Elements with internal discontinuity, behavior CZM_EXP and control of the Summarized

loading:

By leaning on a briefly exposed energy formulation in a first part we present a digital model of two-dimensional crack starting and propagation. This one leans on a finite element with an internal discontinuity of displacement, near to the models known by the name “embedded discontinuity finite element” in the literature.

For this model, we detail the constitutive law used (of Cohesive type exponential Zone Model: key word CZM_EXP), properties of the finite element as well as the resolution of the problem of minimization of total energy. In addition we describe a technique of control of the loading allowing to follow possible instabilities in the total response of structure.
These elements can be used in 2D plane or axisymmetric only with modelizations PLAN_ELDI and AXIS_ELDI. They are validated on the cases tests ssnp128a (documentation [V6.03.128]) and ssna115a (documentation [V6.01.115]).

Note:
- This documentation is largely inspired by works of thesis 23. The interested reader will be able to refer to it to have a more complete vision of the restrain of this model with the energy approach “Frankfurt Marigo” like a certain number of numerical results.
- The purpose of the choice to gather in the same documentation the presentation of a finite element and a constitutive law is related to the specificity of the model and is facilitating its comprehension.

Contents

1 Models energy cracking

the goal of this part is to draw up the general frame of a model of starting and crack propagation based on a principle of minimization of energy. This model takes as a starting point those developed in the literature under name "Models of Cohesive Forces" in the meaning where one seeks to take into account a residual interaction between the lips of crack. However, the major difference in our approach is the formulation of the problem in a broader frame by adopting the energy point of view introduced by Frankfurt and Marigo 23.

The idea consists in taking into account the process of dissipation of energy during cracking thanks to an energy defined on a surface of discontinuity and depend on the jump of displacement through the latter. We will be satisfied here to pose the problem in its continuous form in space and semi-discretized in time without seeking to clarify the form of the energy of surface.

The following part will be devoted to the implementation of the finite element with internal discontinuity leaning on this energy formulation.

1.1 General principle

One considers an elastic *structure* definite by the field Ω and a defined crack as a surface of noted discontinuity Γ through which displacement $\boldsymbol{u}$ admits a jump $\boldsymbol{\delta}$:

$$\boldsymbol{\delta}=[\![\boldsymbol{u}]\!]_{\Gamma} \quad (1)$$

One defines the total energy E_T of this structure as the sum of its noted elastic strain energy Φ , of an energy of noted surface Ψ and W^{ext} potential of the external forces:

$$E_T=\Phi+\Psi-W^{\text{ext}} \quad (2$$

One notes $\Phi=\int_{\Omega}\phi\,.\,d\,\Omega$ and $\Psi=\int_{\Gamma}\psi\,.\,d\,\Gamma$, where ϕ and ψ respectively indicate the density of elastic strain energy and the density of energy of surface. Total energy is a function of displacement $\boldsymbol{u}$ and jump of displacement $\boldsymbol{\delta}$, **the problem of minimization** is written:

$$\text{Chercher } (\boldsymbol{u}^*,\boldsymbol{\delta}^*) \text{ minimum local de l'énergie totale } E_T(\boldsymbol{u},\boldsymbol{\delta}) \quad (3)$$

Let us recall that one carries out a search for local *minimum* because total energy is not limited in a lower position in the presence of external forces. Now, let us see which assumptions one can formulate on the energy of surface in order to take into account the condition of noninterpenetration of the lips of crack and according to whether one or not regards cracking as a reversible process.

1.2 Model reversible

Present a first simple, but not very realistic model from a physical point of view on a macroscopic scale, where it is supposed that the cracking of a material is a reversible process. It is considered that the total reclosing of the lips of a crack makes it possible to find an operational material. The energy of surface can be written as the sum of a function of the euclidian norm of the jump of displacement and of an indicatrix giving an account of the condition of NON-interpenetration of the lips:

$$\Psi(\boldsymbol{\delta})=\Psi_{rev}(\|\boldsymbol{\delta}\|)+I_{\mathbb{R}^+}(\delta_n) \quad (4)$$

with $\delta_n=\boldsymbol{\delta}\cdot\boldsymbol{n}$ where $\boldsymbol{n}$ is the norm with discontinuity and $I_{\mathbb{R}^+}(\delta_n)$ is the indicating function:

$$I_{\mathbb{R}^+}(\delta_n)=\begin{cases}+\infty & \text{si } \delta_n<0\\ 0 & \text{si } \delta_n\geq 0\end{cases} \quad (5)$$

The model developed in *Code_Aster* allows as for him to take into account the irreversibility of cracking, it is presented in the following section.

1.3 Model with memory

By leaning on the writing of the preceding reversible model, one introduces a noted local variable κ allowing to memorize the state of cracking at a given time and to thus translate his irreversible character. One defines his evolution in the following way:

$$\begin{cases} \kappa(0) & = & \kappa_0 \\ \kappa(t) & = & \sup_{\tau \le t} \{\|\delta(\tau)\|, \kappa_0\} \quad \forall t > 0 \end{cases} \tag{6}$$

One can in an equivalent way express this law of evolution by means of a function threshold f defined by:

$$f(\kappa, \boldsymbol{\delta}) = \|\boldsymbol{\delta}\| - \kappa \tag{7}$$

the law of evolution then takes the shape of a classical condition of coherence:

$$f \le 0, \quad \dot{\kappa} \ge 0, \quad f.\kappa = 0 \tag{8}$$

One defines the energy of surface Ψ like a function of the jump of displacement and variable κ :

$$\Psi(\boldsymbol{\delta}, \kappa) = H(\|\boldsymbol{\delta}\| - \kappa).\Psi_{dis}(\|\boldsymbol{\delta}\|) + [1 - H(\|\boldsymbol{\delta}\| - \kappa)].\Psi_{lin}(\|\boldsymbol{\delta}\|, \kappa) + I_{\mathbb{R}^+}(\delta_n) \tag{9}$$

with H function of Heaviside defined by:

$$H(x) = \begin{cases} 1 & \text{si} \quad x \ge 0 \\ 0 & \text{si} \quad x < 0 \end{cases} \tag{10}$$

the energy of surface Ψ will thus be worth Ψ_{dis} if the threshold is positive, representing dissipation when the crack evolves. And it will be worth Ψ_{lin} , representing an evolution without dissipation of energy, if the threshold is negative. Ψ_{lin} takes into account the case where the crack is closed; one makes the choice of a linear discharge, therefore the density of energy of corresponding surface Ψ_{lin} , will be form:

$$\Psi_{lin}(\|\boldsymbol{\delta}\|, \kappa) = \frac{1}{2}.R(\kappa).\|\boldsymbol{\delta}\|^2 + C_0 \tag{11}$$

where $R(\kappa)$ and C_0 are selected to ensure the continuously differentiable character of energy. The density of energy of surface Ψ_{dis} when cracking evolves can take many forms; we will give of it an example for the model finite element of chapter 5. From the densities of energy of surface Ψ_{dis} and Ψ_{lin} one can define the vector forced $\vec{\boldsymbol{\sigma}}$ through discontinuity:

$$\begin{cases} \vec{\boldsymbol{\sigma}} = \dfrac{\partial \Psi_{lin}}{\partial \boldsymbol{\delta}} & \text{si} \quad \|\boldsymbol{\delta}\| < \kappa \\ \vec{\boldsymbol{\sigma}} = \dfrac{\partial \Psi_{dis}}{\partial \boldsymbol{\delta}} & \text{si} \quad \|\boldsymbol{\delta}\| > \kappa \end{cases} \tag{12}$$

It is the model of interface which we will adopt.

1.4 Temporal discretization

Let us present the temporal discretization now. Let us specify first of all that one considers only quasi-static evolutions, time is thus parameterized by increments of loading. By carrying out a semi-discretization in time, one defines the energy of surface Ψ in one time i like a function of the jump of displacement and κ^{i-1} local variable at time $i-1$:

$$\Psi(\boldsymbol{\delta}, \kappa^{i-1}) = H(\|\boldsymbol{\delta}\| - \kappa^{i-1}).\Psi_{dis}(\|\boldsymbol{\delta}\|) + [1 - H(\|\boldsymbol{\delta}\| - \kappa^{i-1})].\Psi_{lin}(\|\boldsymbol{\delta}\|, \kappa^{i-1}) + I_{\mathbb{R}^+}(\delta_n) \tag{13}$$

the problem of minimization is written then at time i :

$$\min_{\boldsymbol{u}, \delta} E_T(\boldsymbol{u}, \boldsymbol{\delta}; \kappa^{i-1}) \tag{14}$$

the local variable at time i is brought up to date once the jump at known i time:

$$\kappa^i = \max\left(\|\boldsymbol{\delta}\|, \kappa^{i-1}\right) \tag{15}$$

2 Models with discontinuity interns

This model allows to take into account the starting and the propagation of cracks in a structure for a given direction. This one is based on finite element particular which one calls *elements with internal discontinuity* (of English embedded discontinuity finite elements). The principal idea rests on the introduction of a discontinuity included into the element and of a local variable of threshold managing the dissipative process as well as the irreversible character of cracking. Moreover, the jump of displacement will be considered constant by element what will facilitate the numerical resolution. One will be able to adopt a static technique of condensation where the computation of the jump during minimization will be done at the elementary level.

