Rate of energy restitution in thermo-élasto-plasticity – Summarized Approach

GTP:

One presents the computation of the rate of refund of total mechanical energy by the method theta in 2D or 3D (approach G_{TP}) for a thermoelastoplastic problem. The behavior models thermoelastoplastic are described in detail in the document [R5.03.02].

This rate of refund of total mechanical energy, called G_{TP}, makes it possible to analyze the nonmonotonous situations of loadings of the default, for irreversible behaviors of material.

Let us note that the problem of the thermoelastoplastic fracture is delicate problems. It is advised to consult the references before a first use.

Caution:

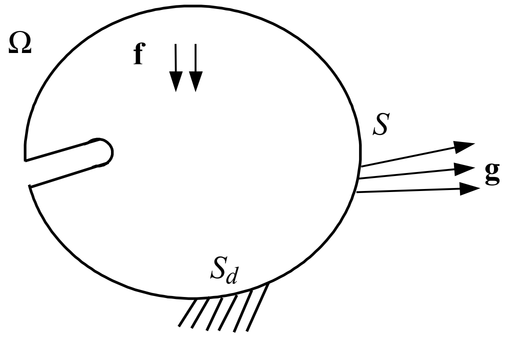
The default must be modelled by a notch and not by a crack [§5]. This approach is thus not valid with the method X-FEM [R7.02.12] because this one does not make it possible to model a notch.

Contents

1 Choices of the formulation of rate of energy restitution in thermo-elastoplasticity

One consider a notched elastic **solid** occupying the field Ω of space $\mathbb{R}^2$ or $\mathbb{R}^3$. That is to say:

$\mathbf{u}$ the field of displacement,
T the field of temperature,
$\boldsymbol{\sigma}$ the tensor of the stresses,
f the field of the volume forces applied to Ω,
$\mathbf{g}$ the field of the surface forces applied to part S of $\partial\Omega$,
$\mathbf{U}$ the field of displacements imposed on part S_d of $\partial\Omega$.



In linear or nonlinear thermoelasticity, the rate of refund of energy G is defined by the opposite of derivative of potential energy compared to the field Ω [bib1] [R7.02.03]:

$$G=-\frac{\partial \mathrm{W}}{\partial \Omega}$$

Total potential energy with the equilibrium of the system is:

$$\mathrm{W}(\mathbf{u})=\int_{\Omega} \Psi \,\mathrm{d}\,\Omega-\int_{\Omega} \mathbf{f}_{\mathbf{i}} \mathbf{u}_{\mathbf{i}} \,\mathrm{d}\,\Omega-\int_{\mathrm{S}} \mathbf{g}_{\mathbf{i}} \mathbf{u}_{\mathbf{i}} \,\mathrm{dG}$$

where Ψ is the density of free energy. In elasticity, Ψ is equal to the density of free energy elastic [R7.02.01]

One extends this definition for the thermoelastoplastic problem (model of Von Mises), while choosing to replace Ψ by total mechanical energy $\tilde{\Psi}$. This choice is justified in the document [bib2].

$\tilde{\Psi}$ is a function of the following variables of state:

$\boldsymbol{\varepsilon}$ the tensor of the total deflections,
$\boldsymbol{\varepsilon}^p$ the tensor of plastic strains,
T the field of temperature,
p the scalar local variable of isotropic hardening (cumulated plastic strain),
$\boldsymbol{\beta}$ one or more tensorial or scalar variables of kinematic hardening.

$$\tilde{\Psi}\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{p},T,p,\boldsymbol{\beta}\right)=\int_{0}^{t}\boldsymbol{\sigma}\frac{d}{dt}\boldsymbol{\varepsilon}(\tau)d\tau=\Psi\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{p},p,T,\boldsymbol{\beta}\right)+\int_{o}^{t}\mathrm{D}^{\mathrm{p}}(\tau)d\tau+\int_{o}^{t}s\frac{d}{d\tau}T(\tau)d\tau$$

where
- Ψ of free energy is the density
- D^{p} power density of plastic dissipation is the density
- $s(T)$ of entropy One is

the density notes that $\tilde{\Psi}$ is the density of increased free energy of the voluminal energy dissipated plastically during all the evolution, and to which energy is added $\int_{o}^{t}\mathrm{s}\frac{d}{d\tau}T()d\tau$ (contribution of the temperature to the variation of free energy).

