Computation of Yield-point load by the method of Norton-Hoff-Friaâ, behavior NORTON_HOFF

Summarized:

The limit analysis makes it possible to determine the acceptable loadings D `a structure, of geometry fixes given, made up of a material having a strength criterion. One considers the case of loadings made up of the sum of a continued load and of another parameterized by the load factor, which one seeks the bearable extreme value.

After a recall of the theoretical formulation, one presents the regularized kinematical approach applied to the strength criterion of Von Mises (method of Norton-Hoff-Friaâ) and put in work in *Code_Aster*. One will be able to refer to [bib4] for the various possible methods of regularization suggested in the literature. One exposes then the computation of the solutions of this nonlinear problem and the post-processing providing an estimate of the Yield-point load (a value by excess in all the cases, and when there is no permanent loading, a value by default).

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1 document Formulation of the limit analysis

1.1 Definition of the Yield-point load

One consider a solid occupying a limited Ω field subjected to surface loadings $\lambda F+F_0$ on edge Γ_f and of the loadings of volume $\lambda f+f_0$ on Ω. One distinguishes the loading (F,f), parameterized by positive reality λ, and the "permanent" loading (F_0,f_0). The homogeneous conditions of Dirichlet which are applied to edge complementary Γ_u to $\partial\Omega$ (an imposed displacement or an initial unelastic strain – thermal, plastic… – do not have an effect on the field of the working loads). One can find in [bib5] several other useful properties.

The material constitutive of solid has a strength criterion expressed by a scalar function of the stresses, negative for working stresses. The criterion used for a material of the perfect elastoplastic type with threshold of von Mises and selected here is:

$$g(\sigma)=J(\sigma)-\sigma_y=\sqrt{3/2}.\sqrt{\sigma^D.\sigma^D}-\sigma_y=\frac{\sqrt{2}}{2}.\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}-\sigma_y$$

σ^D is the deviator of the tensor of the stresses,

σ_y is the threshold of strength in simple tension (like an elastic limit), possibly variable according to the zones of solid considered.

σ_i being principal stresses of σ .

Being given this strength criterion one seeks to calculate the value limits λ, called Yield-point load $\lambda_{\lim}$, for which the structure can support the loadings $\lambda_{\lim}F+F_0$ and $\lambda_{\lim}f+f_0$.
Strictly speaking, the value $\lambda_{\lim}$ indicates the limit of the bearable loadings, but for the materials obeying the Principle of Maximum Plastic work, this value is the limit of the supported loadings.

1.2 Computation of the Yield-point load by a kinematical approach

In yield design two approaches are possible: static approach (in variables of stresses) and kinematical approach (in variables velocities). These approaches provide limits of the Yield-point load: undervaluing for the approach static and raising for the kinematical approach. When both provide the same one result, the Yield-point load obtained is exact.

The kinematical approach is that used in *Code_Aster* using finite elements in displacements. For the loading given $(\mathbf{F},\mathbf{f})$, one defines the space velocities kinematically admissible and standardized by:

$$V_a^1=\left\{\mathbf{v}\ \text{admissible},\ \mathbf{v}=0\ \text{sur}\ \Gamma_u\ ,\ L(\mathbf{v})=\int_\Omega \mathbf{f}.\mathbf{v}\ d\Omega+\int_{\Gamma_f}\mathbf{F}.\mathbf{v}\,ds=1\right\}$$

This standardization forces the work of the loading $(\mathbf{F},\mathbf{f})$ to be unit. The power of the "permanent" loading $(\mathbf{F_0},\mathbf{f_0})$ is noted: $L_0(\mathbf{v})$.

From the strength criterion in stresses $g(\sigma)$, one defines:

all working stresses by: $G_{(\mathbf{x})}=\{\sigma(\mathbf{x})\ ,\ g(\sigma(\mathbf{x}))\le 0\}$
($G_{(\mathbf{x})}$ is convex for the criterion g)

the indicating function: $\Psi_G(\sigma(\mathbf{x}))=\begin{cases}0\ ,\ si\ \sigma(\mathbf{x})\in G_{(\mathbf{x})}\\ +\infty\ ,\ si\ \sigma(\mathbf{x})\notin G_{(\mathbf{x})}\end{cases}$

the function of bearing: $\pi(\varepsilon)=\underset{\sigma\in IR^6}{\mathrm{Sup}}\left[\sigma.\varepsilon-\Psi_G(\sigma)\right]$

