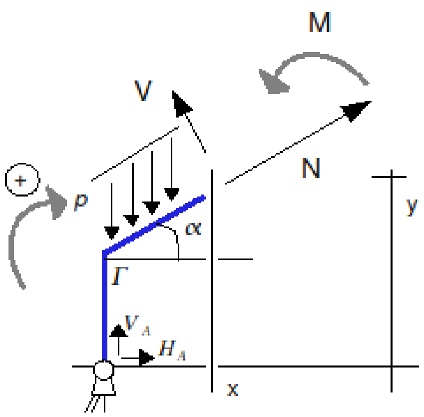
$$N_{iso} = \frac{-3\,pl}{8\cos(\alpha)}$$
$$V_{iso} = \frac{pl}{8\cos(\alpha)}\tan(\beta)$$
$$M_{iso} = \frac{-pl}{8\cos(\alpha)} y.tan(\beta)$$

Beam $C_2 B$



$$N_{iso} = \frac{-pl}{8\cos(\alpha)}$$
$$V_{iso} = \frac{-pl}{8\cos(\alpha)}\tan(\beta)$$
$$M_{iso} = \frac{-pl}{8\cos(\alpha)} y.tan(\beta)$$

Beam $C_1 C$

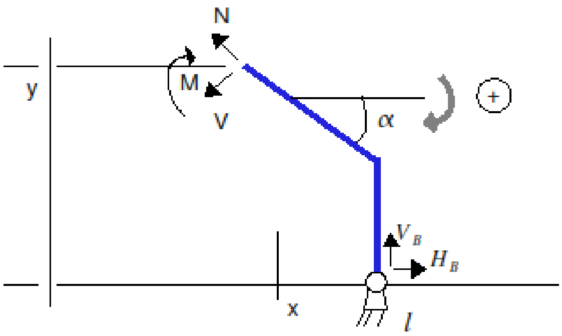


$$N_{iso} = -H_A\cos(\alpha) - V_A\sin(\alpha) + \frac{px}{\cos(\alpha)}\sin(\alpha)$$
$$= -\frac{pl}{8}\left(\tan(\beta) + 3\tan(\alpha) - 8\tan(\alpha)\frac{x}{l}\right)$$
$$V_{iso} = H_A\sin(\alpha) - V_A\cos(\alpha) + \frac{px}{\cos(\alpha)}\cos(\alpha)$$
$$= \frac{pl}{8}\left(\tan(\beta)\tan(\alpha) - 3 + 8\frac{x}{l}\right)$$
$$M_{iso} = -\frac{px^2}{2\cos(\alpha)} + V_A x - H_A y$$
$$= \frac{p}{\cos(\alpha)}\left(\frac{-x^2}{2} + \frac{3\,lx}{8} - \frac{ly\tan(\beta)}{8}\right)$$

with $M_{iso} = 0$ in C

$$M_{iso}=\frac{-pl}{8(a+h)}\left(2s^2\left(\frac{a+h}{b}\right)-s(2a+3h)+bh\right) \text{ with } s=\frac{x}{\cos(\alpha)}\in[0,b]$$

Beam CC_2

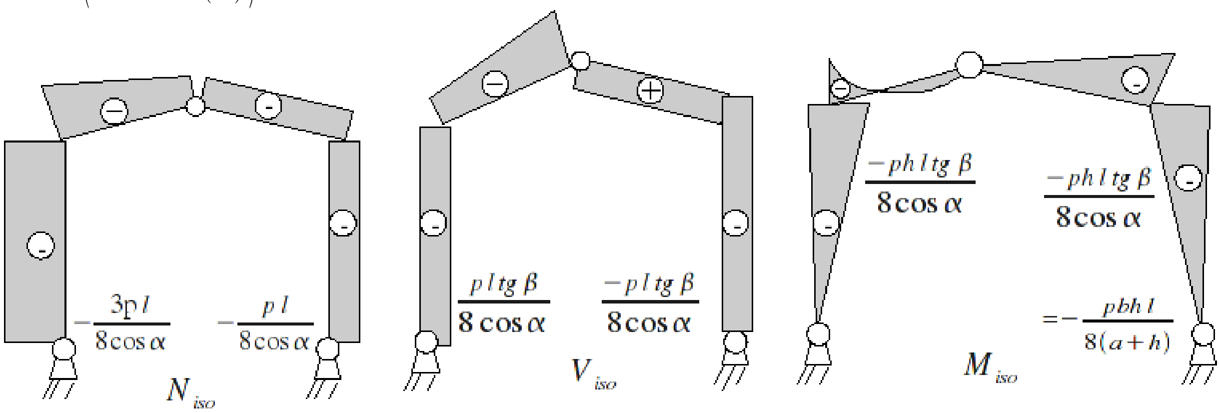


$$N_{iso}=H_B\cos(\alpha)-V_B\sin(\alpha)$$
$$=-\frac{pl}{8}(\tan(\beta)+\tan(\alpha))$$
$$V_{iso}=H_B\sin(\alpha)+V_B\cos(\alpha)$$
$$=-\frac{pl}{8}(\tan(\beta)\tan(\alpha)-1)$$
$$M_{iso}=H_B y-V_B(l-x)$$
$$=-\frac{pl}{8\cos(\alpha)}(y.tan(\beta)-(l-x))$$

with $M_{iso}=0$ in C

5.2.3 Diagrams

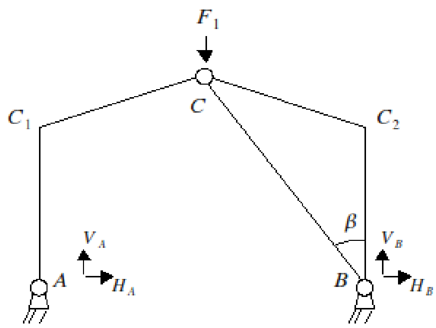
$$\left(b=\frac{l}{2\cos(\alpha)}\right)$$



5.3 Requests under concentrated force F_1 (downwards)

5.3.1 Reactions of bearing

$H_A+H_B=0\,;$
$V_A+V_B=F_1\,;$
$\begin{pmatrix}H_A\\V_A\end{pmatrix}\wedge AC=0=\begin{pmatrix}H_B\\V_C\end{pmatrix}\wedge BC\,;$



