Computation of the factors of intensity of the stresses by extrapolation of the field of displacements

Summarized:

One describes here a méthode de calcul of $K1$, $K2$ and $K3$ in 2D (plane and axisymmetric) and 3D by extrapolation of the jumps of displacements on the lips of crack. It is usable using command POST_K1_K2_K3, as well for a crack with a grid (conventional finite elements) as for a crack nonwith a grid (finite elements nouveau riches: method X-FEM).

If the crack is with a grid, it must necessarily be plane; if the crack is not with a grid (method X-FEM), it can be nonplane (but sufficiently regular). In both cases, the method is applicable only for linear, homogeneous and isotropic materials elastic.

The method used is theoretically less precise than computation from the bilinear form of the rate of refund of energy and singular displacements [R7.02.01 and R7.02.05] (operator CALC_G). It however makes it possible to easily obtain relatively reliable values of the factors of intensity of the stresses. The comparison of the various methods of calculating is useful to estimate the accuracy of the got results.

The accuracy of the results of the method of extrapolation of the jumps of displacement is clearly improved if the mesh is quadratic. For a crack with a grid, it is recommended to use elements known as of "Barsoum" in crack tip (elements whose nodes mediums are located at the quarter of the edges). For a crack nonwith a grid, it is recommended to enrich several layers of elements around the crack's point.

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1 Position of the problem

the method of calculating of the factors of intensity of the stresses by extrapolation of displacement is based on the asymptotic development of the field of displacement in crack tip.

In 2D, in a springy medium, linear, isotropic and homogeneous, the stress fields and of displacement are known analytically for the modes of opening of crack (characterized by $K1$), of plane sliding ($K2$) and sliding antiplan ($K3$), cf [bib1] and [bib2]. In the general case in 3D, one can show that the asymptotic behavior of displacements and stresses is the sum of the solutions correspondings to modes 1 and 2 (in plane strains) and to mode 3 (antiplan), and of four other particular solutions, but which are more regular than the preceding ones [bib3].

In all the cases, the singularity is thus the same one and one can write the following relations in the normal plane with the crack tip, in one point: M

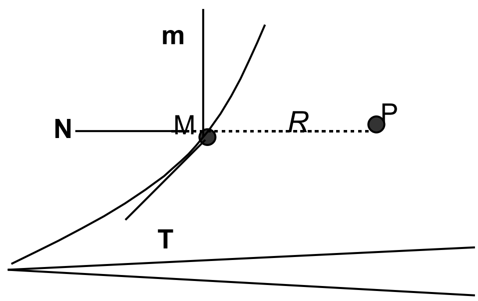
$$K_1(M)=\lim_{r\to 0}\left(\frac{E}{8(1-\nu^2)}[U_m]\sqrt{\frac{2\pi}{r}}\right)$$
$$K_2(M)=\lim_{r\to 0}\left(\frac{E}{8(1-\nu^2)}[U_n]\sqrt{\frac{2\pi}{r}}\right)$$
$$K_3(M)=\lim_{r\to 0}\left(\frac{E}{8(1+\nu)}[U_t]\sqrt{\frac{2\pi}{r}}\right)$$

with:

$\mathbf{t},\mathbf{n}$ the plane of the crack in M,
$\mathbf{t}$ tangent vector with the crack tip in M,
$\mathbf{n}$ normal vector with the crack tip in M,
$\mathbf{m}$ normal vector with the plane of crack in M,
$[U]$ jump of displacement enters the crack lips:

$$[U_{\mathbf{m}}]=\left(U^{\text{lèvre supérieure}}-U^{\text{lèvre inférieure}}\right)\cdot\mathbf{m}$$

$r=\|\mathbf{MP}\|$ where P is a point of the normal plane to the crack tip in M, located on one of the lips.



If the crack is not plane, the three vectors are locally defined at the point M of the bottom considered. The relations precedents thus provide a method to identify numerically $K1$, $K2$ and $K3$. From the factors of intensity of the stresses, the formula of Irwin then makes it possible to calculate the rate of refund of energy G :

$$G=\frac{1}{E}\left(K_1^2+K_2^2\right) \quad \text{in plane stresses}$$
$$G=\frac{1-\nu^2}{E}\left(K_1^2+K_2^2\right)\left(+\frac{1+\nu}{E}K_3^2\right) \quad \text{in plane strains (and 3D)}$$