Sup in $\pi(\varepsilon)$ can be reached only if σ is selected in $G_{(\mathbf{x})}$, such as: $\sigma=\lambda\,\varepsilon^D+\mu\,\mathbf{Id}$ (what ensures $\sigma//\varepsilon^D$). The optimum corresponds to $g(\bar{\sigma})=0\Rightarrow\bar{\lambda}=\sigma_y\sqrt{2/3}.(\varepsilon^D.\varepsilon^D)^{-1/2}$

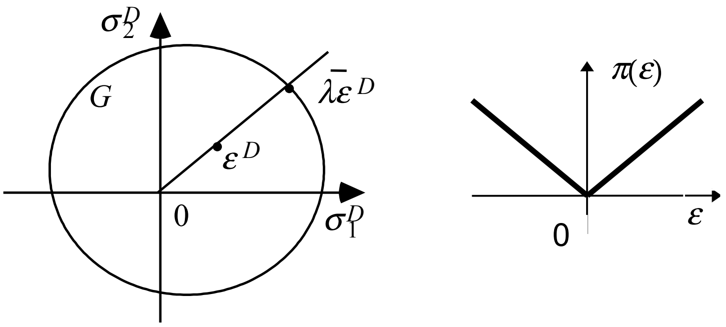

Figure 1.2-a: Optimum $\bar{\sigma}$ and graph of the function $\pi(\varepsilon)$ in 1D

From where the function of bearing:

$$\pi(\varepsilon(\mathbf{v}))=\sigma_y.\sqrt{\frac{2}{3}}\sqrt{\varepsilon(v).\varepsilon(v)}+\underset{\mu\in\mathrm{IR}}{\mathrm{Sup}}(\mu.\mathrm{div}\,\mathbf{v})=\pi_R(\varepsilon(\mathbf{v}))+\underset{\mu\in\mathrm{IR}}{\mathrm{Sup}}(\mu.\mathrm{div}\,\mathbf{v})$$

It is observed that the function $\pi(\varepsilon)$, which is interpreted like the density of power dissipable at the material point, is not differentiable in 0.

One to date does not treat in *Code_Aster* possible internal surfaces of discontinuity within the solid Ω [feeding-bottle 4].

The kinematical approach is defined using the convex functional calculus $S_e(\mathbf{v})$, positively homogeneous of degree one, for $\mathbf{v}\in V_a^1$ definite on the whole field:

$$S_e(\mathbf{v}) \quad = \int_\Omega \pi(\varepsilon(\mathbf{v}))\,d\,\Omega\ -L_0(\mathbf{v})$$

This functional calculus is the integral on the field of the function of bearing π of convex $G_{(\mathbf{x})}$, calculated in $\varepsilon(\mathbf{v})$ and is interpreted as the maximum resistant power in the velocity field $\mathbf{v}$ (the contribution of strength of interface on surfaces of discontinuity is supposed null). The function of bearing π is positively homogeneous of degree 1, and thus the functional calculus $S_e(\mathbf{v})$ also by consequence.

With the criterion of Von Mises the functional calculus of power $S_e(\mathbf{v})$ is:

$$S_e(\mathbf{v})=\int_\Omega\left[\sigma_y.\sqrt{\tfrac{2}{3}}\sqrt{\varepsilon(\mathbf{v}).\varepsilon(\mathbf{v})}+\underset{q\in\mathrm{IR}}{\mathrm{Sup}}(q.\mathrm{div}\,\mathbf{v})\right]d\,\Omega-\mathrm{L}_0(\mathbf{v}) \qquad \text{éq 1.2-1}$$

where it is noted that only the fields $\mathbf{v}$ belonging to $C=\{\mathbf{v}\in V_a^1,\mathrm{div}\,\mathbf{v}=0\ \text{ dans }\ \Omega\}$ provide finished values. The fields $\mathbf{v}$ must thus check the condition known as of incompressibility $\mathrm{div}\,\mathbf{v}=\mathrm{tr}\,\varepsilon(\mathbf{v})=0$. This is why it is necessary to use the quasi-incompressible elements for a computation of Yield-point load with the criterion of Von Mises [R3.06.08].