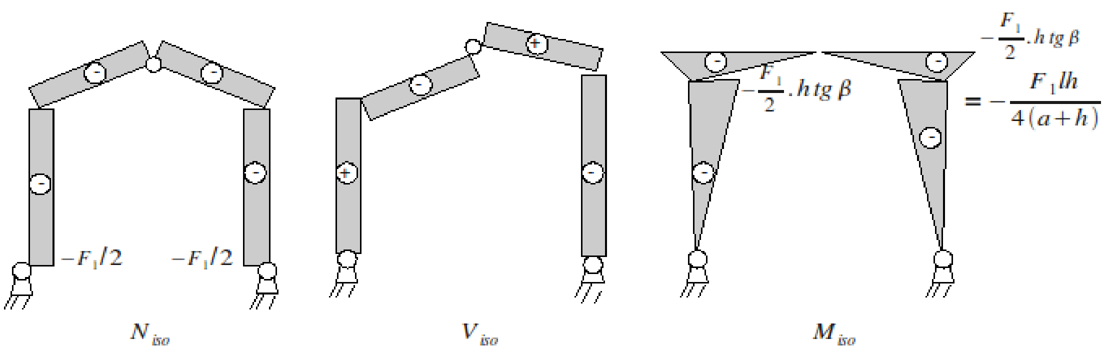
From where:

$$H_A = \frac{1}{2} F_1 \tan(\beta)\ ;\ V_A = \frac{1}{2} F_1\ ;\ H_B = \frac{-1}{2} F_1 \tan(\beta)\ ;\ V_B = \frac{1}{2} F_1$$

5.3.2 Requests

Beam: AC_1	$N_{iso} = \frac{-1}{2} F_1$ $V_{iso} = \frac{1}{2} F_1 \tan(\beta)$ $M_{iso} = \frac{-1}{2} F_1 y \tan(\beta)$
Beam: $C_2 B$	$N_{iso} = \frac{-1}{2} F_1$ $V_{iso} = \frac{-1}{2} F_1 \tan(\beta)$ $M_{iso} = \frac{-1}{2} F_1 y \tan(\beta)$
Beam: $C_1 C$	$N_{iso} = \frac{-1}{2} F_1 (\tan(\beta)\cos(\alpha) + \sin(\alpha))$ $V_{iso} = \frac{1}{2} F_1 (\tan(\beta)\sin(\alpha) - \cos(\alpha))$ $M_{iso} = \frac{-1}{2} F_1 (y \tan(\beta) - x)$
Beam: CC_2	$N_{iso} = \frac{-1}{2} F_1 (\tan(\beta)\cos(\alpha) + \sin(\alpha))$ $V_{iso} = \frac{1}{2} F_1 (\tan(\beta)\sin(\alpha) - \cos(\alpha))$ $M_{iso} = \frac{-1}{2} F_1 (y \tan(\beta) - (l - x))$

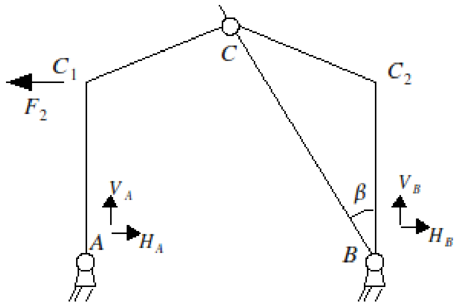
5.3.3 Diagrams (F_1 downwards)



5.4 Requests under the concentrated force F_2 (towards the left)

5.4.1 Reactions of bearing

$$H_A + H_B = F_2;$$
$$V_A + V_B = 0;$$
$$lV_B + hF_2 = 0;$$
$$\begin{pmatrix} H_B \\ V_B \end{pmatrix} ans\ BC = 0$$



From where:

$$H_A = F_1 (1 - \frac{h}{l} \tan(\beta))\ ;\ V_A = F_2 \frac{h}{l}\ ;\ H_B = F_1 \frac{h}{l} \tan(\beta)\ ;\ V_B = -F2 \frac{h}{l}\ ;$$

Note:

$$\frac{h}{l} \tan(\beta) = \frac{h}{2(a+h)} \qquad (1 - \frac{h}{l} \tan(\beta)) = \frac{2\mathrm{a} + h}{2(a+h)}$$
$$\tan(\beta)\sin(\alpha) - \cos(\alpha) = \frac{-hl}{2\mathrm{b}(a+h)},\ \tan(\beta)\cos(\alpha) - \sin(\alpha) = \frac{l^2 - 4(a^2 + ah)}{4\mathrm{b}(a+h)}$$

5.4.2 Requests

Beam: AC_1

$$N_{iso} = -F_2 \frac{h}{l}$$
$$V_{iso} = F_2 (1 - \frac{h}{l} \tan(\beta))$$
$$M_{iso} = -F_2 y (1 - \frac{h}{l} \tan(\beta))$$

Beam: $C_2 B$

$$N_{iso} = F_2 \frac{h}{l}$$
$$V_{iso} = F_2 \frac{h}{l} \tan(\beta)$$
$$M_{iso} = -F_2 y \frac{h}{l} y \tan(\beta)$$