Caution:

One limits oneself to a notched Ω solid (cf [§5]).

2 Behavior model

the behavior of solid is supposed to be thermoelastoplastic associated with a criterion of Von Mises with isotropic or kinematical hardening. This kind of behavior is currently treated in operator `STAT_NON_LINE` [U4.32.01] under the key word factor `COMP_INCR`. The relations treated in this document are:

`VMIS_ISOT_LINE` : Von Mises with linear isotropic hardening,
`VMIS_ISOT_TRAC` : Von Mises with isotropic hardening given by a curve of tension,
`VMIS_ISOT_PUIS` : Von Mises with nonlinear isotropic hardening defined by a function power,
`VMIS_CINE_LINE` : Von Mises with linear kinematic hardening.

For more details, to consult the document [R5.03.02].

$\boldsymbol{\varepsilon}$ is connected to the field of displacement $\mathbf{u}$ by:

$$\boldsymbol{\varepsilon}(\mathbf{u})=\frac{1}{2}\left(\mathbf{u}_{\mathbf{i},\mathbf{j}}+\mathbf{u}_{\mathbf{j},\mathbf{i}}\right)$$

The density of free energy is written:

$$\Psi\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{p},T,p,\boldsymbol{\beta}\right)=\omega^{e}\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{p},T\right)+h(p,\beta,T)+z(T)$$

where

h of energy of hardening an arbitrary function
z of the temperature is the density
ω^{e} the density of energy thermoelastic defined by:

$$\omega_{e}\left(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^{p},T\right)=\frac{1}{2}\boldsymbol{\sigma}_{ij}\boldsymbol{\varepsilon}_{ij}^{e}=\frac{1}{2}\mathbf{A}_{\mathbf{ijkl}}\left(\boldsymbol{\varepsilon}_{ij}-\boldsymbol{\varepsilon}_{ij}^{p}-\alpha\left(T-T_{réf}\right)\boldsymbol{\delta}_{ij}\right)\left(\boldsymbol{\varepsilon}_{kl}-\boldsymbol{\varepsilon}_{kl}^{p}-\alpha\left(T-T_{réf}\right)\boldsymbol{\delta}_{kl}\right),$$

with α the thermal coefficient of thermal expansion, and $\mathbf{A}_{ijkl}$ the elasticity tensor.

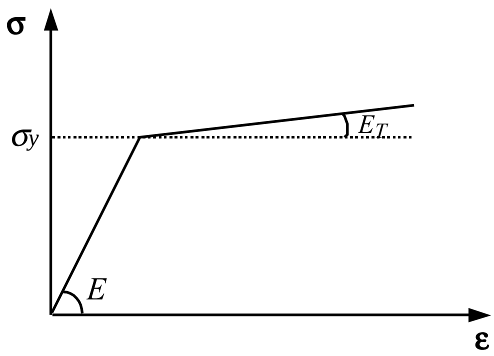
In the typical case where there no was plastic evolution, one finds the statement of the density of elastic strain energy for a linear thermoelastic problem [R7.02.01 §1.1].

The free energy of hardening h is deduced from:

$\frac{\partial \mathrm{h}}{\partial p}(p, T) = R(p, T)$, for isotropic hardening where $\mathrm{R}(p, T)$ is the radius of the surface of load and $\frac{\partial \mathrm{h}}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}, T) = \mathrm{X}(\boldsymbol{\beta}, T)$, for kinematic hardening where $\mathrm{X}(\boldsymbol{\beta}, T)$ is the translation of the surface of load within the space of stresses (in the case of linear kinematic hardening $\boldsymbol{\beta} = \boldsymbol{\varepsilon}^p$)

For the behavior model of Von Mises with linear isotropic hardening:

$$\mathrm{R}(p, T) = \frac{E_T E}{E - E_T} p$$



Curve of tension

characteristics of the material (Young Modulus E and D_SIGM_EPSI : E_T) can depend on the temperature [R5.03.02 §3.2.1].