The Yield-point load $\lambda_{\lim}$ given by the kinematical approach is:

$$\lambda_{\lim}=\underset{\mathbf{v}\in V_a^1}{\mathrm{Inf}}\,S_e(\mathbf{v})=\underset{\substack{\mathbf{v}\in V_a\\ L(v)>0}}{\mathrm{Inf}}\frac{S_e(v)}{L(v)}=\underset{\lambda>0}{\mathrm{Sup}}\,\underset{\mathbf{v}\in V_a^1}{\mathrm{Inf}}\left(S_e(\mathbf{v})-\lambda(L(\mathbf{v})-1)\right)$$

With the optimum one obtains a solution u and the Yield-point load $\lambda_{\lim}$ (not unicity otherwise u unicity of $\lambda_{\lim}$). Thus, any loading $L_0(\mathbf{v})+\lambda L(\mathbf{v})$ with $0\leq\lambda\leq\lambda_{\lim}$ is bearable. Beyond $\lambda_{\lim}$, the problem of equilibrium does not have a solution.

Note:

There exist situations where, even if $L_0(\mathbf{v})$ is not bearable only, the combination $L_0(\mathbf{v})+\lambda L(\mathbf{v})$, for $\lambda_1\leq\lambda\leq\lambda_2$, becomes it on a certain interval, and not only for two loadings colinéaires.

Note:

The Yield-point load calculated for a two-dimensional problem, in plane strains, is necessarily higher than that obtained for this problem modelled in plane stresses. This result thus provides one raising. If one wishes to deal with a problem in plane stresses, it is necessary then to make the kinematical approach on a three-dimensional modelization.

1.3 Regularization of the kinematical approach by the method of Norton-Hoff-Friaâ

the numeric work implementation of the kinematical approach requires the minimization of the NON-differentiable functional calculus $S_e(\mathbf{v})$. Many techniques of regularization exist [bib4]. The method of Norton-Hoff-Friâa is used here [bib2], [bib7]. It rests on precursory works of Casciaro in 1971. It consists in replacing the function of bearing $\pi(\varepsilon)$ by the function of bearing regularized and differentiable $\pi^{NH}(\varepsilon)$. It is adjustable by a parameter of regularization m ($1\leq m\leq 2$), of which the limiting value $m\rightarrow 1^+$ led to convergence towards the function of bearing $\pi(\varepsilon)$:

$$\pi^{NH}(\varepsilon)=\frac{k^{1-m}}{m}\left(\pi(\varepsilon)\right)^m \qquad \text{éq 1.3-1}$$

the scalar k in [éq 1.3-1] is homogeneous with a stress. One notes the space acceptable velocities adapted to the problem of viscous flow for the model of Norton-Hoff of order m :

$$V_a^{m1}=\left\{\mathbf{v}\in L^m(\Omega),\, \text{et}\, \varepsilon(\mathbf{v})\in L^m(\Omega),\, \mathbf{v}=0\,\text{sur}\,\Gamma_u,\, L(\mathbf{v})=1\right\}$$

One defines on this space the regularized functional calculus $S_e^m(\mathbf{v})$:

$$S_e^m(\mathbf{v})=\int_\Omega \frac{k^{1-m}}{m}\pi(\varepsilon(\mathbf{v}))^m d\Omega - L_0(\mathbf{v})$$

The problem of minimization $\underset{\mathbf{v}\in V_a^{m1}}{\text{Inf}}\left[S_e^m(\mathbf{v})\right]$ is well posed thanks to the properties of spaces $L^m(\Omega)$ and admits a single solution $\mathbf{u_m}$, for which the value reached by Inf the east precisely λ_m . One notes the space of the incompressible fields from V_a^{m1} :

$$\tilde{V}_a^{m1}=\left\{\mathbf{v}\in V_a^{m1}\ tel\ que\ \text{div}\,\mathbf{v}=0\right\}$$

It is shown whereas this problem can be also written in the form of the search of POINT-saddles $(\lambda_m,\mathbf{u_m})$ Lagrangian following:

$$\underset{\lambda\in\text{IR}}{\text{Max}}\left[\underset{\mathbf{v}\in\tilde{V}_a^{m1}}{\text{Inf}}\left\{\int_\Omega \frac{A(m)}{m}.\left(\sqrt{\varepsilon(\mathbf{v}).\varepsilon(\mathbf{v})}\right)^m d\Omega - L_0(\mathbf{v})-\lambda\left(L(\mathbf{v})-1\right)\right\}\right] \qquad \text{éq 1.3-2}$$

with: $A(m)=k^{1-m}\left(\frac{2}{3}\right)^{m/2}.\sigma_y^m=\sigma_y^{2-m}.(3\mu)^{m-1}.\left(\frac{2}{3}\right)^{m/2}$.