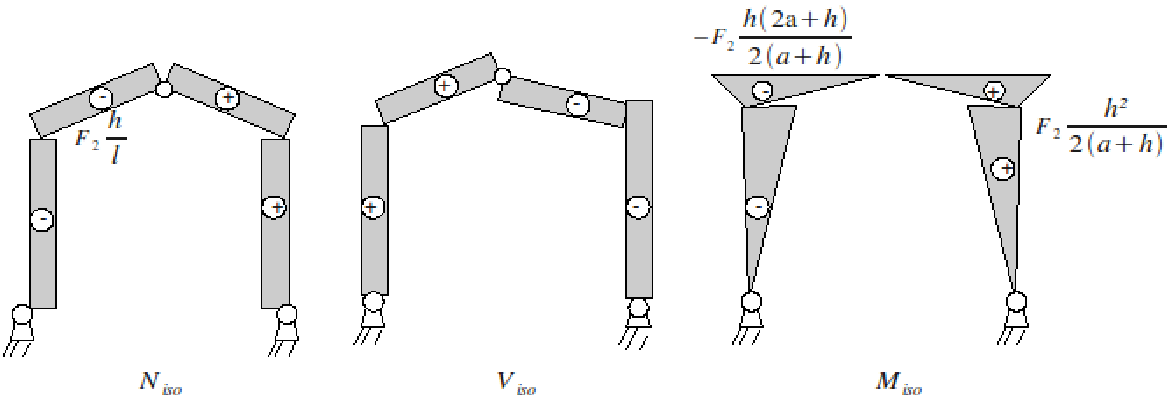
Beam: $C_1 C$

$$N_{iso} = F_2 ((1 - \frac{h}{l} \tan(\beta)) \cos(\alpha) - \frac{h}{l} \cos(\alpha))$$
$$V_{iso} = F_2 ((1 - \frac{h}{l} \tan(\beta)) - \frac{h}{l} \cos(\alpha))$$
$$M_{iso} = F_2 (\frac{h}{l} x - (1 - \frac{h}{l} \tan(\beta)) y)$$

Beam: CC_2

$$N_{iso} = F_2 \frac{h}{l} (\tan(\beta) \cos(\alpha) + \sin(\alpha))$$
$$V_{iso} = F_2 \frac{h}{l} (\tan(\beta) \sin(\alpha) - \cos(\alpha))$$
$$M_{iso} = F_2 \frac{h}{l} (y \tan(\beta) - (l - x))$$

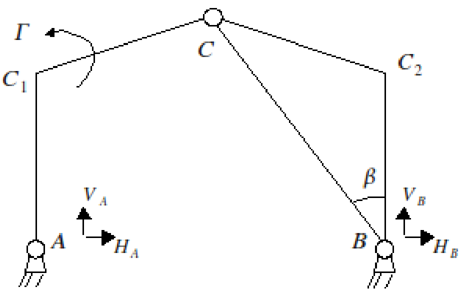
5.4.3 Diagrams



5.5 Requests under the couple concentrated Γ (positive)

5.5.1 Reactions of bearing

$$H_A + H_B = 0\,;$$
$$V_A + V_B = 0\,;$$
$$lV_B + \Gamma = 0\,;$$
$$\begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC = 0\,;$$



From where: $H_A = -\Gamma \tan\frac{\beta}{l}$ $V_A = \frac{\Gamma}{l}$ $H_B = \Gamma \tan\frac{\beta}{l}$, $V_A = \frac{-\Gamma}{l}$

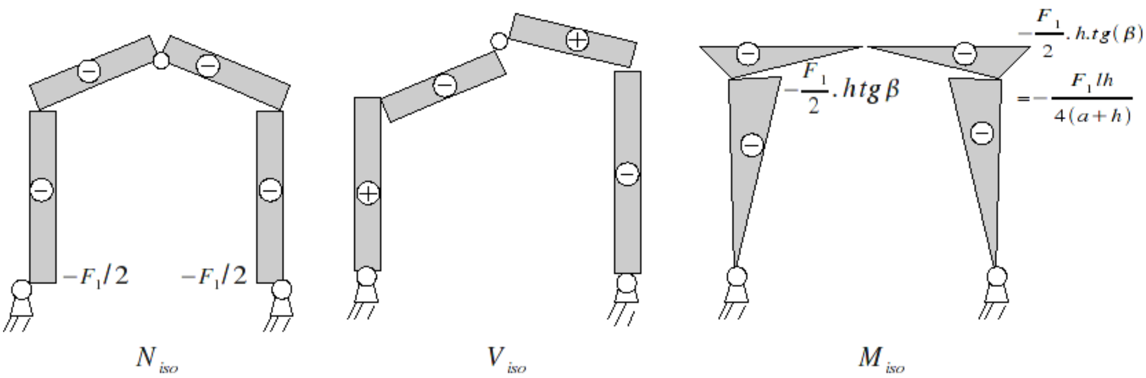
Note:

$$\frac{\tan(\beta)}{l} = \frac{1}{2(a+h)}$$

5.5.2 Requests

Beam: AC_1	$N_{iso}=\frac{-\Gamma}{l}$ $V_{iso}=\frac{-\Gamma\tan(\beta)}{l}$ $M_{iso}=\frac{\Gamma y\tan(\beta)}{l}$
Beam: C_2B	$N_{iso}=\frac{-\Gamma}{l}$ $V_{iso}=\frac{\Gamma\tan(\beta)}{l}$ $M_{iso}=\frac{\Gamma y\tan(\beta)}{l}$
Beam: C_1C	$N_{iso}=\frac{\Gamma}{l}(\tan(\beta)\cos(\alpha)-\sin(\alpha))$ $N_{iso}=\frac{-\Gamma}{l}(\tan(\beta)\sin(\alpha)+\cos(\alpha))$ $N_{iso}=\frac{\Gamma}{l}(x+y\tan(\beta)-l)$
Beam: CC_2	$N_{iso}=\frac{\Gamma}{l}(\tan(\beta)\cos(\alpha)+\sin(\alpha))$ $N_{iso}=\frac{\Gamma}{l}(\tan(\beta)\sin(\alpha)-\cos(\alpha))$ $N_{iso}=\frac{\Gamma}{l}(y\tan(\beta)-(l-x))$

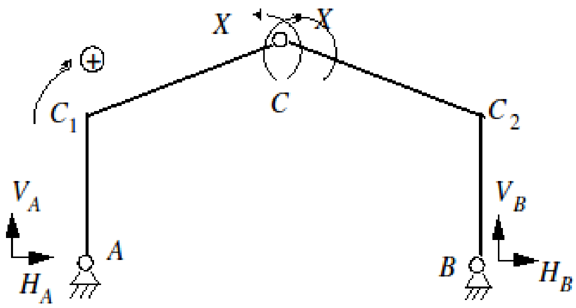
5.5.3 Diagrams (Γ positive)