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- *One can note that the signs of $K2$ and $K3$ depend on the directional sense on $\mathbf{t}$ and $\mathbf{n}$. This is not too awkward insofar as rupture criteria or of fatigue use only the absolute values of $K2$ and $K3$.*
- *One can also give statements according to the stress fields, but the values of the vectors forced on the lips of crack are less precise than displacements (because resulting from a transport from Gauss points to the nodes).*
- *The statement of the asymptotic fields is valid for nonplane cracks (curved cracks for example), but those must nevertheless be sufficiently regular. The user must take care has minimum so that a norm can be defined in any point of the bottom.*
- *The method used here is theoretically less precise than computation from the bilinear form of the rate of refund of energy and singular displacements [R7.02.01 and R7.02.05] (operator `CALC_G`). It however makes it possible to easily obtain relatively reliable values of the factors of intensity of the stresses. The comparison of the various methods of calculating is always useful to estimate the accuracy of the got results.*

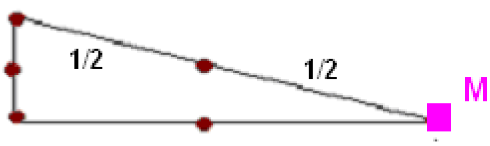
1 Put in work of the methods of extrapolation

the methods of extrapolation of displacements are put in work in operator `POST_K1_K2_K3`, starting from the field of displacement calculated on all structure. The definitions of the factors of intensity of the stresses are not true that asymptotically; extrapolation is thus done while being restricted in the vicinity of the crack tip limited by a maximum distance $dmax$ to the bottom. $dmax$ is parameter `ABSC_CURV_MAXI` of the operator. In the case of a crack with a grid `ABSC_CURV_MAXI` is optional. If it is not noted, $dmax$ is calculated automatically in `POST_K1_K2_K3` and is worth four times the maximum size of meshes connected to the nodes of the bottom.

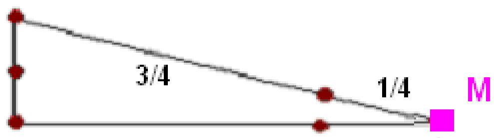
The general principle of computation is the following:

Buckle on the nodes of the crack tip (not running: M)
  Definition of the normal Γ plane to crack and the crack tip, at the point M (plane of norm $\mathbf{t}$)
  Identification of the nodes of the two lips which belong to Γ : $P_i^{\sup}$ and $P_i^{\inf}$
    Buckles on these nodes:
      If $r_i^{\sup}=\|\mathrm{MP}_i^{\sup}\| \mathrm{dmax}$: extraction of displacement in $P_i^{\sup}$
      If $r_i^{\inf}=\|\mathrm{MP}_i^{\inf}\| \mathrm{dmax}$: extraction of displacement in $P_i^{\inf}$
    Computation of the jump of displacement in the three directions
  Extrapolation of the jump of displacement

Three methods of extrapolation are programmed. They are illustrated in this paragraph for a crack with a grid (quadratic mesh), with or without elements of the type "Barsoum". The elements of Barsoum are such as the nodes not tops on the sides of the quadratic elements concerning the crack tip are moved with the quarter of with dimensions [bib4]. They make it possible to better collect the singularity of the stress field in crack tip



Conventional finite element (M being a node of the crack tip)