Plastic dissipation (or mechanical intrinsic dissipation) for a constitutive law of Von Mises is written:
$D^p = \sigma : \dot{\varepsilon}^p - R\,\dot{p} = \sigma_Y\,\dot{p}$
where σ_y is the initial linear elastic limit.
Thus plastic dissipation checks:

$$\int_o^t D^p(\tau)\, d\tau = \sigma_y\, p$$

Finally the density of total mechanical energy $\tilde{\Psi}$ is written:

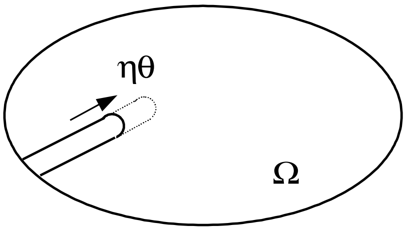
$$\begin{aligned}\tilde{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, T, p, \boldsymbol{\beta}) &= \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, T, p, \boldsymbol{\beta}) + \int_o^t D^p(\tau)\, d\tau + \int_o^t s\,\dot{T}(\tau)\, d\tau \\ &= \frac{1}{2}\sigma_{ij}\left(\boldsymbol{\varepsilon}_{ij} - \boldsymbol{\varepsilon}^p_{ij} - \alpha(T - T_{réf})\boldsymbol{\delta}_{ij}\right) + \int_o^p R(p, T)\, dp + \int_o^b X(\boldsymbol{\beta}, T)\, dc + \sigma_y\, p\end{aligned}$$

For a linear isotropic hardening:

$$\tilde{\Psi} = \tilde{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, T, p, \boldsymbol{\beta}) = \frac{1}{2}\sigma_{ij}\left(\boldsymbol{\varepsilon}_{ij} - \boldsymbol{\varepsilon}^p_{ij} - \alpha(T - T_{réf})\boldsymbol{\delta}_{ij}\right) + \frac{1}{2}\frac{E_T E}{E - E_T} p^2 + \sigma_y\, p$$

3 Lagrangian statement of rate of energy restitution in thermo-élasto-plasticity

the rate of refund is calculated in *Code_Aster* by the method G-theta [R7.02.01 §1.3]. One notes by $\dot{\mathrm{Q}}$ Lagrangian derivative of the quantity Q in a virtual propagation of the notch of $\eta\boldsymbol{\theta}$, η being a small real parameter and $\boldsymbol{\theta}$ a field of vector representing the direction of propagation of the notch (one thus has $\dot{\mathrm{Q}}(\mathbf{x}(\eta),\eta)=\frac{\partial \mathrm{Q}}{\partial \eta}+\nabla Q.\boldsymbol{\theta}$).



The rate of refund of total mechanical energy in this propagation $\eta\boldsymbol{\theta}$ is:

$$-\mathrm{G}(\boldsymbol{\theta})=\int_{\Omega}\left(\overline{\tilde{\Psi}-\mathbf{f}_{\mathrm{i}}\dot{\mathbf{u}}_{\mathrm{i}}}\right)+\left(\tilde{\Psi}-\mathbf{f}_{\mathrm{i}}\mathbf{u}_{\mathrm{i}}\right)\boldsymbol{\theta}_{k,k}\,d\,\Omega$$

$$-\int_{S}\overline{\mathbf{g}_{\mathrm{i}}\dot{\mathbf{u}}_{\mathrm{i}}}+\mathbf{g}_{\mathrm{i}}\mathbf{u}_{\mathrm{i}}\left(\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}}-\frac{\partial\boldsymbol{\theta}}{\partial\mathbf{n}_{\mathbf{k}}}\mathbf{n}_{\mathbf{k}}\right)d\,\Gamma$$

$$\text{Or }\ \dot{\tilde{\Psi}}\left(\boldsymbol{\varepsilon}_{\mathbf{ij}},\boldsymbol{\varepsilon}_{\mathbf{ij}}^{p},T,p,\boldsymbol{\beta}_{i}\right)=\frac{\partial\tilde{\Psi}}{\partial\boldsymbol{\varepsilon}_{\mathbf{ij}}}.\dot{\boldsymbol{\varepsilon}}_{ij}+\frac{\partial\tilde{\Psi}}{\partial\boldsymbol{\varepsilon}_{\mathbf{ij}}^{\mathbf{p}}}.\dot{\boldsymbol{\varepsilon}}_{ij}^{p}+\frac{\partial\tilde{\Psi}}{\partial T}\dot{T}+\frac{\partial\tilde{\Psi}}{\partial p}\dot{p}+\frac{\partial\tilde{\Psi}}{\partial\partial\boldsymbol{\beta}_{\mathbf{i}}}.\partial\dot{\boldsymbol{\beta}}_{i}+s\dot{T}$$