5.6 Requests under momenthyperstatic X

5.6.1 the Reactions of bearing

$H_A+H_B=0$;
$V_A+V_B=0$;
$lV_B=0$;
$H_B(a+h)-X=0$;

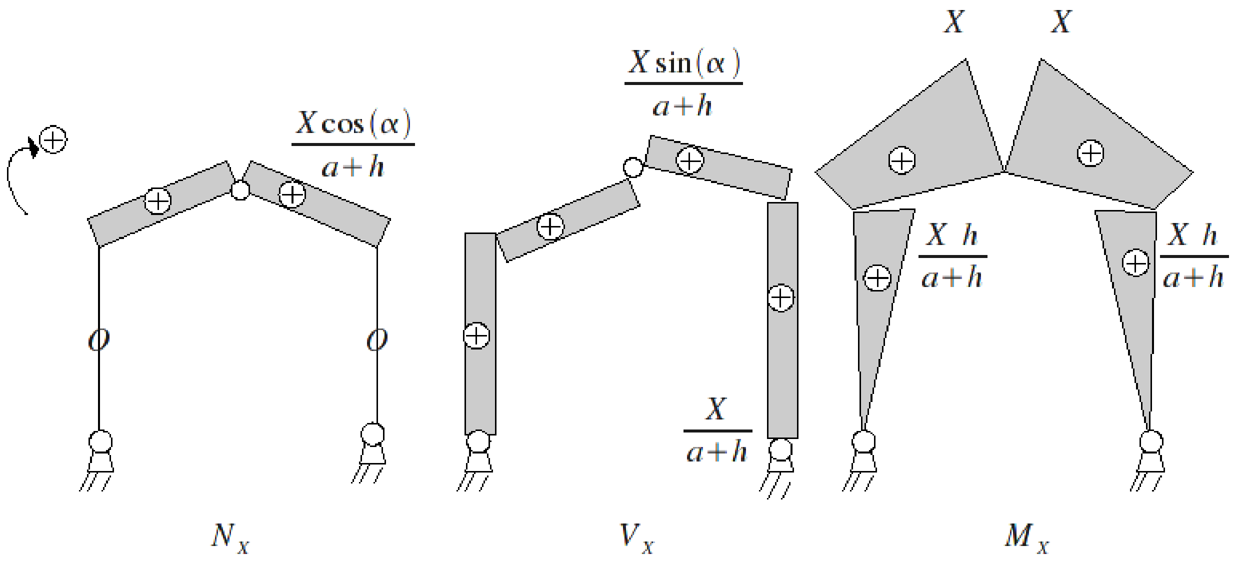


From where reactions: $H_a=\frac{-X}{a+h}$ $V_A=0$ $H_B=\frac{X}{a+h}$, $V_B=0$

5.6.2 Requests

Beam: AC_1	$N_X=0$ $V_X=\frac{-X}{a+h}$ $M_X=\frac{X}{a+h}y$
Beam: C_2B	$N_X=0$ $V_X=\frac{X}{a+h}$ $M_X=\frac{X}{a+h}y$
Beam: C_1C	$N_X=\frac{X}{a+h}\cos(\alpha)$ $V_X=\frac{X}{a+h}\sin(\alpha)$ $M_X=\frac{X}{a+h}y=\frac{X}{a+h}(h+x\tan(\alpha))$
Beam: CC_2	$N_X=\frac{X}{a+h}\cos(\alpha)$ $V_X=\frac{X}{a+h}\sin(\alpha)$ $M_X=\frac{X}{a+h}y$

5.6.3 Diagrams

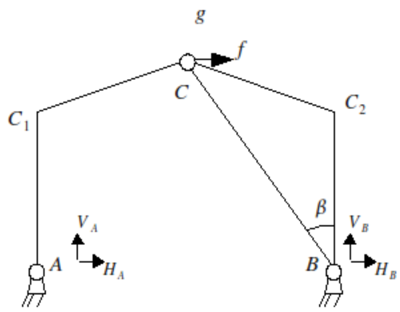


5.7 Requests under specific dummy loads in C

In order to calculating displacement in C , using the Principle of the virtual wors (*cf.* the paragraph $[\S 8]$), it is necessary to establish the diagrams of requests under the action of two “fictitious” forces f etappliquées g in C .

5.7.1 Reactions of bearing

$$H_A+H_B=-f\ ;$$
$$V_A+V_B=-g\ ;$$
$$\begin{pmatrix} H_A \\ V_A \end{pmatrix} \wedge AC=0=\begin{pmatrix} H_B \\ V_B \end{pmatrix} \wedge BC$$



From where:

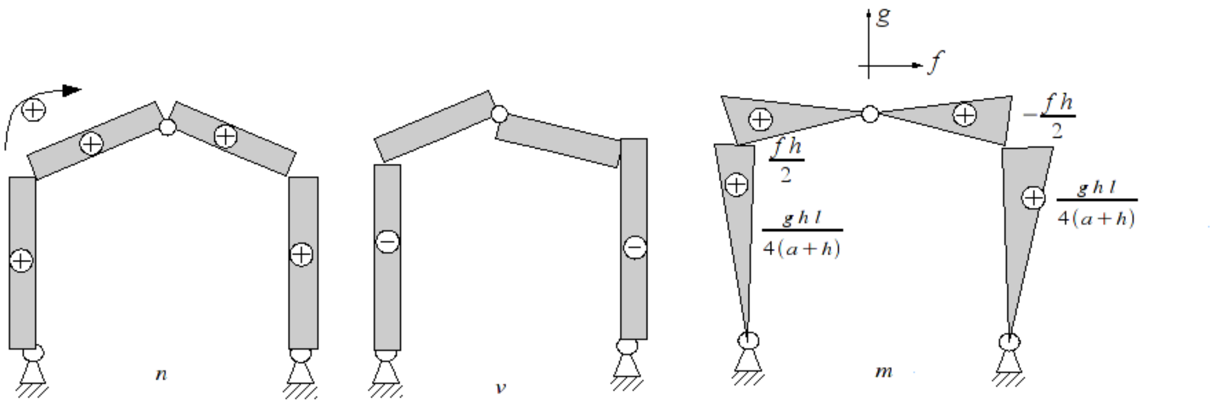
$$H_A=\frac{-1}{2}(f+g\tan(\beta)) \quad V_A=\frac{-1}{2}(g+f\cot(\beta))$$
$$H_B=\frac{-1}{2}(f-g\tan(\beta))\ ,\ V_B=\frac{-1}{2}(g-f\cot(\beta))$$