Finite element of type Barsoum

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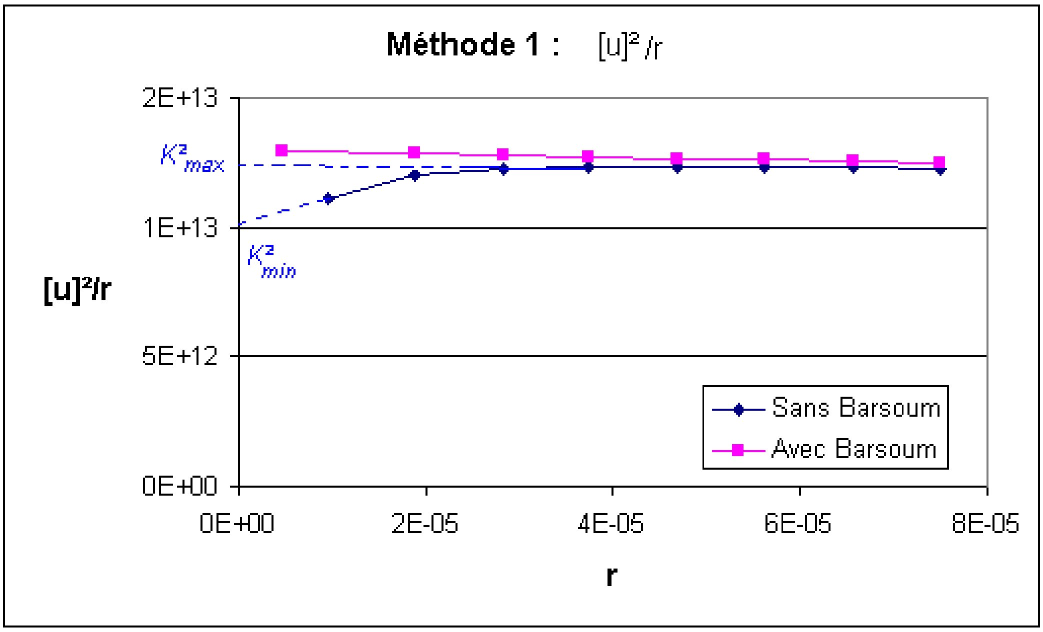
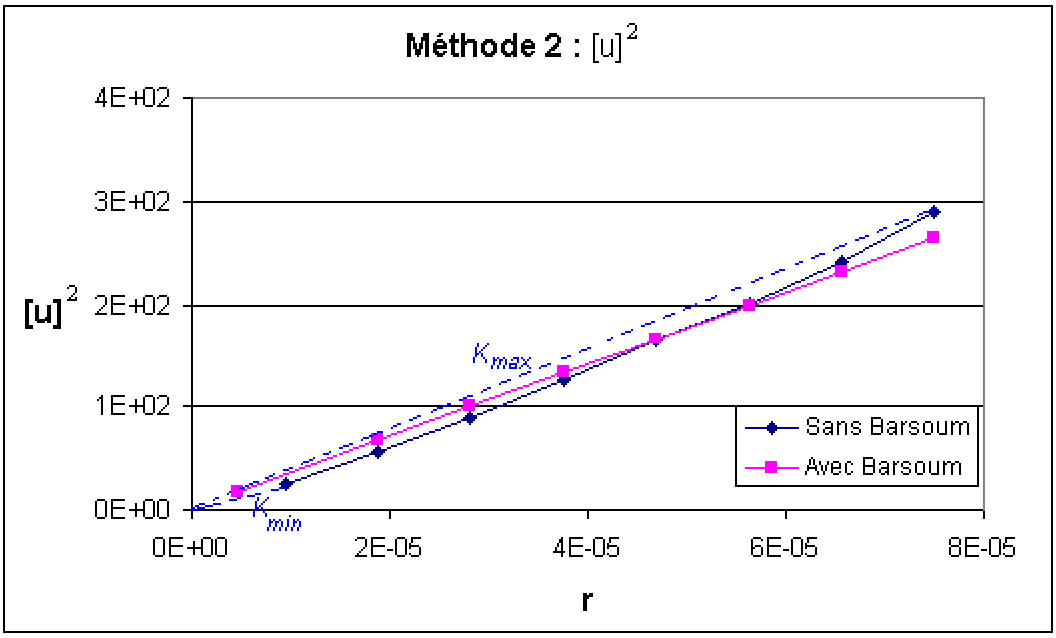
- **Method 1:** one calculates the jump of the field of displacements squared and one divides it par. r
  Various values of K^2 are obtained (except for a multiplicative factor) by extrapolation into cubes $r=0$ line segments thus obtained. If the solution were perfect (analytical asymptotic field everywhere), one should obtain a line. Actually, one obtains almost a line with a mesh of the type "Barsoum", and a nonright curve if not:



- **Method 2:** one traces the jump of the field of displacements squared according to r. The approximations of K are (always with a multiplicative factor near) equal to the root of the slope of the segments connecting the origin to the various points of the curve.



- **Method 3:** one identifies the stress intensity factor K starting from the jump of displacement $[U]$ by a method of the least squares. Retiming is done on a segment length $dmax$, where $dmax$ is the built-in parameter in operand `ABSC_CURV_MAXI` of operator `POST_K1_K2_K3` or in the case of a

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crack with a grid, if ABSC_CURV_MAXI is not indicated, $dmax$ is worth four times the maximum size of meshes connected to the nodes of the bottom :

$$K \text{ minimize } J(k)=\frac{1}{2}\int_0^{\mathrm{dmax}}\left([U(r)]-k\sqrt{r}\right)^2 dr$$

Is thus the explicit formula to compute: K :

$$K=\frac{2}{r_m^2}\int_0^{\mathrm{dmax}}[U(r)]\sqrt{r}\,dr=\frac{1}{r_m^2}\sum_{i=0}^{nbno-1}\left(r_{i+1}-r_i\right)\left([U]_{i+1}\sqrt{r_{i+1}}-[U]_i\sqrt{r_i}\right)$$

where $nbno$ is the number of nodes on the segment of retiming $[0, dmax]$. It is noticed that in this statement K is, for $dmax$ built-in, the linear shape of the field of displacement.