$$\text{avec}\begin{cases}\dfrac{\partial\tilde{\Psi}}{\partial\partial\boldsymbol{\varepsilon}_{ij}}=\boldsymbol{\sigma}_{ij}\\[2mm]\dfrac{\partial\tilde{\Psi}}{\partial\boldsymbol{\varepsilon}_{ij}^{\mathbf{p}}}=-\boldsymbol{\sigma}_{ij}\\[2mm]\dfrac{\partial\tilde{\Psi}}{\partial T}=-s\\[2mm]\dfrac{\partial\tilde{\Psi}}{\partial p}=\mathrm{R}(p,T)+\sigma_{Y}p\\[2mm]\dfrac{\partial\tilde{y}}{\partial\boldsymbol{\beta}_{\mathbf{i}}}=X_{i}\left(\partial\boldsymbol{\beta}_{\mathbf{i}},T\right)\end{cases}$$

$$\text{soit } -G(\boldsymbol{\theta})=\int_{\Omega}\boldsymbol{\sigma}_{\boldsymbol{ij}}\dot{\boldsymbol{\varepsilon}}_{ij}-\boldsymbol{\sigma}_{\boldsymbol{ij}}\dot{\boldsymbol{\varepsilon}}_{ij}^{p}+\frac{\partial\tilde{\Psi}}{\partial T}\dot{T}+\mathrm{R}\,\dot{p}+\frac{\partial\tilde{\Psi}}{\partial\boldsymbol{\beta}_{\mathbf{i}}}\dot{\beta}_i+\tilde{\Psi}\,\boldsymbol{\theta}_{\boldsymbol{k},\boldsymbol{k}}\,d\,\Omega$$

$$+\ \text{termes classiques}\ (f\,,g)$$

$$\text{avec}\ \begin{cases}\dot{\boldsymbol{\varepsilon}}_{ij}=\dfrac{1}{2}\left(\dfrac{\partial\,\mathbf{u}_{\mathbf{i},\mathbf{j}}}{\partial\eta}+\dfrac{\partial\,\mathbf{u}_{\mathbf{j},\mathbf{i}}}{\partial\eta}\right)-\dfrac{1}{2}\left(\mathbf{u}_{\mathbf{i},\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k},\mathbf{j}}+\mathbf{u}_{\mathbf{j},\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k},\mathbf{i}}\right)\\ \dot{\boldsymbol{\varepsilon}}_{ij}^{p}=\dfrac{\partial\,\boldsymbol{\varepsilon}_{\boldsymbol{ij}}^{\mathbf{p}}}{\partial\eta}+\boldsymbol{\varepsilon}_{\boldsymbol{ij},\mathbf{k}}^{\mathbf{p}}\boldsymbol{\theta}_{\mathbf{k}}\\ \dot{T}=\dfrac{\partial T}{\partial\eta}+\mathbf{T}_{,\mathbf{k}}\boldsymbol{\theta}_{\boldsymbol{k}}\\ \dot{p}=\dfrac{\partial\,p}{\partial\eta}+\boldsymbol{p}_{,\boldsymbol{k}}\boldsymbol{\theta}_{\boldsymbol{k}}\\ \dot{\boldsymbol{\beta}}_{ij}=\dfrac{\partial\,\boldsymbol{\beta}_{\boldsymbol{ij}}}{\partial\,\eta}+\boldsymbol{\beta}_{\boldsymbol{ij},\boldsymbol{k}}\boldsymbol{\theta}_{\boldsymbol{k}}\end{cases}$$

One can eliminate $\dot{u}$ from the statement from $\mathrm{G}(\boldsymbol{\theta})$ by noticing that $\dot{u}$ is kinematically admissible and by means of the balance equation [R7.02.01 §1.3]. In the same way, the terms $(-\boldsymbol{\sigma}.\dfrac{\partial\boldsymbol{\varepsilon}^{\mathbf{p}}}{\partial\eta}+\mathrm{R}\,\dfrac{\partial\,p}{\partial\eta}+X.\dfrac{\partial\boldsymbol{\beta}}{\partial\,\eta})+\sigma_{Y}\dfrac{\partial\,p}{\partial\eta}$ are eliminated as well as the terms $\dfrac{\partial\tilde{\Psi}}{\partial T}.\dfrac{\partial T}{\partial\eta}-s\,\dfrac{\partial T}{\partial\eta}$.