The element with discontinuity was developed with an aim of freeing itself from the regularization of energy into zero (penalized dependancy) from model CZM_EXP_REG developed for the element from joint (confer to the documentation [R3.06.09] for the element and [R7.02.11] the model). Indeed, this one is prejudicial for the models based on a principle of minimum of energy since it results in cancelling derivative of the density of energy of surface in the zero i.e. stress criticizes model. In such a situation, the starting of a crack occurs as of loading, also weak is it (let us note however that does not prevent from getting results interesting with such a model, since one sets the way of cracking).

To take into account discontinuity one carries out an enrichment of displacements and strains which makes it possible to ensure their compatibility.

In this part one will initially present the constitutive law which controls the element with discontinuity. One will be brought to define a vector forced in the element which will be useful during minimization of energy. One will detail then the properties of the finite element then the numerical resolution of the problem of minimization of energy.

2.1 Constitutive law of the element to discontinuity : CZM_EXP

the elements with discontinuity have a mixed behavior: elastic and dissipative. The total energy of the element is written as the sum of an elastic strain energy and an energy of surface of the type CZM. The following parts will be devoted to the presentation of the latter like to the stress which in drift.

2.1.1 Total energy in the element with discontinuity

One chooses total energy E_T as the sum of an elastic strain energy defined on the noted element Ω_e and of an energy of surface defined on the discontinuity of the element noted Γ_e :

$$E_T(\boldsymbol{U}, \boldsymbol{\delta}; \kappa) = \Phi(\boldsymbol{U}, \boldsymbol{\delta}) + \Psi(\boldsymbol{\delta}, \kappa) \tag{16}$$

where Φ corresponds to elastic strain energy. Let us note that this one depends on nodal displacement $\boldsymbol{U}$ and the jump of constant $\boldsymbol{\delta}$ displacement (in the element) which one will see that it induces an additional strain. The energy of surface Ψ depends on the jump of displacement and κ local variable making it possible to treat the irreversibility of cracking.

Note: The way of cracking is known *a priori*, the norm with discontinuity will thus be fixed by the directional sense of the elements in the mesh (see figure 1).

2.1.1.1 Elastic strain energy

to define elastic strain energy we will need to introduce as of now the interpolation of the field of displacement into the element with discontinuity, the properties of this last will be however detailed in a part to follow (see §2.22.2). One represents on figure 1 the element with discontinuity noted Ω_e, it is a quadrangle with four nodes with an internal discontinuity which one notes Γ_e.

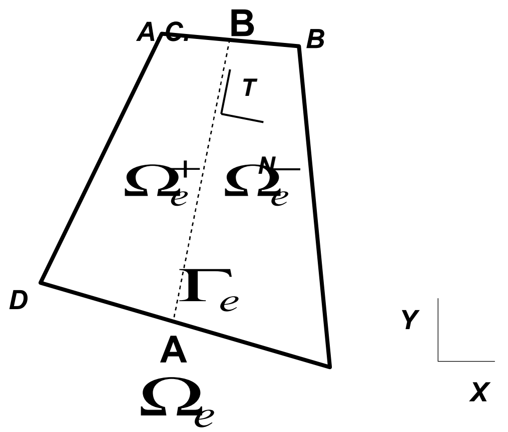


Figure 1: Element with discontinuity

the interpolation of the field of displacement $\boldsymbol{u}(x)$ in the element with discontinuity is written:

$$\boldsymbol{u}(x)=\boldsymbol{N}(x).\begin{bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C-\boldsymbol{\delta}\\ \boldsymbol{U}_D-\boldsymbol{\delta}\end{bmatrix}+H_{\Gamma_e}(x).\boldsymbol{\delta} \quad (17)$$

with $\boldsymbol{N}$ classical matrix of the shape functions bilinear of type $Q1$, H_{Γ_e} the function of Heaviside on the discontinuity of the element Γ_e being worth 1 if $x\in\Omega_e^+$ and 0 if $x\in\Omega_e^-$. The vector $\boldsymbol{\delta}$ corresponds to the jump of displacement on discontinuity. One has well $[\![\boldsymbol{u}]\!]=\boldsymbol{\delta}$ since $\boldsymbol{N}$ is a continuous function and $H_{\Gamma_e}=1$ on Γ_e. Moreover, them $\boldsymbol{U}$ (subscripted by the tops) correspond to nodal displacements, indeed one can write:

$$\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C\\ \boldsymbol{U}_D\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C+\boldsymbol{\delta}-\boldsymbol{\delta}\\ \boldsymbol{U}_D+\boldsymbol{\delta}-\boldsymbol{\delta}\end{Bmatrix} \quad (18)$$

From the approximate field (17) one can define the strain associated on the element ***out of discontinuity*** :

$$\boldsymbol{\epsilon}=\nabla^s\boldsymbol{N}(x).\begin{Bmatrix}\boldsymbol{U}_A\\ \boldsymbol{U}_B\\ \boldsymbol{U}_C-\boldsymbol{\delta}\\ \boldsymbol{U}_D-\boldsymbol{\delta}\end{Bmatrix}+\underbrace{\nabla^s H_{\Gamma_e}(x)}_{=0}\cdot\boldsymbol{\delta} \quad (19)$$

∇^s indicates the symmetrized gradient. One can récrire (19) in the synthetic form:

$$\boldsymbol{\epsilon}=\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta} \quad (20)$$

where $\boldsymbol{B}$ is the matrix of the symmetrized gradients of the shape functions, $\boldsymbol{U}=(\boldsymbol{U}_A,\boldsymbol{U}_B,\boldsymbol{U}_C,\boldsymbol{U}_D)^T$ and $\boldsymbol{D}.\boldsymbol{\delta}=\boldsymbol{B}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{\delta},\boldsymbol{\delta})^T$. In the statement (20) one distinguishes part of the strain related to displacements $\boldsymbol{B}.\boldsymbol{U}$ and another part $-\boldsymbol{D}.\boldsymbol{\delta}$ related to the jump. The stress for an elastic constitutive law is written:

$$\boldsymbol{\sigma}=\boldsymbol{E}.\boldsymbol{\epsilon} \quad (21)$$

With $\boldsymbol{E}$ the elasticity tensor. Elastic strain energy in the element is thus defined by:

$$\Phi(\boldsymbol{U},\boldsymbol{\delta})=\frac{1}{2}.\int_{\Omega_e}\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}\right)^T.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}\right).d\Omega \quad (22)$$

2.1.1.2 Energy of surface

As we presented in the model theoretical (see §11), one chooses an energy of surface depending on the norm of the jump of displacement $\|\boldsymbol{\delta}\|$, κ positive local variable (its initial value κ_0 is null) and of the directional sense of crack (only for the condition of contact). One takes:

$$\Psi(\boldsymbol{\delta},\kappa)=H\left(\|\boldsymbol{\delta}\|-\kappa\right).\Psi_{dis}\left(\|\boldsymbol{\delta}\|\right)+\left[1-H\left(\|\boldsymbol{\delta}\|-\kappa\right)\right].\Psi_{lin}\left(\|\boldsymbol{\delta}\|,\kappa\right)+I_{\mathbb{R}^+}\left(\boldsymbol{\delta}\cdot\boldsymbol{n}\right) \quad (23)$$

the indicating function $I_{\mathbb{R}^+}$ makes it possible to take into account the condition of noninterpenetration of the lips of crack:

$$I_{\mathbb{R}^+}\left(\delta_n\right)=\begin{cases}+\infty & \text{si } \boldsymbol{\delta}\cdot\boldsymbol{n}<0\\ 0 & \text{si } \boldsymbol{\delta}\cdot\boldsymbol{n}\geq 0\end{cases} \quad (24)$$

According to the value of the threshold, the energy of surface will be worth Ψ_{dis} or Ψ_{lin} (more the indicating function). In the first case one will speak about dissipative mode, in the second linear case of mode. One can write two energies in the form:

$$\Psi_{dis}\left(\|\boldsymbol{\delta}\|\right)=\int_{\Gamma_e}\psi_{dis}\left(\|\boldsymbol{\delta}\|\right).d\Gamma \text{ and } \Psi_{lin}\left(\|\boldsymbol{\delta}\|,\kappa\right)=\int_{\Gamma_e}\psi_{lin}\left(\|\boldsymbol{\delta}\|,\kappa\right).d\Gamma \quad (25)$$

with ψ_{dis} and ψ_{lin} densities of energy of surface. Let us present in detail the values of these densities.