It is noticed that $A(m)$ is increasing with m (if $E\geq\sigma_y$, which is the case in practice) and homogeneous with a stress, and remains limited when $m\to1^+$. If one chooses $k=\sigma_y$, then $A(m)=\sigma_y\left(\frac{2}{3}\right)^{m/2}$ and one finds the incompressible elastic problem when $m=2$, if a Young modulus is chosen $E=\sigma_y$.

One thus notes that this potential [éq 1.3-2] defines a constitutive law giving the tensor of the stresses $\sigma(\mathbf{u})$ by the behavior model of Norton-Hoff, to see [§2.1].

One builds thus a decreasing continuation of λ_m and the Yield-point load $\lambda_{\lim}$ is the limit of this continuation when $m\to1^+$ (either $n\to+\infty$):

$$\lambda_{\lim}=\lim_{m\to1}\left(\operatorname{Inf}_{\mathbf{v}\in V_a^{ml}}\left[S_e^m(\mathbf{v})\right]\right)=\lim_{m\to1}\left(S_e^m(\mathbf{u_m})\right) \quad \text{éq 1.3-3}$$

For the demonstration one will refer to [bib4] and [bib7].

Note:

If the intensity of the loading is amplified $L\to\beta L$ (whereas one does not consider a permanent loading $L_0=0$), the solutions depend on the factor β according to the following relations:
$\mathbf{u_m}(\beta)=\beta^{-1}\mathbf{u_m}(1)$; $\sigma^D(\mathbf{u_m}(\beta))=\beta^{1-m}\sigma^D(\mathbf{u_m}(1))$; $S_e^m(\mathbf{u_m}(\beta))=\beta^{-m}S_e^m(\mathbf{u_m}(1))$.
With convergence for $m\to1^+$, the conclusion given by the solution $\mathbf{u_m}(\beta)$ is well the same one as that given by $\mathbf{u_m}(1)$, since $\lambda_{\lim}(\beta)=\lambda_{\lim}(1)/\beta$.

2 Numerical aspects of the computation of the Yield-point load

2.1 Behavior model of Norton-Hoff

the tensor of the stresses $\sigma(\boldsymbol{u})$ checks the behavior model of Norton-Hoff. The deviator of the stresses associated at the strainrate is:

$$\sigma^D(\mathbf{u})=A(m).\left(\sqrt{\varepsilon^D(\mathbf{u}).\varepsilon^D(\mathbf{u})}\right)^{m-2}.\varepsilon^D(\mathbf{u})\Leftrightarrow\sigma^D(\mathbf{u})=A(m)^n.\left(\sqrt{\sigma^D(\mathbf{u}).\sigma^D(\mathbf{u})}\right)^{1-n}.\varepsilon^D(\mathbf{u}) \quad \text{éq 2.1-1}$$

with $\operatorname{tr}\varepsilon(\mathbf{u})=0$, and: $A(m)=k^{1-m}\left(\frac{2}{3}\right)^{m/2}\sigma_y^m$ and $n=\frac{1}{m-1}$.

This behavior is integrated in the same way that the incremental behaviors elastoplastic ones of Von Mises [R5.03.02]. Let us notice however that, in a point of integration, the computation of the tensor of the stresses according to the tensor of the strains is explicit, no iterative diagram being used. Moreover, no local variable is necessary to the integration of this behavior.

In *Code_Aster*, the computation of the Yield-point load being independent of the moduli of elasticity, one chooses $k=\sigma_y$, from where $A(m)=\sigma_y\left(\frac{2}{3}\right)^{m/2}$. One finds the incompressible elastic problem thus when $m=2$, for a Young modulus $E=\sigma_y$.

Moreover, the continuation of the scalars m is directly deduced from the list of times (fictitious) of computation by:

$$m=1+10^{1-t} ,$$

so that when time increases, m tends towards 1, and the behavior approaches a behavior perfect rigid-plastic, to see into unidimensional the curves [fig. 2.1-a]. In practice, one chooses the continuation of the values of [. 2.1-a].