The following statement then is obtained:

$$-G(\boldsymbol{\theta})=\int_{\Omega}\tilde{\Psi}\,\theta_{k,k}-\sigma_{ij}u_{i,k}\,\sigma_{k,j}-\sigma_{ij}\,\varepsilon_{ij,k}^{p}\theta_k+\frac{\partial\tilde{\Psi}}{\partial T}T_{,k}\,q_k$$

$$+\left(\mathrm{R}(p\,,T)+\sigma_y\right)p_{,k}\,\theta_k+\frac{\partial\tilde{\Psi}}{\partial\,\beta_{ij}}\,\beta_{ij,k}\theta_k\,d\,\Omega$$

$$+\text{termes classiques}(\mathbf{f}\,,\mathbf{g})$$

et finalement :

$$G(\boldsymbol{\theta})=\int_{\Omega}-\tilde{\Psi}\,\theta_{k,k}+\sigma_{ij}u_{i,k}\,\theta_{k,j}-\left(\frac{\partial\tilde{\Psi}}{\partial T}T_{,k}+\left(\mathrm{R}+\sigma_y\right)p_{,k}+\frac{\partial\tilde{\Psi}}{\partial\,\beta_{ij}}\,\beta_{ij,k}-\sigma_{ij}\,\varepsilon_{ij,k}^{p}\right)\theta_k\,d\,\Omega$$

$$+\text{termes classiques}(\mathbf{f}\,,\mathbf{g})$$

For a radial and monotonous loading: $\sigma_{ij}\,\varepsilon_{ij,k}^{p}=\left(\mathrm{R}+\sigma_{y}\right)p_{,k}+\dfrac{\partial\tilde{\Psi}}{\partial\,\beta_{ij}}\,\beta_{ij,k}$ and one finds the statement of $\mathrm{G}(\boldsymbol{\theta})$ in nonlinear thermoelasticity [R7.02.03].

In the general frame, the invariance of $\mathrm{G}(\boldsymbol{\theta})$ according to integration contour is not shown.

4 Establishment in *Code_Aster*

the comparison of the formulations of $G(\boldsymbol{\theta})$ in linear thermoelasticity and thermo - elastoplasticity shows that the terms of the two formulations differ only by terms from transport from the local variables.

The presence of the key word factor COMP_INCR, and the key word factor RELATION = "VMIS_ISOT_LINE" (or "VMIS_ISOT_TRAC" or "VMIS_ISOT_PUIS" or "VMIS_CINE_LINE") indicates that it is necessary to recover the field of displacements, $\mathbf{u}$ the stresses $\boldsymbol{\sigma}$, and the characteristic of the elastoplastic material.

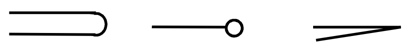
Element types finished which supports these options are the same ones as in elasticity [R7.02.01 §2.4]. They are the isoparametric elements 2D and 3D.

The supported loadings are the same ones as in the elastic case.

5 Restrictions

Attention:

This formulation of G for a thermoelastoplastic relation is valid only for one notched solid and not for a fissured solid. One will choose for example (but the user will be able to choose its own regular notch):



Indeed, the principal difficulty in the establishment of this formulation is impossibility of showing the existence of derivative of total mechanical energy for a field comprising a crack, and this mainly by the absence of knowledge of the singularities of the fields in plasticity. To circumvent the problem, one regularizes the field by representing the default in the form of notch. For more details, it is advised to consult [bib2].

The validation of this formulation is carried out in test SSNP102 [V6.03.102] - Computation of rate of energy restitution for an elastoplastic problem – Approach GTP.

6 Bibliography

1) BUI H.D.: Brittle fracture mechanics, Masson, 1977.

2) DEBRUYNE G.: Proposal for an energy parameter of ductility fracture in thermo - plasticity, HI-74/95/027/0, 2/23/96

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
4	G.DEBRUYNE , E.VISSE-	initial Text

	R&D/AMA	