Density of energy of surface in linear mode

If an existing crack evolves (opening or reclosing) without dissipating energy i.e., $\|\boldsymbol{\delta}\|<\kappa$ the element will be in a linear phase (load or discharge). One thus chooses a density of quadratic energy function of the norm of the jump: (26

$$\Psi_{lin}\left(\|\boldsymbol{\delta}\|,\kappa\right)=\frac{1}{2}.P(\kappa).\|\boldsymbol{\delta}\|^2+C_0 \quad)26$$

is $P(\kappa)=\frac{\sigma_c}{\kappa}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right)$ selected in order to ensure the continuity of derivative of in Ψ (i.e κ the continuity of the stress in the element from one mode to another) and constant C_0 allowing to ensure the continuity of in. Ψ Density κ

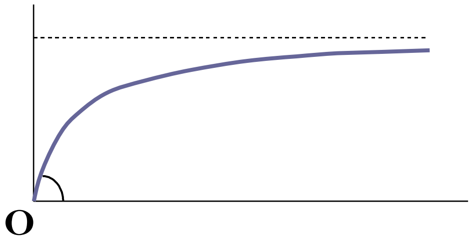
of energy in dissipative mode While

taking as a starting point the idea of Barenblatt [1] 23 account of the process of fracture of interatomic connections, one supposes that the energy of surface east null into zero and grows towards the energy of Griffith when the value of the jump becomes important in front of the length characteristic of the atomic scale. Moreover by leaning on the form of the interatomic potentials one chooses a concave increasing energy and with convex derivative. It is known in particular that concavity plays a crucial role in dimension 1 to limit the number of cracks (see Charlotte et al *. [2]).* 23, in the case where, one $\|\boldsymbol{\delta}\|<\kappa$ chooses one density of energy of surface of the following form, illustrated on figure 2: (2)

$$\Psi_{dis}\left(\|\boldsymbol{\delta}\|\right)=G_{c.}\left[1-\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)\right]$$

represents 27

G_c the critical rate of energy restitution (or tenacity) of the material and the critical stress $\sigma_c=\Psi'_{rev}(0)$. Figure



2: Density 2of energy of surface according to the norm of the jump Vector

2.1.2 forced in the element with discontinuity Is

the vector $\vec{\boldsymbol{\sigma}}=(\sigma_n, \sigma_t)$ forced in the element. When the element is healthy, $(\kappa=0)$ the energy of surface east null. The vector forced is defined starting from the tensor of the stresses of the elastic constitutive law and the norm at line of discontinuity: (28)

$$\vec{\boldsymbol{\sigma}}=\boldsymbol{\sigma}\cdot\boldsymbol{n}$$ It28

thereafter (see § 2.3.1.12.3.1.1 the criterion of starting) that the jump remains null as long as the norm of the vector forced does not reach the critical stress. On the other hand σ_c , if the threshold is not κ null, the vector forced in the element is given by derivative of the density of energy of surface compared to the jump (the vector forced remains equal to since $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ the element ensures the continuity of the normal stress, seeing §2.2.22.2.2

2.1.2.1 mode In linear

mode (), $\|\boldsymbol{\delta}\|<\kappa$ the vector forced is worth: (29)

$$\vec{\boldsymbol{\sigma}}=\frac{\partial\Psi_{lin}}{\partial\boldsymbol{\delta}}=P(\kappa).\boldsymbol{\delta}$$ And29

derivative compared to the jump: (30)

$$\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial\boldsymbol{\delta}}=\boldsymbol{Id}\,.\,P(\kappa)$$ dissipative 30

2.1.2.2 Mode In dissipative

mode (), $\|\boldsymbol{\delta}\|>\kappa$ the vector forced is worth: (31)

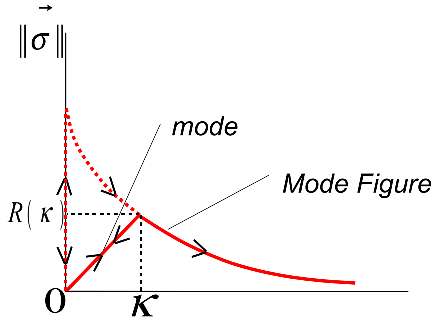
$$\vec{\boldsymbol{\sigma}}=\frac{\partial\Psi_{dis}}{\partial\boldsymbol{\delta}}=\sigma_c.\frac{\boldsymbol{\delta}}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)$$ And31

derivative compared to the jump: (32)

$$\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial\boldsymbol{\delta}}=\sigma_c.\left[\frac{\boldsymbol{Id}}{\|\boldsymbol{\delta}\|}-\frac{\boldsymbol{\delta}}{\|\boldsymbol{\delta}\|}\otimes\frac{\boldsymbol{\delta}}{\|\boldsymbol{\delta}\|}.\left(\frac{\sigma_c}{G_c}+\frac{1}{\|\boldsymbol{\delta}\|}\right)\right].\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)$$ graphic32

2.1.2.3 Illustration One represents

on figure 3 3 of the norm of the vector forced in the element according to the norm of the jump. The deflections represent the possible evolutions of the vector forced following the case (healthy element, linear mode or dissipative mode). Linear



3: 3Vector forced according to the norm of the sautAu threshold in

jump corresponds κ a threshold in stress normalizes which one notes. This $R(\kappa)$ last will determine from which level of stress the crack will dissipate energy. This threshold will evolve with the opening of crack, it depends on and κ is defined by: (33)

$$R(\kappa)=\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right)$$

Let us note33

that when this threshold $\kappa=0$ corresponds to the criterion of starting which we will present to the § 2.3.1.12.3.1.1

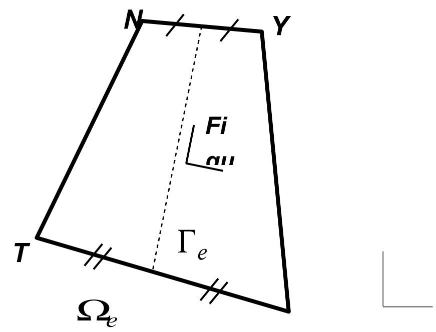
2.2 of the element to discontinuity the element

with discontinuity is a quadrangle with four nodes with a jump of internal displacement. A first part will be devoted to the geometrical description of the element like to the presentation of a parameter setting by leaning on the element of reference. Then, we will see that the element ensures the continuity of the normal stress through the jump of displacement. Then, we will show the unicity of the jump of displacement provided that the size of the element is sufficiently small. To finish, we will see that the choice of a constant jump introduces a parasitic energy of surface which tends towards zero when the mesh is refined. Geometry

2.2.1 of the element and parameter setting Geometry

2.2.1.1 Is

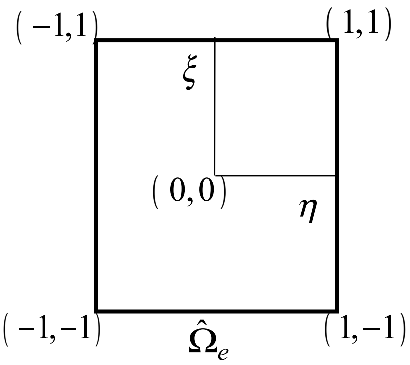
the Cartesian $(\boldsymbol{X},\boldsymbol{Y})$ base constituting a total reference of. $\mathbb{R}^2$ The element with discontinuity, noted is Ω_e a finite element quadrangle with four nodes. It consists of an elastic region and a discontinuity passing Γ_e by the center of the element (segment passing by the mediums on the sides and) $[AD]$ and $[BC]$ length (see l figure 4). D 4 B



4: Element 4with discontinuitéL' directional sense

of discontinuity defines a local coordinate system in the element. $(\boldsymbol{n},\boldsymbol{t})$ The corresponding element of reference is a square defined $\hat{\Omega}_e$ by the field (see $[-1,1]\times[-1,1]$ figure 5). Each 5element has four Gauss points. Those of the element of reference have as coordinates. And the $\left(\pm\sqrt{(3)}/3,\pm\sqrt{(3)}/3\right)$ let us note $\boldsymbol{\omega}_g$

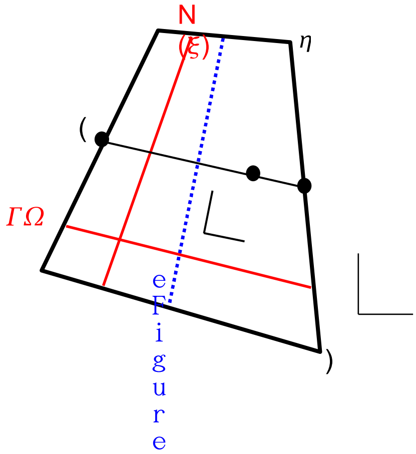
weight $\hat{\omega}_g$ of Gauss points respectively in the real configuration and the reference configuration. Lastly, let us recall that the approximation of the field of displacement in the element was presented in the part § 2.1.1.15