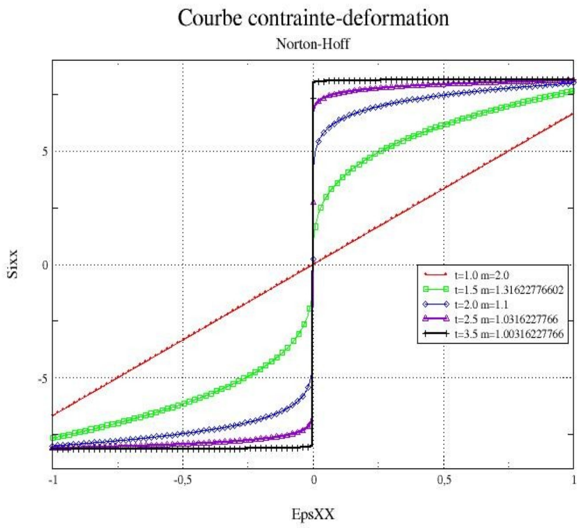

Fig. 2.1-a. Stress-strain curve for various values of time t .

t	1.1.5.1.7	2	…	∞
$m=1+10^{1-t}$	2.1.3.1.2	1.1	…	1

. 2.1-a. Continuation of the values of time t , and values m the corresponding ones.

The tangent operator, used in option FULL_MECA of the method of Newton, is written thanks to [éq 2.1-1]:

$$\left.\frac{d\sigma^D}{d\varepsilon^D}\right|_{\varepsilon(\mathbf{u})} = A(m)\,\|\varepsilon^D\|^{m-2}\left(\mathbf{Id}\otimes\mathbf{Id}+(m-2)\,\|\varepsilon^D\|^{-2}\varepsilon^D\otimes\varepsilon^D\right) \qquad \text{éq 2.1-2}$$

with ε^D, σ^D vectors of the strains and deviatoric stresses writings in vectorial notations of WALPOLE-COWIN:

$$\varepsilon^D=\left(\varepsilon^D_{11},\varepsilon^D_{22},\varepsilon^D_{33},\sqrt{2}\varepsilon^D_{12},\sqrt{2}\varepsilon^D_{23},\sqrt{2}\varepsilon^D_{31}\right)$$
$$\sigma^D=\left(\sigma^D_{11},\sigma^D_{22},\sigma^D_{33},\sqrt{2}\sigma^D_{12},\sqrt{2}\sigma^D_{23},\sqrt{2}\sigma^D_{31}\right)$$

2.2 Control

the problem is written in variational form in the following way on the space of the incompressible fields.

For m given, therefore at a given t time, knowing the solution at previous time (noted $\mathbf{u}^-,\lambda^-$), to find $(\Delta\lambda,\Delta\mathbf{u})\in IR\times\tilde{V}_a$ such as:

$$\begin{cases}\int_\Omega \sigma(\mathbf{u}^-+\Delta\mathbf{u}).\varepsilon(\mathbf{v})\,d\Omega=L_0(\mathbf{v})+(\lambda^-+\Delta\lambda)\,L(\mathbf{v}) & \forall\,\mathbf{v}\in\tilde{V}_0\\ L(\mathbf{v})=\int_\Omega \mathbf{f}.\mathbf{v}\,d\Omega+\int_{\Gamma_f}\mathbf{F}.\mathbf{v}\,dS=1\end{cases} \qquad \text{éq 2.2-1}$$

L_0 is the permanent loading and L the loading controlled by the parameter λ , cf [§1.1],

$\tilde{V}_0$ is a space of functions discretized on the basis of incompressible finite elements, and is thus defined by a vector $(\mathbf{U})$ of degrees of freedom.

This problem admits a single solution for all $1\le m\le 2$ (see [bib4]). For $m=2$ the problem is of standard incompressible linear elasticity.

The problem discretized at time t (thus for a value of m, cf [. 2.1-a]) can be written (by omitting the boundary conditions to simplify):

$$\begin{cases}\mathbf{F_{int}}(\Delta\mathbf{U}\,;\,\mathbf{U}^-;..)=\mathbf{F_0^{ext}}+\lambda\,\mathbf{F}^{ext}\\ L(\mathbf{U}^-+\Delta\mathbf{U})=1\end{cases}$$

The search for λ ensuring the condition $L(\mathbf{U})=1$ is ensured by an algorithm of control [R5.03.80].