1 Accuracy of the methods suggested

the method of extrapolation of the jumps of displacement was validated on tests whose analytical solutions are known. One has certain results below of them, in 2D and 3D, for a crack with a grid or not. One also compares the results with the method theoretically more precise founded on computation of the rate of refund of energy and on the singular functions (operator CALC_G: method theta).

1.1 Test SSLP313: 2D C_PLAN (crack with a grid)

It acts of a crack inclined in an infinite medium subjected to a uniform stress field in a direction (analytical reference solution in plane stresses, exact in infinite medium). The crack opens in mixed mode ($K1$ and $K2$) [V3.02.313].

For the test, the crack is with a grid in a rather large plate. The mesh quadratic is very fine. The results are the following:

Reference solution (analytical solution)

$K1$	$K2$	G
3.58E+06	2.69E+06	1.00E+02

Computation with the method theta (CALC_G)

	$K1$	$K2$	G
CALC_G without node to the quarter	3.60E+06	2.70E+06	1.01E+02
Variation/ref.	*0.8%*	*0.2%*	*1.1%*
CALC_G with nodes with the quarter	3.60E+06	2.70E+06	1.01E+02
Variation/ref.	*0.8%*	*0.2%*	*1.2%*

POST_K1_K2_K3 : mesh without node of edges to quarter

method	$K1_{max}$	$K1_{min}$	$K2_{max}$	$K2_{min}$	G_{max}	G_{min}	Variation G_{max} /ref.	Variation G_{min} /ref.
1	3.54E+06	3.19E+06	2.63E+06	1.92E+06	9.73E+01	6.94E+01	– 3,33%	– 30,70%
2	3.51E+06	3.33E+06	2.61E+06	2.25E+06	9.57E+01	8.08E+01	– 4,50%	– 19,32%
3	3.50E+06		2.59E+06		9.47E+01		-5,47%	

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`POST_K1_K2_K3` *: mesh with nodes of edges to quarter*

method	$K1_{max}$	$K1_{min}$	$K2_{max}$	$K2_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ref.**	**Variation** G_{min} **/ref.**
1	3.61E+06	3.60E+06	2.70E+06	2.69E+06	1.01E+02	1.01E+02	1,29%	1,07%
2	3.60E+06	3.53E+06	2.69E+06	2.65E+06	1.01E+02	9.75E+01	1,02%	– 2,67%
3	3.56E+06		2.66E+06		9.88E+01		-1,42%	

On this test one note that the mesh of the type "Barsoum" is essential if one wants results precise. With "Barsoum" method 1 is more stable. It provides values of G (from $K1$ and $K2$) to approximately 1% of the analytical solution. Methods 2 and 3 lead to errors from 1 to 2,5%. It is noted that in this case, the method by extrapolation of displacements is as precise as the method theta.

On the other hand, with a normal mesh, the results of the method by extrapolation vary much (between – 3% and -30% of the solution). It is the same with linear elements. In the case of a mesh without elements of "Barsoum", method 3 is most precise.

1.2 Test SSLV134: 3D (crack with a grid)

It is about a plane crack in the shape of disc in an infinite medium 3D subjected to a uniform stress field in a direction (known analytical reference solution under the name of "penny shape ace"). The crack opens in mode 1 pure, and $K1$ is constant along the crack tip [V3.04.134].

For this test, the crack is with a grid in a block parallelepiped. The mesh is relatively coarse.

Analytical reference solution:

$K1$	G room
1,59 106	11,59

Computation with the method theta (`CALC_G` *)*

	G
`CALC_G` with nodes with quarter	11.75
Variation/ref.	*1.3%*

`POST_K1_K2_K3` *: mesh without nodes of edges to quarter*

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ref.**	**Variation** G_{min} **/ref.**
1	1.56E+06	1.45E+06	1.11E+01	9.63E+00	– 4,32%	– 16,91%
2	1.53E+06	1.49E+06	1.06E+01	1.01E+01	– 8,35%	– 13,08%
3	1.52E+06		1.05E+01		– 9,51%	

`POST_K1_K2_K3` : mesh with nodes of edges to quarter

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ref.**	**Variation** G_{min} **/ref.**
1	1.61E+06	1.59E+06	1.18E+01	1.16E+01	1,32%	– 0,06%
2	1.59E+06	1.53E+06	1.15E+01	1.07E+01	– 0,42%	– 7,87%
3	1.55E+06		1.10E+01		– 5,16%	

On this test one still notes that the mesh of the type "Barsoum" is essential if one wants results precise. With "Barsoum" method 1 is most stable, with a variation with the reference solution lower

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than 1,5% for G. The mesh is relatively coarse, which explains why the method theta is more precise.