5: Element 5of reference Parameter setting

2.2.1.2 T N YXMPQBCDAT

the geometrical T transformation which at a point of (x, y) the element to discontinuity associates a point of element (η, ξ) of reference. Let us seek to parameterize the position of an unspecified point M of the element to discontinuity by its coordinates (η, ξ) in the reference of the element of reference. Are



, and the P Q points M (see figui

6: Parameter setting 6of the element with discontinuitéSoit

$$\begin{cases} \vec{AP} = \frac{1}{2}.(1+\xi).\vec{AB} \\ \vec{DQ} = \frac{1}{2}.(1+\xi).\vec{DC} \\ \vec{PM} = \frac{1}{2}.(1-\eta).\vec{PQ} \end{cases}$$

By means of34

the sum of the vectors, one a: (35)

$$\vec{AM} = \vec{AP} + \vec{PM} = \frac{1}{2}.(1+\xi).\vec{AB} + \frac{1}{2}.(1-\eta).\vec{PQ}$$ One35

- writing the vector: (36 $\vec{PQ}$)

$$\vec{PQ} = \vec{PA} + \vec{AD} + \vec{DQ}$$ While36

with the statements of (34):3437)

$$\overrightarrow{PQ}=\frac{1}{2}.(1+\xi).\overrightarrow{BA}+\overrightarrow{AD}+\frac{1}{2}.(1+\xi).\overrightarrow{DC}$$ By37

the terms, one obtains finally: (38)

$$\overrightarrow{AM}=\frac{1+\eta}{2}.\frac{1+\xi}{2}.\overrightarrow{AB}+\frac{1-\eta}{2}.\overrightarrow{AD}+\frac{1-\eta}{2}.\frac{1+\xi}{2}.\overrightarrow{DC}$$ And38

: (39)

$$d\,\boldsymbol{M}=\begin{Bmatrix}dx\\dy\end{Bmatrix}=\frac{\boldsymbol{N}(\xi)}{2}.d\eta+\frac{\boldsymbol{T}(\eta)}{2}.d\xi$$ With39

the two vectors of the reference on discontinuity: (40)

$$\begin{cases}\boldsymbol{N}(\xi)=\begin{Bmatrix}N_x\\N_y\end{Bmatrix}=\frac{(1+\xi)}{2}.\overrightarrow{AB}-\frac{(1-\xi)}{2}.\overrightarrow{DC}-\overrightarrow{AD}\\ \boldsymbol{T}(\eta)=\begin{Bmatrix}T_x\\T_y\end{Bmatrix}=\frac{(1+\eta)}{2}.\overrightarrow{AB}+\frac{(1-\eta)}{2}.\overrightarrow{DC}\end{cases}$$ There40

then the parameter setting: (41)

$$\begin{Bmatrix}dx\\dy\end{Bmatrix}=[\boldsymbol{J}].\begin{Bmatrix}d\eta\\d\xi\end{Bmatrix}$$ Where41

is $[\boldsymbol{J}]$ the jacobian matrix of the transformation: (42)

$$[\boldsymbol{J}]=\frac{1}{2}.\begin{bmatrix}N_x & T_x\\N_y & T_y\end{bmatrix}$$ Note:42

: Discontinuity being at the center of the element, one can define the vectors of the local coordinate system in this one like its length starting from the vector in: $\boldsymbol{T}$ (43 $\eta=0$)

$$\begin{cases}\boldsymbol{n}=R_{-\pi/2}\,\boldsymbol{T}(0)/\|\boldsymbol{T}(0)\|\\ \boldsymbol{t}=\boldsymbol{T}(0)/\|\boldsymbol{T}(0)\|\\ l=\|\boldsymbol{T}(0)\|\end{cases}$$ Continuity 43

2.2.2 of the normal stress By

leaning on the preceding parameter setting, one shows in [6] how 23 the element with discontinuity ensures the continuity of the normal stress through the jump of displacement when the stress is homogeneous in the element. Known as differently, it is shown that the vector forced of $\vec{\sigma}=\psi'(\delta)$ constitutive law CZM_EXP (see §2.1.2)2.1.2 equal to the normal stress in the element. Condition $\boldsymbol{\sigma}.\boldsymbol{n}$

2.2.3 of existence and unicity of the jump in the element From

a numerical point of view it is important to be placed in a case where the search of the jump in the element led to a single solution. In other words it is necessary that the solution of the problem of minimization (44) has44 only one solution with and built-in $\boldsymbol{U}$ κ . (44)

$$\min_{\delta\in\mathbb{R}^2} E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)$$ In44

[6] it 23 that the existence and the unicity of the jump are assured as soon as the following condition, bearing on the geometry like on the material parameters of the element, is assured: (45)

$$\frac{\mu}{16}.\sum_g\left(\frac{1}{\omega_g}.\min_{\eta_g}\|\boldsymbol{T}(\eta_g)\|^2\right)>l.\frac{\sigma_c^2}{G_c}$$ Note:45

: In the typical case where the element with discontinuity is rectangular, the weights of Gauss points are equal to a quarter of the surface of the element. Moreover, the length of the element is equal to the length of discontinuity. If l the width of e the element is noted, one has for all: (46 g)

$$\omega_g=\frac{l.e}{4}$$ Moreover4 6

on each $\|\boldsymbol{T}\|=l$ Gauss point, therefore the condition of unicity becomes a condition over the width of the element: (47)

$$e<\mu.\frac{G_c}{\sigma_c^2}$$ It47 is

this last condition which makes it possible to ensure the unicity of the jump for the elements discontinuity in Code_Aster *. It is thus necessary* to handle with precaution the latter when they are not rectangular form and to make sure that the condition given by (45)45 is well checked. Energy

2.2.4 parasitizes the constant

jump in the element with discontinuity results in introducing a parasitic energy with the interface between two adjacent elements. One shows in [6] that 23 energy tends towards zero when the mesh is refined. Minimization

2.3 of total energy By adopting

the principle of minimization of energy, the goal of this part is to present the computation of the jump of displacement in the elements to discontinuity like that of the field of displacement. Total energy (see §2.1.1)2.1.1 is not convex with respect to the couple. The search $(\boldsymbol{U},\boldsymbol{\delta})$ for a total minimum for such a functional calculus is not possible with the numerical method which we will use. Moreover, with imposed force, total energy not being limited in a lower position, the total minimum does not exist. These two arguments bring us has to make the choice to search a local minimum. The problem of minimization of total energy is written: (48)

$$\text{Chercher } (\boldsymbol{U}^*,\boldsymbol{\delta}^*) \text{ minimum local de l'énergie totale } E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)$$ In48

a first section (§2.3.1)12 to calculate minimum $\boldsymbol{\delta}^*$ room of total energy to built-in $\boldsymbol{U}$: (49)

$$\boldsymbol{\delta}^*(\boldsymbol{U})=\underset{\delta}{\operatorname{argmin}}\, E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)$$ In49

the second section (§2.3.2)16 will be satisfied to seek a requirement so that is $\boldsymbol{U}^*$ minimum room of total energy with: (50 $\boldsymbol{\delta}=\boldsymbol{\delta}^*(\boldsymbol{U})$)

$$\boldsymbol{U}^*=\underset{U}{\operatorname{argmin}}\, E_T(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)$$ This50

second stage amounts solving the balance equations by taking account of possible works of the external forces and the displacements imposed on structure. Note:

The digital processing of the indicatrix translating $I_{\mathbb{R}^+}(\boldsymbol{\delta}\cdot\boldsymbol{n})$ the condition of NON-interpenetration of the lips of crack, will be carried out by adding the stress to the problem $\boldsymbol{\delta}\cdot\boldsymbol{n}\geq 0$ of minimization. Minimization

2.3.1 of total energy compared to the jump the object

of this section is to calculate the jumps of displacement, on each element to discontinuity of a mesh, which minimizes total energy. If one index by the elements i with discontinuity and that one notes the jumps $\boldsymbol{\delta}_i$ of displacement on each one of these elements, the problem of minimization is written: (51)

$$\text{Chercher les } \boldsymbol{\delta}_i \text{ minima locaux de l'énergie totale, solutions de } \min_{\boldsymbol{\delta}_i,\boldsymbol{\delta}_i\cdot\mathbf{n}\geq 0}\sum_i E_T(\boldsymbol{U},\boldsymbol{\delta}_i,\kappa)$$ Two51

important points will make it possible to simplify the problem appreciably. First of all, the choice of a jump of displacement independent from one element to another makes it possible to minimize total energy compared to the jump on an elementary level (technique of static condensation). There is the following equality: (52)

$$\min_{\boldsymbol{\delta}_i,\boldsymbol{\delta}_i\cdot\boldsymbol{n}\geq 0}\sum_i E_T(\boldsymbol{U},\boldsymbol{\delta}_i,\kappa)=\sum_i\min_{\boldsymbol{\delta}_i,\boldsymbol{\delta}_i\cdot\boldsymbol{n}\geq 0}E_T(\boldsymbol{U},\boldsymbol{\delta}_i,\kappa)$$ In addition52