5.7.2 Requests

Beam: AC_1	$n=\frac{1}{2}(g+f\cot(\beta))$ $v=\frac{-1}{2}(f+g\tan(\beta))$ $m=\frac{1}{2}(f+g\tan(\beta))$
Beam: C_2B	$n=\frac{1}{2}(g-f\cot(\beta))$ $v=\frac{-1}{2}(f-g\tan(\beta))$ $m=\frac{-1}{2}(f-g\tan(\beta))\,y$
Beam: C_1C	$n=\frac{1}{2}(f+g\tan(\beta))\cos(\alpha)+\frac{1}{2}(g+f\cot(\beta))\sin(\alpha)$ $v=\frac{-1}{2}(f+g\tan(\beta))\sin(\alpha)+\frac{1}{2}(g+f\cot(\beta))\cos(\alpha)$ $m=\frac{1}{2}(f+g\tan(\beta))\,y-\frac{1}{2}(g+f\cot(\beta))\,x$
Beam: CC_2	$n=\frac{-1}{2}(f-g\tan(\beta))\cos(\alpha)+\frac{1}{2}(g-f\cot(\beta))\sin(\alpha)$ $v=\frac{-1}{2}(f-g\tan(\beta))\sin(\alpha)-\frac{1}{2}(g-f\cot(\beta))\cos(\alpha)$ $m=\frac{-1}{2}(f-g\tan(\beta))\,y-\frac{1}{2}(g-f\cot(\beta))(l-x)$

5.7.3 Diagrams

Here diagrams of requests under the action of the two "fictitious" forces f and g. One considers here: $f\geq 0,\ g\geq f\cot(\beta)$.



5.8 Determination of momenthyperstatic X

One is placed in elasticity; one considers only the energy of bending, the beams being slender. The natural state is supposed to be virgin (not prestressings nor of displacement of bearing).
The complementary potential is then:

$$F*(X)=\int_{poteaux}\frac{(M_{iso}+M_1X)^2}{EI_1}+\int_{charpentes}\frac{(M_{iso}+M_1X)^2}{EI_2}$$

It is steady with the equilibrium, from where:

$$\delta . X = \left[\int_{pot} \frac{M_1^2}{EI_1} + \int_{charp} \frac{M_1^2}{EI_2} \right] . X = -\int_{pot} \frac{M_1 M_{iso}}{EI_1} - \int_{charp} \frac{M_1 M_{iso}}{Ei_2} = S$$

The coefficient of flexibility δ is the sum of:

$$\int_{pot} \frac{M_1^2}{EI_1} = \frac{2\text{h}}{3\text{EI}_1} \left(\frac{h}{a+h} \right)^2$$

$$\int_{charp} \frac{M_1^2}{EI_2} = \frac{2\text{b}}{EI_2} [(\frac{h}{a+h})^2 + \frac{1}{3} (\frac{a}{a+h})^2 + \frac{ah}{(a+h)^2}]$$

that is to say:

$$E . \delta = \frac{2}{(a+h)^2} [\frac{h^3}{3\text{I}_1} + \frac{b(3\text{h}^2 + a^2 + 3\text{ah})}{3\text{I}_2}]$$

Numerical application:

In the example considered:

$$I_1 = 2\text{I}_2 = 5.0\,E-4\,m^4 \quad h = 2\text{a} = 8\,m \quad l = 20\ m\ , \ b = \frac{l}{2} \sqrt{1.16}$$

From where: $\gamma = \frac{2}{E(a+h)_1^{2\text{I}}} \underbrace{\frac{h^2}{3} (h + \frac{19\text{b}}{2})}_{2353.45347\,m^3}$

One to compute: studies one after the other the various loadings the second members S.

5.8.1 Charge répartiesur p $C_1 C$

the second membredû S àest f:

$$-\int_{pot} \frac{M_1 M_{iso}}{EI_1} = \frac{3\text{h}}{3\text{EI}_1} (\frac{h}{a+h}) (\frac{pblh}{8(a+h)}) = \frac{2}{E(a+h)^2 I_1} \frac{ph^3 bl}{24}$$

$$-\int_{CC_2} \frac{M_1 M_{iso}}{EI_2} = \frac{pb^2 hl}{8(a+h) EI_2} (\frac{1}{2} \frac{h}{a+h}) + (\frac{a}{6} \frac{a}{a+h}) = \frac{2}{E(a+h)^2 I_2} \frac{phb^{2\text{l}}(3\text{h}+a)}{48}$$

$$-\int_{C_1 C} \frac{M_1 M_{iso}}{EI_2} = \frac{1}{EI_2} \frac{pl}{8(a+h)^2} \int_0^b \left[2\text{s}^2 \frac{a+h}{b} - s(2\text{a}+3\text{h}) + bh \right] \left[h + s\frac{a}{b} \right] ds$$

$$= \frac{1}{E(a+h)^2 I_2} \frac{plb^2}{48} (h^2 + 2\text{ah} + a^2)$$

From where:

$$S = \frac{2}{E(a+h)^2} \frac{plb}{96} \left[\frac{4\text{h}^3}{I_1} + \frac{hb(3\text{h}+a)}{I_2} + \frac{b(h^2 - 2\text{ah} - a^2)}{I_2} \right]$$

Numerical application:

$I_1 = 2\,I_2$; $h = 2\text{a}$; $p = 3000\,N.m^1$ (downwards)

$$S = \frac{2}{E(a+h)^2 I_1} \underbrace{\frac{plbh^2}{96} \left[4\text{h} + \frac{13}{2} b \right]}_{43946\,021.89\,N.m^4}$$

From where:

- moment in C :
  $X=18672994\,N.m$
- reaction in A :

$$H_A=p\frac{bl}{8(a+h)}-\frac{X}{a+h}=\frac{\frac{pbl}{8-X}}{a+h}\quad H_A=5175.37\,N$$

$$V_A=\frac{3\mathrm{pb}}{4}-0\ ,\ V_A=24233.24\,N$$

5.8.2 ponctuelleen F_1 C

the second member Charges is obtained using:

$$-\int_{pot}\frac{M_1M_{iso}}{EI_1}=\frac{2\mathrm{h}}{3\mathrm{EI}_1}\left(\frac{h}{a+h}\right)\left(\frac{F_1lh}{4(a+h)}\right)=\frac{2}{E(a+h)^2I_1}\frac{F_1lh^3}{12}$$

$$-\int_{charp}\frac{M_1M_{iso}}{EI_2}=\frac{2\mathrm{b}}{EI_2}\frac{F_1lh}{4(a+h)}\left(\frac{1}{2}\left(\frac{h}{a+h}\right)+\frac{1}{6}\left(\frac{a}{a+h}\right)\right)=\frac{2}{E(a+h)^2I_2}\frac{F_1blh(3\mathrm{h}+a)}{24}$$

From where:

$$S=\frac{2}{E(a+h)^2}\frac{F_1lh}{24}\left[\frac{2\mathrm{h}^2}{I_1}+\frac{b(3\mathrm{h}+a)}{I_2}\right]$$

Numerical application:

$I_1=2I_2$; $h=2\mathrm{a}$; $F_1=20000\,N$ (downwards)

$$S=\frac{2}{E(a+h)^2I_1}\underbrace{\frac{F_1lh^2}{24}[2\mathrm{h}+7b]}_{97485127.76\,N.m^4}$$

From where:

- moment in C :
  $X=41422.161\,N.m$
- reaction in A :

$$H_A=\frac{1}{4}F_1\frac{l}{a+h}-\frac{X}{a+h}=\frac{\frac{F_1l}{4-X}}{a+h}\quad H_A=4881.4866\,N$$

$$V_A=\frac{1}{2}F_1-0\ ,\ V_A=10000.0\,N$$

5.8.3 ponctuelleen F_2 C_1

the second member Charges is obtained using:

$$-\int_{AC_1}\frac{M_1M_{iso}}{EI_1}=\frac{h}{3\mathrm{EI}_1}\left(\frac{h}{a+h}\right)\frac{F_2h(2\mathrm{a}+h)}{2(a+h)}=\frac{2}{E(a+h)^2I_1}\frac{F_2h^3(2\mathrm{a}+h)}{12}$$

$$-\int_{C_2B}\frac{M_1M_{iso}}{EI_1}=\frac{h}{3\mathrm{EI}_1}\left(\frac{h}{a+h}\right)\frac{(-F_2h^2)}{2(a+h)}=\frac{2}{E(a+h)_1^{2\mathrm{I}}}\frac{F_2h^3(2\mathrm{a}+h)}{12}$$

$$-\int_{C_1C}\frac{M_1M_{iso}}{EI_2}=\frac{b}{EI_2}\frac{F_2h(2\mathrm{a}+h)}{2(a+h)}\left[\frac{1}{2}\left(\frac{h}{a+h}\right)+\frac{1}{6}\left(\frac{a}{a+h}\right)\right]=\frac{2}{E(a+h)^2I_2}\frac{F_2bh(3\mathrm{h}^2+7\mathrm{ah}+2\mathrm{a}^2)}{24}$$

$$-\int_{CC_2}\frac{M_1 M_{iso}}{EI_2}=\frac{b}{EI_2}\frac{-F_2 h^2}{2(a+h)}\left[\frac{1}{2}(\frac{h}{a+h})+\frac{1}{6}(\frac{a}{a+h})\right]=\frac{2}{E(a+h)^2 I_2}\frac{-F_2 bh^2(3\mathrm{h}+a)}{24}$$

$$S=\frac{2}{E(a+h)^2}\frac{F_2 ha}{12}\left[\frac{2\mathrm{h}^2}{I_1}+\frac{b(3\mathrm{h}+a)}{I_2}\right]$$

Numerical application:

$I_1=2I_2$; $h=2\mathrm{a}$; $F_2=10\,000\,N$ (towards the left)

$$S=\frac{2}{E(a+h)^2 I_1}\underbrace{\frac{F_2 h^2 a}{12}[2\mathrm{h}+7b]}_{19\,497\,025.55\,N.m^4}$$

From where:

- moment in C :

$X=8\,284.4321\,N.m$

- reaction in A :

$$H_A=F_2\frac{2\mathrm{a}+h}{2(a+h)}-\frac{X}{a+h}=\frac{F_2(a+\frac{h}{2})-X}{a+h}\quad H_A=5976.297\,N$$

$$V_A=\frac{F_2 h}{l}\ ,\ V_A=4\,000.0\,N$$

5.8.4 ponctuelen Couples Γ C_1

the second member is obtained using:

$$-\int_{pot}\frac{M_1 M_{iso}}{EI_1}=\frac{-2\mathrm{h}}{3\mathrm{EI}_1}(\frac{h}{a+h})\frac{\Gamma h}{2(a+h)}=\frac{2}{E(a+h)^2 I_1}\frac{-\Gamma h^3}{6}$$

$$-\int_{C_1C}\frac{b}{EI_2}=\frac{\Gamma(h+2\mathrm{a})}{2(a+h)}\frac{b}{EI_2}\left[\frac{1}{2}(\frac{h}{a+h}+\frac{1}{6}(\frac{a}{a+h}))\right]=\frac{2}{E(a+h)^2 I_2}\frac{\Gamma(h+2\mathrm{a})(3\mathrm{h}+a)b}{24}$$

$$-\int_{CC_2}\frac{M_1 M_{iso}}{EI_2}=\frac{b}{EI_2}\frac{-\Gamma h}{2(a+h)}\left[\frac{1}{2}(\frac{h}{a+h})+\frac{1}{6}(\frac{a}{a+h})\right]=\frac{2}{E(a+h)^2 I_2}\frac{-\Gamma hb(3\mathrm{h}+a)}{24}$$

From where:

$$S=\frac{2}{E(a+h)^2}\frac{-\Gamma}{12}\left[\frac{2\mathrm{h}^3}{I_1}+\frac{ab(3\mathrm{h}+a)}{I_2}\right]$$