Briefly, the principle is the following: by linearization of the equations relating to the internal forces, one obtains, for the iteration n of the algorithm of Newton, cf [R5.03.01]:

$$\underbrace{\left[\frac{\partial\mathbf{F_{int}}}{\partial\mathbf{U}}(\Delta\mathbf{U^n})\right]}_{\mathbf{K_T}}[\delta\mathbf{U}]=\underbrace{\left[\mathbf{F_0^{ext}}-\mathbf{F_{int}}(\Delta\mathbf{U^n})\right]}_{\mathbf{R^{cst}}}+\lambda\underbrace{\left[\mathbf{F}^{ext}\right]}_{\mathbf{R^{pilo}}} \qquad \text{éq 2.2-2}$$

One can now express the corrections of displacements $\delta\mathbf{U}$ and Lagrange multipliers $\delta\lambda$ according to λ with the help of the resolution of this linear system:

$$[\delta\mathbf{U}]=[\delta\mathbf{U}^{\mathbf{cst}}]+\lambda[\delta\mathbf{U}^{\mathbf{pilo}}] \quad \text{où} \quad [\delta\mathbf{U}^{\mathbf{cst}}]=\mathbf{K}_{\mathbf{T}^{-1}}\mathbf{R}^{\mathbf{cst}} \quad \text{et} \quad [\delta\mathbf{U}^{\mathbf{pilo}}]=\mathbf{K}_{\mathbf{T}^{-1}}\mathbf{R}^{\mathbf{pilo}} \qquad \text{éq 2.2-3}$$

One can substitute the correction of displacement $\delta\mathbf{U}$ according to its statement [éq 2.2-2] in the equation of control of the control of the system $L(\mathbf{U})=1$; it results a scalar equation from it in $\Delta\lambda$:

$$L\left(\mathbf{U}^{-}+\Delta\mathbf{U}^{\mathbf{n}}+\delta\mathbf{U}^{\mathbf{cst}}+\lambda\delta\mathbf{U}^{\mathbf{pilo}}\right)=1 \qquad \text{that is to say}$$

$$\int_{\Omega}\mathbf{f}.\left(\mathbf{U}^{-}+\Delta\mathbf{U}^{\mathbf{n}}+\delta\mathbf{U}^{\mathbf{cst}}+\lambda\delta\mathbf{U}^{\mathbf{pilo}}\right)d\Omega+\int_{\Gamma_f}\mathbf{F}.\left(\mathbf{U}^{-}+\Delta\mathbf{U}^{\mathbf{n}}+\delta\mathbf{U}^{\mathbf{cst}}+\lambda\delta\mathbf{U}^{\mathbf{pilo}}\right)dS=1 \qquad \text{éq 2.2-4}$$

what in discretized form returns to:

$$\sum_e\mathbf{F}^{ext}.\left(\mathbf{U}^{-}+\Delta\mathbf{U}^{\mathbf{n}}+\delta\mathbf{U}^{\mathbf{cst}}+\lambda\delta\mathbf{U}^{\mathbf{pilo}}\right)=1$$

what leads to:

$$\lambda=\frac{1-\sum_e\mathbf{F}^{ext}.\left(\mathbf{U}^{-}+\Delta\mathbf{U}^{\mathbf{n}}+\delta\mathbf{U}^{\mathbf{cst}}\right)}{\sum_e\mathbf{F}^{ext}.\delta\mathbf{U}^{\mathbf{pilo}}} \qquad \text{éq 2.2-5}$$

2.3 Postprocessing of the computation of the Yield-point load

having obtained the solution $(\lambda_m,\mathbf{u_m})$, for each time, therefore each m given, it remains to use the continuation of λ_m to build the approximation of the Yield-point load. For that one exploits the properties [éq 1.3-2], [éq 1.3-3], the fact that $A(m)$ is increasing and the property resulting from minimization [éq 1.3-2] (see [bib7]).