, one saw in the §2.2.32.2.3 for sufficiently small elements, the jump of displacement is single. Thus, the problem is brought back to a search for total minimum on each element: (53)

$$\text{Sur chaque élément chercher }\ \boldsymbol{\delta}\in S\ \text{ minimum global de } E_T(\boldsymbol{U},\boldsymbol{\delta}_i,\kappa)$$ with53

the group S of the jump acceptable: (54)

$$S=\{\boldsymbol{s}/\boldsymbol{s}\ \text{constant par élément et }\ \boldsymbol{s}\cdot\boldsymbol{n}\geq 0\}$$ 54

known as condition of optimality of order one becomes necessary and sufficient to determine the solution of the problem. This one is written in the form: (55)

$$E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)\leq E_T(\boldsymbol{U},\boldsymbol{\delta}+h\boldsymbol{\phi},\kappa)$$ For55

very constant $\boldsymbol{\phi}$ by element and sufficiently h small so that. While $\boldsymbol{\delta}+h\boldsymbol{\phi}\in S$ passing in extreme cases when tends h towards zero, becomes a condition on directional derivative: (56)

$$\lim_{h\to 0}\frac{1}{h}.\left(E_T(\boldsymbol{U},\boldsymbol{\delta}+h\boldsymbol{\phi},\kappa)-E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)\right)\geq 0$$ For56

very constant $\boldsymbol{\phi}$ by element such as. Generally $\boldsymbol{\delta}+h\boldsymbol{\phi}\in S$, the directional derivative is a function positively homogeneous of degree one in. Thereafter $\boldsymbol{\phi}$

we will present the computation of the jump displacement on an element, by leaning on the condition of optimality. One will start by highlighting a requirement and sufficient, which one will call criterion of starting, so that the null jump is solution of the problem. Then we will detail the computation of the jump after starting, for the two types of behavior of the element: linear and dissipative. Criterion

2.3.1.1 of starting In

this part we will seek to determine with which condition the jump of null displacement is solution of the problem (53). Let us note53 that the choice to make depend energy on surface of the norm of the jump implies that total energy does not admit a derivative into zero. This fact the derivative of energy in the direction will not be $\boldsymbol{\phi}$ a function linear but positively homogeneous of degree one in. By means of $\boldsymbol{\phi}$ the definition of total energy (see §2.1.1)5 the element is healthy (i.e when), one calculates $\kappa=0$ its derivative in the direction into zero $\boldsymbol{\phi}$: (57)

$$\lim_{h\to 0}\frac{1}{h}.\left(E_T(\boldsymbol{U},h\boldsymbol{\phi},\kappa)-E_T(\boldsymbol{U},0,\kappa)\right)=-\sum_g\omega_g.\phi^t.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}+l.\sigma_c.\|\phi\|$$ the condition57

of optimality (55) thus55 led to the inequality: (58) For

$$-\sum_g\omega_g.\phi^t.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}+l.\sigma_c.\|\phi\|\geq 0$$ 58

. However, $\boldsymbol{\phi}\in S$ to show the continuity of the normal stress (see [6]), it 23 that: (59) Moreover

$$\sum_g\omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}=l.\vec{\boldsymbol{\sigma}}$$ 59

is constant $\boldsymbol{\phi}$ on the element, therefore (58) becomes58: (60) While

$$-\vec{\boldsymbol{\sigma}}\cdot\boldsymbol{\phi}+\sigma_c.\|\boldsymbol{\phi}\|\geq 0$$ posing60

with and $\begin{cases}\phi_n=\rho.\cos\theta\\ \phi_t=\rho.\sin\theta\end{cases}$ so that $\rho>0$ $\frac{-\pi}{2}\leq\theta\leq\frac{\pi}{2}$, and by means of $\phi_n\geq 0$ the definition, one obtains $\vec{\boldsymbol{\sigma}}=(\sigma_n,\sigma_t)$ the condition: (61)

$$\sigma_n.\cos\theta+\sigma_t.\sin\theta\leq\sigma_c \qquad \forall\,\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$ a study61

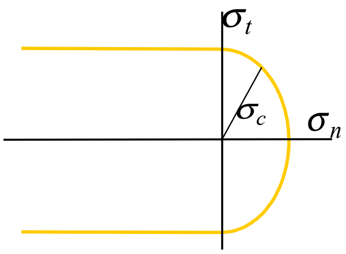
of the function gives: $f(\theta)=\sigma_n.\cos\theta+\sigma_t.\sin\theta$ (62) With

$$\sup_{-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}} f(\theta)=\sqrt{<\sigma_n>_+^2+\sigma_t^2} \qquad 62$$

the definition of the positive part of a quantity. The inequality $<.>_+=max(.,0)$ (61) thus61 led to the criterion of starting in stress according to: (63) One

$$\sqrt{<\sigma_n>_+^2+\sigma_t^2}\leq\sigma_c$$ represents 63

this criterion in the plane on figure (σ_n,σ_t) 7. Figure 7



7: Criterion 7of starting in stress of the element to discontinuity Note:

One places figure in the case of where the directional sense of discontinuity is built-in $\boldsymbol{n}$ (by the mesh). That explains why one obtains a condition on the vector forced and not on the principal components of the stress. Computation of

2.3.1.2 the jump Let us seek

to solve the problem of minimization if the jump is not null. In this case, the directional derivative is a linear function of, the condition $\boldsymbol{\phi}$ of optimality (55) is written55: (64) For

$$\lim_{h\to 0}\frac{1}{h}.\left(E_T(\boldsymbol{U},\boldsymbol{\delta}+h\boldsymbol{\phi},\kappa)-E_T(\boldsymbol{U},\boldsymbol{\delta},\kappa)\right)=\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}\geq 0 \qquad 64$$

constant $\boldsymbol{\phi}$ by element such as. While exploiting $\boldsymbol{\delta}+h\boldsymbol{\phi}\in S$ the acceptable directions one from of deduced the conditions: (65) In addition

$$\begin{cases}\dfrac{\partial E_T}{\partial\delta_n}.\phi_n\geq 0 & \forall\,\phi_n\geq 0 \quad \text{si} \quad \delta_n=0\\ \dfrac{\partial E_T}{\partial\delta_n}.\phi_n=0 & \forall\,\phi_n \qquad \text{si} \quad \delta_n>0\\ \dfrac{\partial E_T}{\partial\delta_t}.\phi_t=0 & \forall\,\phi_t\end{cases} \qquad 65$$

, by approaching the integral by a discrete sum on Gauss points, the derivative of total energy in the direction is written $\boldsymbol{\phi}$: (66) With

$$\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}=-\sum_g\omega_g.\phi^t.\boldsymbol{D}_g^t.\boldsymbol{E}.\left(\boldsymbol{B}_g.\boldsymbol{U}-\boldsymbol{D}_g\boldsymbol{\delta}\right).\boldsymbol{\phi}+l.\frac{\partial\psi}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi} \qquad 66$$

the density ψ of energy of surface, equalizes with in linear ψ_{lin} mode and with dissipative ψ_{dis} mode. To reduce the notations one defines in the following way $\boldsymbol{S}$: (67) and

$$\boldsymbol{S}=\begin{Bmatrix}S_n\\S_t\end{Bmatrix}=-\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}$$:6768

) There $\boldsymbol{Q}$

$$\boldsymbol{Q}=\begin{bmatrix}Q_{nn} & Q_{nt}\\Q_{nt} & Q_{tt}\end{bmatrix}=-\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{D}_g$$ is68

the writing simplified of (66): (6966 now

$$\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.\frac{\partial\psi}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}$$ 69