Numerical application:

$I_1=2I_2$; $h=2\mathrm{a}$; $\Gamma=-100\,000\,N.m$ (meaning hands clock)

$$S=\frac{2}{E(a+h)^2 I_1}\underbrace{\frac{-\Gamma}{6}[h^3-ab(3\mathrm{h}+a)]}_{11\,571\,281.93\,N.m^4}$$

From where:

- moment in C :

$X=4\,916.7243\,N.m$

- reaction in A :

$$H_A = \frac{-\Gamma}{2(a+h)} - \frac{X}{a+h} = \frac{\frac{-\Gamma}{2} - X}{a+h} \quad H_A = 4576.394\,N$$

$$V_A = \frac{\Gamma}{l} \ , \ V_A = 5\,000.0\,N$$

5.8.5 Summary

CAS	Moment in C $(N.m)$	Reactions in $A(N)$	
		H_A	V_A
p on C_1C	18672.994	5175.37	24233.240
F_1 in C	41422.161	4881.487	10000.000
F_2 in C_1	8284.432	5976.297	4000.000
Γ in C_1	4916.724	4576.394	5000.000
TOTAL	73296.311	22033.31	43233.24

Remark

Recall: in the column AC_1 : normal force = $-V_A$, shears = H_A .

5.9 Computation of displacement in C

One considers also only the elastic strain energy of bending (slender beams). By applying the Principle of the Virtual wors on the structure subjected to the fictitious forces of the paragraph $[\S 6]$, working in sought displacements, one calculates nombresetdépendant them w d linearly deet f g :

$$f\,u_c + g\,v_c = \int_{pot} \frac{m(M_{iso} + XM_1)}{EI_1} + \int_{charp} \frac{m(M_{iso} + XM_1)}{EI_2} = w + Xd \,,\, \forall (f\,,g)$$

5.9.1 Charge répartiesur p C_1C

$$\int_{pot} \frac{m\,M_{iso}}{EI_1} = \frac{2\mathrm{h}}{3\mathrm{EI}_1} \frac{ghl}{4(a+h)} \frac{-pbhl}{8(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{-gpbh^3 l^2}{96}$$

$$\int_{C_1C} \frac{m\,M_{iso}}{EI_2} = \frac{2}{E(a+h)^2 I_2} \frac{-plhb^2}{384} (2\mathrm{f}\,(a+h) + gl)(h-a)$$

$$\int_{CC_2} \frac{M_{iso}}{EI_2} = \frac{b}{3\mathrm{EI}_2} \frac{pbhl}{8(a+h)} \left(\frac{fh}{2} - \frac{glh}{4(a+h)} \right) = \frac{2}{E(a+h)^2 I_2} \frac{-pb^2 lh^2 (gl - 2\mathrm{f}\,(a+h))}{192}$$

From where:

$$w = \frac{2}{E(a+h)^2} \frac{-pbhl}{384} \left(\frac{4\mathrm{glh}^2}{I_1} + \frac{glb(3\mathrm{h}+a) - efb(a+h)^2}{I_2} \right)$$

Numerical application:

$I_1 = 2\mathrm{I}_2$; $h = 2\mathrm{a}$; $p = 3000\,N.m^{-1}$ (downwards)

$$w = \frac{2}{E(a+h)^2 I_1} \left(gl\left(2\mathrm{h} + \frac{5}{2}b\right) - \frac{9}{2} fbh\right) \underbrace{\frac{-pbh^2 l}{192}}_{-215\,406.5922\,N.m^3}$$

5.9.2 concentrated Loading F_1 in C

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{2\mathrm{h}}{3\mathrm{EI}_1} \frac{ghl}{4(a+h)} \frac{-F_1 hl}{4(a+h)} = \frac{2}{E(a+h)^2 I_1} \frac{-F_1 gh^3 l^2}{48}$$

$$\int_{charp} \frac{m M_{iso}}{EI_2} = \frac{2\mathrm{b}}{3\mathrm{EI}_2} \frac{ghl}{4(a+h)} \frac{-F_1 hl}{4(a+h)} = \frac{2}{E(a+h)^2 I_2} \frac{-F_1 gbh^2 l^2}{48}$$

From where (one notes quene w does not depend depour f this loading):

$$w = \frac{2}{E(a+h)^2} \frac{-F_1 gh^2 l^2}{48} \left(\frac{h}{I_1} + \frac{b}{I_2} \right)$$

Numerical application:

$I_1 = 2\mathrm{I}_2 \quad h = 2\mathrm{a}$, $F_1 = 20\,000\,N$ (downwards)

$$w = \frac{2\mathrm{g}}{E(a+h)^2 I_1} \underbrace{\frac{-F_1 h^2 l^2}{48} (h+2\mathrm{b})}_{-3155100365.0\,N.m^5}$$

5.9.3 concentrated Loading F_2 in C_1

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{h}{3\mathrm{EI}_1} \frac{F_2 h}{2(a+h)} \left[-(2\mathrm{a}+h)\left(\frac{fh}{2} + \frac{ghl}{4(a+h)}\right) + h\left(\frac{-fh}{2} + \frac{ghl}{4(a+h)}\right) \right]$$

$$= \frac{2}{E(a+h)^2 I_1} \frac{-F_2 h^3}{24} (agl + 2\mathrm{f}(a+h)^2)$$

$$\int_{charp} \frac{m M_{iso}}{EI_2} = \frac{2}{E(a+h)^2 I_2} \frac{-F_2 bh^2}{24} (agl + 2\mathrm{f}(a+h)^2)$$

From where:

$$w = \frac{2}{E(a+h)^2} \frac{-F_2 h^2}{24} (agl + 2\mathrm{f}(a+h)^2) \left(\frac{h}{I_1} + \frac{b}{I_2} \right)$$

Numerical application:

$I_1 = 2\mathrm{I}_2 \quad h = 2\mathrm{a}$, $F_2 = 10\,000\,N$ (towards the left)

$$w = \frac{2}{E(a+h)^2 I_1} (gl + 9\mathrm{gh}) \underbrace{\frac{-F_2 h^3 (h+2\mathrm{b})}{48}}_{-3151003.65\,N.m^4}$$