Of these two last, with $1\le r\le s$, one deduces that for $\mathbf{u_r}$ and $\mathbf{u_s}$ respective solutions (also checking the condition of incompressibility and standardization) of [éq 1.3-2] for $m=r$ and $m=s$:

$$\left(\int_{\Omega}A(r)\left(\varepsilon(\mathbf{u_r}).\varepsilon(\mathbf{u_r})\right)^{r/2}d\Omega\right)\le\left(\int_{\Omega}A(s)\left(\varepsilon(\mathbf{u_s}).\varepsilon(\mathbf{u_s})\right)^{s/2}d\Omega\right)$$

Associated with the property [éq 1.3-3], one draws for $1\le r\le s$, while noting $\|\Omega\|_r=\int_{\Omega}A(r)\,d\Omega$:

$$\int_{\Omega}A(r)\sqrt{\varepsilon(\mathbf{u_r}).\varepsilon(\mathbf{u_r})}\,d\Omega\le\|\Omega\|_r^{1-\frac{1}{r}}\left(\int_{\Omega}A(r)\left(\varepsilon(\mathbf{u_r}).\varepsilon(\mathbf{u_r})\right)^{r/2}d\Omega\right)^{\frac{1}{r}}\le\|\Omega\|_s^{1-\frac{1}{s}}\left(\int_{\Omega}A(s)\left(\varepsilon(\mathbf{u_s}).\varepsilon(\mathbf{u_s})\right)^{s/2}d\Omega\right)^{\frac{1}{s}}$$

éq 2.3-1

Indeed, this property results from the inclusion of functional spaces $L^r\supset L^s$, for $1\le r\le s$, that is to say also, where $h(\mathbf{x})$ the role of a variable measurement plays (conditioned by the limit of strength σ_y):

$$\|\Omega\|_r^{-\frac{1}{r}}\left(\int_{\Omega}h(\mathbf{x})\,|f(\mathbf{x})|^r\,d\Omega\right)^{\frac{1}{r}}\le\|\Omega\|_s^{-\frac{1}{s}}\left(\int_{\Omega}h(\mathbf{x})\,|f(\mathbf{x})|^s\,d\Omega\right)^{\frac{1}{s}}$$

One notes the terms $\tilde{\lambda}_m$ of the continuation below, which one calculates in practice by postprocessing using $\mathbf{u_m}$ (the external power being unit:

$$\tilde{\lambda}_m = \|\Omega\|_m^{1-\frac{1}{m}} \left(\int_\Omega A(m) \left(\varepsilon(\mathbf{u_m}).\varepsilon(\mathbf{u_m}) \right)^{m/2} d\,\Omega \right)^{\frac{1}{m}} - L_0(\mathbf{u_m}) \qquad \text{éq 2.3-2}$$

This continuation $\tilde{\lambda}_m$ is thus decreasing for $m \to 1^+$ and it is shown [bib7] that it converges towards, which $\lambda_{\lim}$ allows a good control. As one can undervalue (knowing that $A(1) = \sigma_y \sqrt{\frac{2}{3}}$) the first term of [éq 2.3-1]:

$$\lambda_{\lim} \leq \int_\Omega \sigma_y \sqrt{\frac{2}{3} \varepsilon(\mathbf{u_m}).\varepsilon(\mathbf{u_m})}\, d\,\Omega \quad - L_0(\mathbf{u_m}) \leq \tilde{\lambda}_m$$

one thus calculates for each value of m (thus of time t) the Yield-point load per $\tilde{\lambda}_m$ excess also converging towards $\lambda_{\lim}$:

$$\lambda_{\lim} \leq \hat{\lambda}_m = \int_\Omega \sigma_y \sqrt{\frac{2}{3} \varepsilon(\mathbf{u_m}).\varepsilon(\mathbf{u_m})}\, d\,\Omega \quad - L_0(\mathbf{u_m}) \qquad \text{éq 2.3-3}$$

One judges quality of the approximation of the Yield-point load $\lambda_{\lim}$ by comparison of the various values of $\tilde{\lambda}_m$ which converge towards $\lambda_{\lim}$ by (en) excess $m \to 1^+$. These terms are calculated by numerical integration with Gauss points of the finite elements.

Another interpretation of the interest which this continuation brings lies in the fact that it directly exploits the statement of the function of bearing of convex of strength, i.e. the power dissipable in the modes of potential failure, steady to the incompressible and standardized solutions calculated $\mathbf{u_m}$.

If the permanent loading is null: $L_0 = 0$, one can easily exploit the stress field (almost statically admissible) calculated with the solution $\mathbf{u_m}$ and obtain a value by estimate of the Yield-point load, which would be necessarily a true lower limit if the equilibrium were checked exactly (see [bib4]).