Let us seek to calculate the jump displacement starting from the conditions by means of the derivative of the energy written in the form (69). One will distinguish69 computation in the linear mode from that in the dissipative mode and for each one of them one will distinguish the case or the normal jump is null of that where it is not it. Computation of

the jump in linear mode In linear

mode density of energy of surface east given by: (70)

$$\psi=\psi_{lin}=\frac{1}{2}.P(\kappa).\boldsymbol{\delta}\cdot\boldsymbol{\delta}$$ the derivative 70

of total energy compared to the jump (69) becomes69: (71) In the case of

$$\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.P(\kappa).\boldsymbol{\delta}\cdot\boldsymbol{\phi}$$ 71

a normal jump no one, the conditions $(0,\delta_t)$ of optimality (65) lead65 to: (72)

$$\begin{cases}S_n+Q_{nt}.\delta_t\geq 0\\ \delta_t=-\dfrac{S_t}{Q_t+l.P(\kappa)}\end{cases}$$ the tangent72

jump is given explicitly by the second condition. The first imposes that the normal stress in the element is negative or null. If such were not the case, there would be necessarily a normal jump. Known as differently, the jump in the element will be null as soon as the element is put in compression. This translated the taking into account of the NON-interpenetration of the lips of crack. In the case of

a strictly positive normal jump, the conditions $(\delta_n>0,\delta_t)$ of optimality (65) give65 the jump of displacement directly: (73) In

$$\boldsymbol{\delta}=-\left(\boldsymbol{Q}+\boldsymbol{Id}.l.P(\kappa)\right)^{-1}.\boldsymbol{S}$$ linear73

mode the local variable does not evolve κ . Computation of

the jump in dissipative mode In dissipative

mode density of energy of surface east given by: (74)

$$\psi=\psi_{dis}=G_c.\left[1-\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)\right]$$ the derivative 74

of total energy compared to the jump (69) becomes69: (75) In the case of

$$\frac{\partial E_T}{\partial\boldsymbol{\delta}}\cdot\boldsymbol{\phi}=(\boldsymbol{S}+\boldsymbol{Q}.\boldsymbol{\delta})\cdot\boldsymbol{\phi}+l.\sigma_c.\frac{\boldsymbol{\delta}\cdot\boldsymbol{\phi}}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right)$$ 75

a normal jump no one, the conditions $(0,\delta_t)$ of optimality (65) lead65 to: (76)

$$\begin{cases} S_n + Q_{nt}.\delta_t \geq 0 \\ S_t + Q_{tt}.\delta_t + l.\sigma_c.\text{sign}(\delta_t).\exp\left(-\frac{\sigma_c}{G_c}.|\delta_t|\right) = 0 \end{cases}$$ the first76

condition implies that the normal jump is null when the normal stress in the element is negative. What translates the taking into account of the NON-interpenetration of the lips of crack. Second condition, one deduces that, therefore $\text{sign}(\delta_t) = -\text{sign}(S_t)$ with is $\delta_t = -\text{sign}(S_t).\beta$ solution $\beta > 0$ of the following nonlinear scalar equation: (77) that

$$|S_t| + Q_n.\beta + l.\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\beta\right) = 0$$ one77

solves by an algorithm of Newton. In the case of
a strictly positive normal jump, the conditions $(\delta_n > 0, \delta_t)$ of optimality (65) lead65 to the following nonlinear equation: (78) One

$$\boldsymbol{S} + \boldsymbol{Q}.\boldsymbol{\delta} + l.\sigma_c.\frac{\boldsymbol{\delta_t}}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right) = 0$$ notes78

with and $\boldsymbol{\delta} = r.\tilde{\boldsymbol{\delta}}$ where $r > 0$ so that $\tilde{\boldsymbol{\delta}} = (\cos\theta, \sin\theta)$ $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. According to $\tilde{\boldsymbol{\delta}} \cdot \boldsymbol{n} \geq 0$ (78) one a:78 (79) It

$$\tilde{\boldsymbol{\delta}} = -\left(r.\boldsymbol{Q} + l.\sigma_c.\boldsymbol{Id}.\exp\left(-\frac{\sigma_c}{G_c}.r\right)\right)^{-1}.\boldsymbol{S}$$ remains79

to calculate. For that r it is enough to solve the nonlinear scalar equation by an algorithm $\|\tilde{\boldsymbol{\delta}}\| = 1$ of Newton. In dissipative mode the local variable evolves, κ it corresponds to the norm of the jump calculated. Calculation algorithm $\kappa = \|\tilde{\boldsymbol{\delta}}\|$

2.3.1.3 In this

part, one presents the calculation algorithm of the jump on an element $\boldsymbol{\delta}^*$ given as well as the evolution of the local variable. Beginning κ of

1.computation Test of existence
2.and unicity of the solution (see §2.2.3) If2.2.3
3.) then $\kappa = 0$ $\sqrt{<\sigma_n>_+^2 + \sigma_t^2} \leq \sigma_c$, the threshold
- $\boldsymbol{\delta}^* = \boldsymbol{0}$ So then κ remains null → Fine **computation**

1.Computation $\kappa > 0$ of
- the threshold in linear mode (see 2.3.1.2 2.3.1.2 , then $\boldsymbol{\delta}^c$
  - $\|\boldsymbol{\delta}^c\| < \kappa$, the threshold $\boldsymbol{\delta}^* = \boldsymbol{\delta}^c$ does not evolve κ → Fine of computation **Computation of**

1.the jump in dissipative mode (see § 2.3.1.2 2.3.1.2 actualization $\boldsymbol{\delta}^c$
- $\boldsymbol{\delta}^* = \boldsymbol{\delta}^c$ of the Fine threshold → $\kappa = \|\boldsymbol{\delta}^*\|$ of the Fine **computation of computation**

1.Minimization

2.3.2 of total energy compared to displacements After

having calculated the jumps of displacement on each $\boldsymbol{\delta}^*(\boldsymbol{U})$ element to discontinuity, the purpose is to calculate the field of displacements which minimizes total energy. The problem is formulated in the following way: (80)

$$\text{Chercher } \boldsymbol{U}^* \text{ minimum local de l'énergie totale } E_T(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)$$ the functional calculus80

is not $E_T(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)$ convex in. To be $\boldsymbol{U}$ minimum room, the solution must check the equilibrium conditions and of stability. The second condition relates to derivative second of the functional calculus and requires the use of a method specifically dedicated to minimization. The algorithm of Newton used does not make it possible to check this second condition. One will thus be satisfied to check the equilibrium condition, requirement so that displacement is solution of the problem (80). Let us consider80

a structure with Ω the following loading: density

- $\boldsymbol{f}$ of volume force on; density Ω
- $\boldsymbol{F}$ of force surface on; displacements Γ_N
- $\boldsymbol{U}^d$ imposed on and are Γ_D

Γ_D Γ_N disjoined parts of the border of. The work Ω of the external forces is written: (81)

$$W^{ext}(\boldsymbol{U})=\int_\Omega \boldsymbol{f}.\boldsymbol{U}.d\Omega+\int_{\Gamma_N}\boldsymbol{F}.\boldsymbol{U}.d\Gamma_N$$ the total81

energy of structure is written then: (82) With

$$E_T(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)=\Phi(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}))+\Psi(\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)-W^{ext}(\boldsymbol{U}) \quad 82$$

$\boldsymbol{U}$ to the space of the kinematically admissible fields of displacement. The condition of optimality of order one, requirement so that is to say minimum $\boldsymbol{U}$ room, is written: (83) For

$$\frac{\partial E_T}{\partial \boldsymbol{U}}(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa).\boldsymbol{V}\geq 0 \quad 83$$

field acceptable test $\boldsymbol{V}$. It is known that: (84) What

$$\frac{\partial E_T}{\partial \boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{\delta}^*(\boldsymbol{U}),\kappa)=0$$ 84gives

: (85) This

$$\int_\Omega \boldsymbol{B}^t.\boldsymbol{E}.(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}^*(\boldsymbol{U})).\boldsymbol{V}.d\Omega-\int_\Omega \boldsymbol{f}.\boldsymbol{V}.d\Omega-\int_{\Gamma_N}\boldsymbol{F}.\boldsymbol{V}.d\Gamma_N\geq 0 \quad \forall \boldsymbol{V} \quad 85$$

becomes an equality by taking good chosen $\boldsymbol{V}$, and, as this equality is true for very acceptable $\boldsymbol{V}$, one obtains the equilibrium between the internal forces and outsides: (86) With

$$\boldsymbol{F}^{\text{int}}(\boldsymbol{U})=\boldsymbol{F}^{\text{ext}} \quad 86$$

(87) Let us note

$$\begin{cases}\boldsymbol{F}^{\text{int}}(\boldsymbol{U})=\int_\Omega \boldsymbol{B}^t.\boldsymbol{E}.(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}^*).d\Omega\\ \boldsymbol{F}^{\text{ext}}=\int_\Omega \boldsymbol{f}.d\Omega+\int_{\Gamma_N}\boldsymbol{F}.d\Gamma_N\end{cases} \quad 87$$

the linear $\boldsymbol{C}$ operator translating the imposed conditions of displacement. The dualisation of these conditions leads to the following system: (88)

$$\begin{cases}\boldsymbol{F}^{\text{int}}(\boldsymbol{U})+\boldsymbol{C}^t.\lambda=\boldsymbol{F}^{\text{ext}}\\ \boldsymbol{C}.\boldsymbol{U}=\boldsymbol{U}^d\end{cases}$$ the unknowns 88

are now at any times the couple where represents $(\boldsymbol{U},\boldsymbol{\lambda})$ $\boldsymbol{\lambda}$ the Lagrange multipliers associated with the conditions with Dirichlet. The method used to solve is a method of Newton (confer to the documentation of STAT_NON_LINE [R5.03.01 }). The computation of the tangent matrix is explained in appendix. The purpose of continuation method