5.9.4 Couples specific Γ in C_1

$$\int_{pot} \frac{m M_{iso}}{EI_1} = \frac{h}{3\mathrm{EI}_1} \frac{\Gamma h}{2(a+h)} \left[\left(\frac{fh}{2} + \frac{glh}{4(a+h)}\right) + \left(\frac{-fh}{2} + \frac{glh}{4(a+h)}\right) \right]$$

$$= \frac{2}{E(a+h)^2 I_1} \frac{\Gamma h^3 lg}{24}$$

$$\int_{charp} \frac{m M_{iso}}{EI_2} = \frac{b}{3\mathrm{EI}_2} \frac{\Gamma h}{2(a+h)} \left[-(2\mathrm{a}+h)\left(\frac{fh}{2}+\frac{glh}{4(a+h)}\right)+h\left(\frac{-fh}{2}+\frac{glh}{4(a+h)}\right) \right]$$
$$= \frac{-2}{E(a+h)^2 I_2} \frac{\Gamma bh}{24}\left(agl+2\mathrm{f}(a+h)^2\right)$$

numerical Application:

$$I_1 = 2\mathrm{I}_2 \quad ; \quad h = 2\mathrm{a} \quad ; \quad \Gamma = -100\,000\, N.m$$
$$w = \frac{2}{E(a+h)^2 I_1} \underbrace{\frac{\Gamma h^2}{24}}_{-3151003.65\, N.m^4} \left(gl(h-b)-9\mathrm{fhb}\right)$$

5.9.5 Computation of $d = \int \frac{m \cdot M_1}{EI}$

$$\int_{pot} \frac{mM_1}{EI_1} = \frac{2\mathrm{h}}{3\mathrm{EI}_1} \frac{glh}{4(a+h)} \frac{h}{a+h} = \frac{2}{E(a+h)^2 I^1} \frac{glh^3}{12}$$
$$\int_{charp} \frac{m M_1}{EI_2} = \frac{2\mathrm{b}}{EI_2}\left[\frac{1}{2}\left(\frac{h}{a+h}\right)+\frac{1}{6}\left(\frac{a}{a+h}\right)\right]\frac{glh}{4(a+h)} = \frac{2}{E(a+h)^2 I_2} \frac{glbh(3\mathrm{h}+a)}{24}$$

From where (one notes quene d does not depend on f):

$$d = \frac{2}{E(a+h)^2} \frac{glh}{24}\left(\frac{2\mathrm{h}^2}{I_1}+\frac{b(3\mathrm{h}+a)}{I_2}\right)$$

Numerical application:

$$I_1 = 2 I_2 ,\ h = 2\mathrm{a}$$
$$d = \frac{2}{E(a+h)^2 I_1} g \frac{lh^2}{24} \underbrace{(2\mathrm{h}+7\mathrm{b})}_{-4874.2564\, N.m^4}$$

5.9.6 Summary of displacements u_c and v_c

$$I_1 = 5.0\, E-4\, m^4$$
$$E = 210\,000\, MPA$$

CAS	X	$X\bar{d}$	w_v
pressure on $C_1 C$	18672.994	91016960.3	– 184930109.4
F_1 in C	41422.161	201902233.4	– 315100365.0
F_2 in C_1	8284.432	40380445.6	– 63020073.0
Γ in C_1	4916.724	23965373.4	14775091.25

CAS	w_h	$u_c(m)$	$v_c(m)$
pressure out of $C_1 C$	83519999.94	0.0110476	– 0.012422374
F_1 in C	0.00	0.00	– 0.01497330
F_2 in C_1	– 226872262.8	– 0.03000956	– 0.00299466
Γ in C_1	206790328.5	0.0273532	– 0.001215646

Note:

$$d=\frac{2}{E(a+h)^2 I_1} g\,\bar{d}$$, with: $\bar{d}=4\,874.2564\,m^4$

$$w=\frac{2}{E(a+h)^2 I_1}(g\,w_v+f\,w_h)$$ to see higher

$$u_c=\frac{2}{E(a+h)^2 I_1} w_H \quad ; \quad v_c=\frac{2}{E(a+h)^2 I_1}(w_v+X\,\bar{d})$$

$$\frac{2}{E(a+h)^2 I_1}=1.32275132\,E-10\,N^{-1} m^{-4}$$

Comparison *Aster* - reference analytical (R.)

CAS		**Moment in** $C(N.m)$	**Reaction** $H_A(N)$	**Reaction** $V_A(N)$	**Displacement** $u_c(m)$	**Displacement** $v_c(m)$
P on $C_1 C$	R:	18672.994	5175.37	24233.24	0.0110476	– 0.012422374
	Aster :	18673.20	5175.36	24233.2	0.0110472	– 0.0124233
F_1 in C	R:	41422.161	4881.487	10000.00	0.00000	– 0.01497330
	Aster :	41422.40	4881.47	10000.0	0.0000	– 0.0
F_2 in C_1	R:	8284.432	5976.297	4000.00	– 0.03000956	– 0.00299466
	Aster :	8284.34	5976.31	4000.0	– 0.0300098	– 0.00299450
Γ in C_1	R:	4916.724	4576.394	5000.00	0.0273532	– 0.001215646
	Aster :	4916.62	4576.38	5000.0	0.0273536	– 0.00121583

Foot-note:

The computation *Aster* was carried out by taking very slender elements, so that: $Sl^2 \ll I$. Thus, the energy of bending is prevalent. The values of Aster computation result from the case-testappelé $VPCS\ \ SSLL14$, with the following data:

$I_1=5.0\,E-4\,m^4$; $I_2=2.5\,E-4\,m^4$; $E=210\,000\,MPa$

$h=2\text{a}=8\,m$; $l=20\,m$; $b=\frac{l}{2}\sqrt{1.16}$

$p=3000\,N.m$ (downwards),
$F_1=20\,000\,N$ (downwards),
$F_2=10\,000\,N$ (towards the left),
$\Gamma=-100\,000\,Nm$ (meaning switches of watch).