1.3 Test SSLV134: 3D (crack nonwith a grid)

the case considered is the same one as that of the preceding paragraph, but this time the crack is not with a grid. It is directly defined in the command file, by means of the method X-FEM [R7.02.12]. The mesh not being regular with respect to crack, the values of K and of G calculated vary along the crack tip. For the comparison below, one retains the value corresponding to an arbitrarily selected particular point (medium of the crack tip represented).

The mesh is **linear** and relatively **coarse**. In the method X-FEM, the user can choose the zone on which the elements around the crack tip are nouveau riches with asymptotic displacements (key words RAYON_ENRI and NB_COUCHES of DEFI_FISS_XFEM). This enrichment aims at improving the accuracy of computation. One compares here the results got with an enrichment limited to the only elements containing the crack tip and with an enrichment on four layers of elements around the crack's point.

Computation with the method theta (CALC_G – lissage owing to lack of type LEGENDRE of degree 5)

	G
CALC_G with enrichment on a layer	11.42
Variation/ref.	*-1.4%*
CALC_G with enrichment on four layers	11.61
Variation/ref.	*0.2%*

POST_K1_K2_K3 : enrichment on only one layer

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ref.**	**Variation** G_{min} **/ref.**
1	1.65E+06	1.43E+06	12.4	9.34	6,99%	– 19,41%
2	1.52E+06	1.44E+06	10.5	9.45	– 9,41%	– 18,46%
3	1.47E+06		9.81		– 15,35%	

POST_K1_K2_K3 : enrichment on four layers

method	$K1_{max}$	$K1_{min}$	G_{max}	G_{min}	**Variation** G_{max} **/ref.**	**Variation** G_{min} **/ref.**
1	1.58E+06	1.58E+06	11.3	11.3	– 2,51%	– 2,51%
2	1.55E+06	1.47E+06	10.9	9.88	– 5,95%	– 14,65%
3	1.51E+06		10.4		– 10,26%	

On this test, one note that it is essential to enrich on several layers by elements around the crack tip to have satisfactory results. To note that the mesh used here is linear and relatively coarse: with a finer mesh, the results are significantly improved. A study of convergence on a similar case is presented in [bib5].

With an enrichment on four layers, method 1 is that which leads to the most precise results. Maximum curvilinear X-coordinate corresponds, in both cases, with the distance from four elements approximately. The method theta is here as for it less sensitive to the parameter of enrichment.

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2 Conclusion

the results got with the method of extrapolation of displacement are as a whole satisfactory, with less than 5% of error of G, especially if the elements of the crack tip are of type Barsoum (case fissures with a grid) or if several layers of elements are nouveau riches around the bottom (case fissures nonwith a grid, method X-FEM). In both cases, it is a question as well as possible of collecting the asymptotic behavior of displacement.

It should indeed be noticed that the asymptotic statement of displacements is valid only for r tending towards 0. This is why it is necessary to take care not to choose a too large field of extrapolation (distance $dmax$ from operator `POST_K1_K2_K3` of about 4 to 5 elements).

On the tests presented for a crack with a grid, method 1 gives the most precise results and the more stable, than it is in 2D or 3D, if there are elements of Barsoum. If the mesh does not comprise elements of Barsoum, one then advises to use the results of method 3. For a crack nonwith a grid, method 1 seems also most precise.

On a study for which one does not know a reference solution, it is possible to estimate the quality of computation a posteriori. Indeed, `POST_K1_K2_K3` systematically provides for the first two methods the values minimum maximum and the values (on all the calculated points) of the factors of intensity of the stresses, as well as the value of G recomputed by the formula of Irwin. Method 3 provides as for it only one value for each stress intensity factor. This method is a weighted average of the factors of intensity of the stresses extrapolated in each node.

One result can be regarded as satisfactory if the 5 values thus provided (min and max of the methods 1 and 2, and method 3) are close. One can also recommend to compare the results got with this method with those resulting from computation from the rate of refund from energy and the singular functions (operator `CALC_G`).

3 Bibliography

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1 of the document Index Version Aster

Author	**(S) or contributor**	**(S), organization Description of the modifications**	**A5 J.M.Proix EDF/R & D /AMA initial**
		Text B 7.4 E.Galenne	EDF/R & D /AMA
Method		of the least squares	7.2.24 C 9.4 E.Galenne EDF/R & D /AMA
Fissures		nonplane with XFEM	9.2.8

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