3 of the loading the continuation method

of the loading is taking into account instabilities of structure, and to thus follow possible "back return" of the total response forces displacement. A typical case of this method, initially developed by Lorentz and Badel [7], 23was adapted to the preceding model of cracking. The principle
of the continuation method as well as the resolution of the new total system which results from this is explained in detail in [R5.03.80]. We will detail the part of control specific to model CZM_EXP by explaining the choice of the equation of control of control as well as the method to solve it. Equation

3.1 of control of control the goal of

this part is to present the equation of control of control like its resolution for the model to internal discontinuity CZM_EXP. The unknown of this equation is the intensity of the loading at the time η_i^n of computation and i the iteration of Newton which one n will note henceforth to simplify η the notation. As we saw with the §2.1 the constitutive law2.1 of the element to discontinuity is controlled by a threshold. Let us note the function F_{el} threshold in following stress: (89) With

$$F_{el}(\sigma_n,\sigma_t)=\sqrt{<\sigma_n>_+^2+\sigma_t^2}-R(\kappa) \quad 89$$

the vector jump of stress which is worth: (90) And

$$\vec{\sigma}=\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\left(\boldsymbol{B}_g.\boldsymbol{U}-\boldsymbol{D}_g\boldsymbol{\delta}\right) \quad 90$$

the variable $R(\kappa)$ of threshold in stress defined by (33): (9133 Let us note

$$R(\kappa)=\sigma_c.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right) \quad 91$$

the function G_{el} threshold in jump: (92) With

$$G_{el}(\delta_n,\delta_t)=\sqrt{\delta_n^2+\delta_t^2}-\kappa \quad 92$$

the threshold $\delta_n\geq 0$ κ in jump. Let us choose an equation of control of control so that the intensity of the loading makes leave η the criterion at least an element to discontinuity of structure of a quantity proportional to. The functions $\Delta\tau$ thresholds depend then on and will note η we them and. $\tilde{F}_{el}(\eta)$ The equation $\tilde{G}_{el}(\eta)$ of control of control is written: For

- the criterion in stress: (93) For

$$\tilde{P}(\eta)\underset{\text{def}}{=}\max_j \tilde{F}_{el}^j(\eta)=R(\kappa^-).\Delta\tau \quad 93$$

- the criterion in jump: (94) Where

$$\tilde{P}(\eta)\underset{\text{def}}{=}\max_j \tilde{G}_{el}^j(\eta)=\kappa^-.\Delta\tau \quad 94$$

j the index of the elements to discontinuity of the mesh and the variable κ^- threshold at time. Note: $i-1$

In the case where

- , the jump $\kappa^-=0$ being null, we use the equation of control of the control expressed with the criterion in stress (93). In the case where93
- , we use $\kappa^->0$ the equation of control of the control expressed with the criterion in jump (94). Indeed94, the jump not being known, one cannot use the stresses which depend (σ_n,σ_t) explicitly on this one. Resolution

3.1.1 of the equation with the criterion in stress We

thus place ourselves if. The goal $\kappa^- = 0$ here is to explain the resolution of (93) for93 a given element, without taking account of the elements max on all the (the taking into account of the east explained max in [R5.03.80]). One thus seeks to solve: (95) initially

$$\tilde{F}_{el}(\eta) = R(0) . \Delta\tau \quad 95$$

Let us reveal explicitly in η the statement of the criterion before explaining $\tilde{F}_{el}(\eta)$ the resolution of the equation. At time one seeks i the displacement which U_i is expressed with the iteration of Newton: (96) One n

$$\mathbf{U}_i = \mathbf{U}_{i-1} + \Delta\,\mathbf{U}_i^n + \delta\,\mathbf{U}_{impo,i}^n + \eta_i . \delta\,\mathbf{U}_{pilo,i}^n$$ separates96

the known part (within the meaning of non-pilotéee) of displacement: (97) And

$$\mathbf{U}_{impo} = \mathbf{U}_{i-1} + \Delta\,\mathbf{U}_i^n + \delta\,\mathbf{U}_{impo,i}^n \quad 97$$

part: (98) Directly

$$\mathbf{U}_{pilo} = \delta\,\mathbf{U}_{pilo,i}^n \quad 98$$

: (99)

$$\mathbf{U}_i = \mathbf{U}_{impo} + \eta . \mathbf{U}_{pilo}$$ the jump99

of displacement being no one a: (100) By means of

$$\begin{Bmatrix} \sigma_n \\ \sigma_t \end{Bmatrix} = \frac{1}{l} . \sum_g \omega_g . \boldsymbol{D}_g^t . \boldsymbol{E} . \boldsymbol{B}_g . \boldsymbol{U}_i \quad 100$$

decomposition (99): (10199) With

$$\begin{Bmatrix} \sigma_n \\ \sigma_t \end{Bmatrix} = \mathbf{S}_{impo} + \eta . \mathbf{S}_{pilo} \quad 101$$

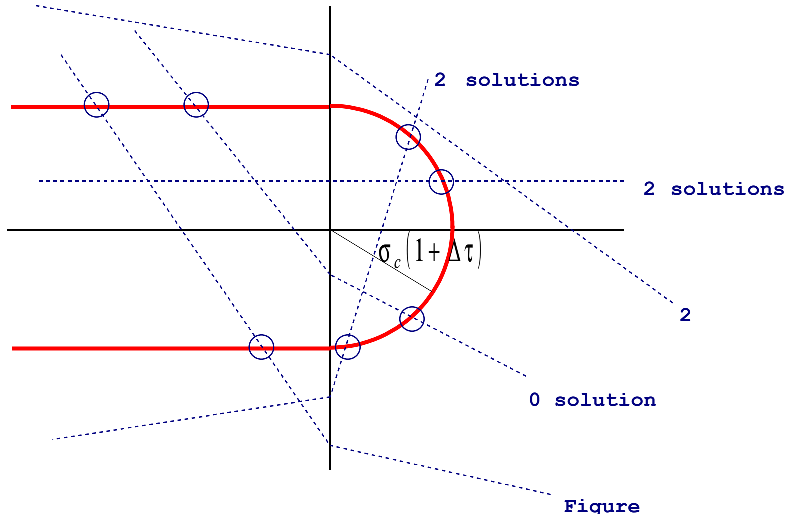
(102)

$$\mathbf{S}_{impo} = \frac{1}{l} . \sum_g \omega_g . \boldsymbol{D}_g^t . \boldsymbol{E} . \boldsymbol{B}_g . \boldsymbol{U}_{impo}$$
$$\mathbf{S}_{pilo} = \frac{1}{l} . \sum_g \omega_g . \boldsymbol{D}_g^t . \boldsymbol{E} . \boldsymbol{B}_g . \boldsymbol{U}_{pilo}$$ the equation102

of control of control is thus written, by taking account owing to the fact that: (103) $\kappa^- = 0$ This

$$\tilde{F}_{el}(\eta) \underset{\text{def}}{=} \sqrt{\langle(\mathbf{S}_{\text{impo}} + \eta . \mathbf{S}_{\text{pilo}}) \cdot \boldsymbol{n}\rangle_+^2 + ((\mathbf{S}_{\text{impo}} + \eta . \mathbf{S}_{\text{pilo}}) \cdot \boldsymbol{t})^2} - \sigma_c = \sigma_c . \Delta\tau \quad 103$$

corresponds to a polynomial of degree two, which one notes, if and p_1 with $(\mathbf{S}_{\text{impo}} + \eta . \mathbf{S}_{\text{pilo}}) \cdot \boldsymbol{n} > 0$ a polynomial of degree two that one notes in p_2 the contrary case. 1 solution



8: Left 8the criterion in stress to find

, these η two polynomials are solved. That returns, in space, to find (σ_n, σ_t) the intersection of two half-lines with the criterion in stress. The figure (8) represents8 the "output of criterion" controlled by the parameter. Each one $\Delta\tau$ of

these two polynomials has 0,1 or 2 solutions, one will admit the solutions of which check p_1 solutions and of the $(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}>0$ which checks p_2 . The allowed $(\mathbf{S}_{\text{impo}}+\eta.\mathbf{S}_{\text{pilo}})\cdot\boldsymbol{n}\leq 0$ solutions will be to more both, one notes them. Resolution $\eta_{k=0,1,2}$