One thus calculates the continuation $\underline{\lambda}_m$, which does not have on the other hand properties of monotony:

$$\underline{\lambda}_m = \int_\Omega \frac{A(m)}{m} . \left(\sqrt{\varepsilon(\mathbf{u_m}) . \varepsilon(\mathbf{u_m})} \right)^m d\,\Omega \;\; . \left(\operatorname*{Sup}_{\mathbf{x} \in \Omega} \left(\frac{\sqrt{\frac{3}{2} \sigma^D(\mathbf{u_m}) \,.\, \sigma^D(\mathbf{u_m})}}{\sigma_y} \right) \right)^{-1} \leq \hat{\lambda}_m \qquad \text{éq 2.3-4}$$

This maximization (of the function called gauge of convex of strength) is calculated only with Gauss points of the finite elements. Also the value obtained, for each m, lower than $\tilde{\lambda}_m$ [bib4], can be regarded only as one indication.

On the other hand, always if the continued loads are null, it allows, with the value by excess $\tilde{\lambda}_m$, to provide a framing of the Yield-point load of the discretized problem.

3 Features and checking

to carry out a computation in *Code_Aster* in limit analysis with the method of regularization of Norton-Hoff-Friaâ with the strength criterion of Von Mises, it is necessary:

to define the model 2D (plane or axis) or 3D with the quasi-incompressible finite elements, modelizations `3D_INCO`, `D_PLAN_INCO`, or `AXIS_INCO` ;
to ensure the condition of incompressibility: `GONF=0` in `AFFE_CHAR_MECA` ;
to define only the characteristic of the material s_y , the Yield-point load being independent of E and ν ,
to define the permanent loading and that which is parameterized by λ ;
to define the discretization in time, (in practice enters $t_{min}=1$ and $t_{max}=2$ to 5);
to carry out a nonlinear computation with behavior model `NORTON_HOFF` with the command `STAT_NON_LINE` [U4.51.03], and control `ANA_LIM`. One can use the linear search in practice to improve convergence, and the subdivision of time step,
post-to treat computation to obtain the Yield-point load with command `POST_ELEM` [U4.81.22].

The use of these commands is detailed in the document [U2.05.04].

With regard to postprocessing, operator `POST_ELEM` then produces an array which gives for each time of computation, i.e. for increasingly weak regularizations, 2 parameters:

parameter "`CHAR_LIMI_SUP`" contains a higher limit of the Yield-point load, by integration on each finite element and a sum on all the elements of the model: $\hat{\lambda}_m=\int_\Omega \sigma_y \sqrt{\frac{2}{3}\varepsilon(\mathbf{u_m}).\varepsilon(\mathbf{u_m})}\,d\Omega \quad -L_0(\mathbf{u_m})$

and, in the absence of constant loading, (`CHAR_CSTE = ' NON'`), parameter "`CHAR_LIMI_ESTIMEE`" contains an estimate of a lower limit $\underline{\lambda}_m$ corresponding to:

$$\underline{\lambda}_m=\int_\Omega \frac{A(m)}{m}.\left(\sqrt{\varepsilon(\mathbf{u_m}).\varepsilon(\mathbf{u_m})}\right)^m d\Omega \quad . \left(\underset{\mathbf{x}\in\Omega}{\mathrm{Sup}}\left(\frac{\sqrt{\frac{3}{2}\sigma^D(\mathbf{u_m}).\sigma^D(\mathbf{u_m})}}{\sigma_y}\right)\right)^{-1} \le \hat{\lambda}_m$$

If a constant loading is present, (to inform then imperatively `CHAR_CSTE = "OUI'`), parameter `PUIS_CHAR_CSTE` represents the power of the constant loading in the velocity field solution of the problem.

Several tests of checking are available, in particular test SSNV124 [V6.04.124]. On this very simple problem, an analytical computation makes it possible to obtain the exact Yield-point load in the direction of the loading, as well as the estimates produced by the method of regularization. For more details one will refer to [bib4] and [bib5].

In addition complementary validations were carried out in the frame of comparative studies, like the benchmark European LISA [bib8, bib10]: on computations of Yield-point loads in 2D, 2D axis and 3D, the regularized kinematical method presented here makes it possible to gain a factor from 6 to 10 over time computation compared to an incremental elastoplastic computation, and makes it possible to obtain a framing of the Yield-point load, contrary to the methods of the other participants.

4 Bibliography

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5 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	F.VOLDOIRE EDF-R&D/AMA	initial Text
8.4	E.LORENTZ, S.MICHEL-PONNELLE EDF-R&D/AMA	Modification of the management of the exhibitor of the model of Norton-Hoff