3.1.2 of the equation with the criterion in jump the goal here

is to solve the equation of control of control for a given element in the case where, always $\kappa^{-}>0$ without taking account of the elements max on all the. One seeks to solve: (104) initially

$$\tilde{G}_{el}(\eta)=\kappa^{-}.\Delta\tau \qquad 104$$

Let us reveal explicitly in η the statement of the criterion then detail $\tilde{G}_{el}(\eta)$ the computation of (104). One104 like in the case of the criterion in stress (§3.1.1). 18 relates to the writing of the jump of stress in the general case (when the jump is not null): (105) That

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.(\boldsymbol{B}_g.\boldsymbol{U}_i-\boldsymbol{D}_g.\boldsymbol{\delta}) \qquad 105$$

breaks up into three parts: (106) With

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=\mathbf{S}_{impo}+\eta.\mathbf{S}_{pilo}+\boldsymbol{Q}.\boldsymbol{\delta} \qquad 106$$

(107) In addition

$$\begin{aligned}\mathbf{S}_{impo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{impo}\\ \mathbf{S}_{pilo}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{B}_g.\boldsymbol{U}_{pilo}\\ \mathbf{Q}&=\frac{1}{l}.\sum_g \omega_g.\boldsymbol{D}_g^t.\boldsymbol{E}.\boldsymbol{D}_g\end{aligned} \qquad 107$$

for a threshold in built-in jump one $\kappa^{-}>0$ has, according to constitutive law CZM_EXP : (108) With

$$\begin{Bmatrix}\sigma_n\\ \sigma_t\end{Bmatrix}=P(\kappa^-).\boldsymbol{\delta}$$ 108

From (106 $P(\kappa^-)=\frac{\sigma_c}{\kappa^-}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa^-\right)$) and (106) one deduces108: (109) What

$$\mathbf{S}_{impo}+\eta.\mathbf{S}_{pilo}+\boldsymbol{Q}.\boldsymbol{\delta}=P(\kappa^-).\boldsymbol{\delta}$$ 109makes it possible

to know according to $\boldsymbol{\delta}$ the parameter of control: (110) η With

$$\boldsymbol{\delta}=\begin{Bmatrix}\delta_n\\ \delta_t\end{Bmatrix}=\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}}$$ 110

(111)

$$\boldsymbol{\delta}_{\text{impo}}=(P(\kappa^-).\boldsymbol{Id}-\boldsymbol{Q})^{-1}.\boldsymbol{S}_{\text{impo}}$$
$$\boldsymbol{\delta}_{\text{pilo}}=(P(\kappa^-).\boldsymbol{Id}-\boldsymbol{Q})^{-1}.\boldsymbol{S}_{\text{pilo}}$$ the equation111

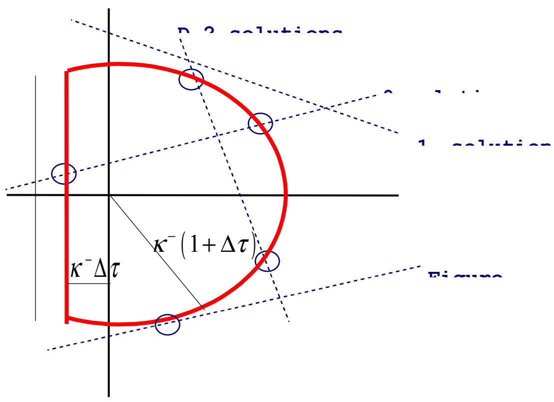
of control of control is thus written: (112) With

$$\tilde{G}_{el}(\eta)\underset{\text{def}}{=}\sqrt{((\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n})^2+((\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{t})^2}-\kappa^-=\kappa^-.\Delta\tau$$ 112

the jump condition normal positive: (113)

$$(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}\geq 0$$ the criterion113

is reduced to a half rings in space. To be able (δ_n,δ_t) to use control when the normal jump tends to being negative (compression of the element) we let us authorize a small output of criterion for such jumps. The "output of criterion", controlled by the parameter, can be $\Delta\tau$ represented in space by an arc of a circle (δ_n,δ_t) and a segment (see figure 9). 2 solutions 9



9: Left 9the criterion in jump Note

: It is understood that only the prediction of the algorithm of Newton will leave the criterion, therefore in particular could violate the condition of contact. The solution of the problem, as for it, will respect it. The resolution
of the equation of control of control thus amounts finding the intersection between a line and an arc of a circle or the intersection between a line and a segment. In short

, one solves: The polynomial

- (112), of112 two in (right η intersection/circle): one admits which checks η ; The polynomial $(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}\geq-\kappa^-.\Delta\tau$

•, of degree $(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\cdot\boldsymbol{n}=-\kappa^{-}.\Delta\tau$ one in: (right η $(\boldsymbol{\delta}_{p}+\eta\boldsymbol{\delta}_{d})\cdot\boldsymbol{n}=-\kappa^{-}\Delta\tau$ intersection/segment), one admits which checks η with. $\left|(\boldsymbol{\delta}_{\text{impo}}+\eta.\boldsymbol{\delta}_{\text{pilo}})\right|\cdot\boldsymbol{t}\leq d$ $d=\kappa^{-}.\sqrt{1+2.\Delta\tau}$

The allowed solutions $\eta_{k=0,1,2}$ will be noted. As regards choice among these solutions one will refer to [R5.03.80]. APPENDIX

4 : Computation of the tangent matrix for the element with discontinuity In this

part we will detail the computation of derivative of the internal forces compared to displacement: who appears $\frac{\partial \boldsymbol{F}^{int}}{\partial \boldsymbol{U}}$ in tangent K_T matrix of the system intervening in the algorithm of Newton. One a: (114) Thus

$$\boldsymbol{F}^{\text{int}}(\boldsymbol{U})=\int_{\Omega}\boldsymbol{B}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}^{*}\right).d\,\Omega \qquad 114$$

(115) It

$$\frac{\partial \boldsymbol{F}^{\text{int}}(\boldsymbol{U})}{\partial \boldsymbol{U}}=\int_{\Omega}\boldsymbol{B}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}-\boldsymbol{D}.\frac{\partial \boldsymbol{\delta}^{*}}{\partial \boldsymbol{U}}\right).d\,\Omega$$

is necessary 115

to calculate the term on each $\frac{\partial \boldsymbol{\delta}^{*}}{\partial \boldsymbol{U}}$ element with discontinuity (static Ω_e condensation). Minimization compared to total $\boldsymbol{\delta}$ energy led us to calculate solution $\boldsymbol{\delta}^{*}$ of the equation: (116) Let us derive

$$\int_{\Omega_e}\boldsymbol{D}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\boldsymbol{\delta}^{*}\right).d\,\Omega=l.\psi'\left(\delta^{*}\right) \qquad 116$$

this equation compared to: (117) $\boldsymbol{U}$ Thus

$$\int_{\Omega_e}\boldsymbol{D}^{t}.\boldsymbol{E}.\left(\boldsymbol{B}.\boldsymbol{U}-\boldsymbol{D}.\frac{d\,\boldsymbol{\delta}^{*}}{d\,\boldsymbol{U}}\right).d\,\Omega=l.\psi''\left(\delta^{*}\right).\frac{d\,\boldsymbol{\delta}^{*}}{d\,\boldsymbol{U}} \qquad 117$$

(118) It

$$\frac{d\,\boldsymbol{\delta}^{*}}{dU}=\left[l.\Psi''(\boldsymbol{\delta}*)+\int_{\Omega_e}\boldsymbol{D}^{t}.\boldsymbol{E}.\boldsymbol{D}.d\,\Omega\right]^{-1}.\int_{\Omega_e}\boldsymbol{D}^{t}.\boldsymbol{E}.\boldsymbol{B}.d\,\Omega$$

is then enough118

to calculate, one distinguishes Ψ'' three cases: If, then

1. $\boldsymbol{\delta}=\boldsymbol{0}$; If the element $\Psi''=0$

2.is in linear mode; If the element $\Psi''=\Psi''_{lin}=\frac{\sigma_c}{\kappa}.\exp\left(-\frac{\sigma_c}{G_c}.\kappa\right).\boldsymbol{Id}_{2\times 2}$

3.is in dissipative mode, with (119 $\Psi''=\Psi''_{dis}=C.\boldsymbol{Id}_{2\times 2}+\tilde{C}.\boldsymbol{\delta}\otimes\boldsymbol{\delta}$) Bibliographical

$$\begin{cases} C=\frac{\sigma_c}{\|\boldsymbol{\delta}\|}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right) \\ \tilde{C}=-\left(\frac{1}{\|\boldsymbol{\delta}\|}+\frac{\sigma_c}{G_c}\right).\frac{\sigma_c}{\|\boldsymbol{\delta}\|^{2}}.\exp\left(-\frac{\sigma_c}{G_c}.\|\boldsymbol{\delta}\|\right) \end{cases} \qquad 119$$

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