Computation of the stress intensity factors in linear thermoelasticity

Summarized:

One presents the method of calculating of the stress intensity factors K_I, K_{II} and K_{III} in linear thermoelasticity. The formulation regards rate of energy restitution as the symmetric bilinear shape of the field of displacement $\boldsymbol{u}$ and uses the explicit statements of the fields of singular displacements known in plane linear elasticity.

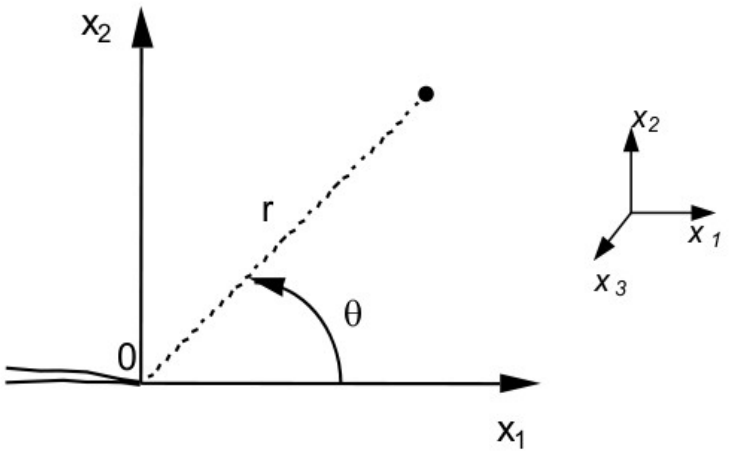
This method is usable using option CALC_K_G of the command CALC_G, as well for a crack with a grid (conventional finite elements) as for a crack nonwith a grid (finite elements nouveau riches: method X-FEM).

This method is also usable to compute: the factors of intensity of the stresses associated with the eigen modes with vibration with a structure.

Contents

1 Statements of the stress intensity factors in linear thermoelasticity

1.1 Presentation in linear thermoelasticity planes



Are the axes of Cartesian coordinates Ox_1 in the prolongation of crack and Ox_2 perpendicular to crack. The problem is plane. We will express the Cartesian components of displacements and the stresses according to the polar coordinates r and θ.

In linear elasticity, the system of the equilibrium equations, without volume force, and the homogeneous boundary conditions on crack, the null stresses ad infinitum, admit a nontrivial solution of the form $\boldsymbol{u_i}=\sqrt{(r)}\,g_i(\theta)$. The stresses are infinite at the bottom of crack like $r^{-1/2}$ [bib3].

For an unspecified problem in plane linear elasticity (plane strains or plane stresses), the field of displacement $\boldsymbol{u}$ can break up into a singular part and a regular part. The singular part, also called singularity, is that clarified above Ci -. It is associated with the factors of intensity of the stresses K. In linear elasticity, the modes of fracture I (opening) and II (sliding plane) are separate:

$$\boldsymbol{u}=\boldsymbol{u_R}+K_I\boldsymbol{u_S^I}+K_{II}\boldsymbol{u_S^{II}}$$

with:

$$\begin{cases} u_{S1}^{I}=\dfrac{1+\nu}{E}\left(\dfrac{r}{2\pi}\right)^{1/2}\cos\left(\dfrac{\theta}{2}\right)(k-\cos\theta) \\ u_{S2}^{I}=\dfrac{1+\nu}{E}\left(\dfrac{r}{2\pi}\right)^{1/2}\sin\left(\dfrac{\theta}{2}\right)(k-\cos\theta) \end{cases}$$

$$\begin{cases} u_{S1}^{II}=\dfrac{1+\nu}{E}\left(\dfrac{r}{2\pi}\right)^{1/2}\sin\left(\dfrac{\theta}{2}\right)(k+\cos\theta-2) \\ u_{S2}^{II}=-\dfrac{1+\nu}{E}\left(\dfrac{r}{2\pi}\right)^{1/2}\cos\left(\dfrac{\theta}{2}\right)(k+\cos\theta-2) \end{cases}$$

where:

$k = 3 - 4\nu$ in plane strains D_PLAN

$k = (3-\nu)/(1+\nu)$ in plane stresses C_PLAN

and:

E modulate YOUNG

ν Poisson's ratio

the distribution of the singular stresses in the vicinity of crack is given by the formulas:

$$\begin{cases} \sigma_{11}^{S} = K_I \sigma_{11}^{I} + K_{II} \sigma_{11}^{II} \\ \sigma_{12}^{S} = K_I \sigma_{12}^{I} + K_{II} \sigma_{12}^{II} \\ \sigma_{22}^{S} = K_I \sigma_{22}^{I} + K_{II} \sigma_{22}^{II} \end{cases}$$

with:

$$\begin{cases} \sigma_{11}^{I} = \frac{1}{(2\pi r)^{1/2}} \cos(\frac{\theta}{2})(1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \\ \sigma_{12}^{I} = \frac{1}{(2\pi r)^{1/2}} \cos(\frac{\theta}{2})\sin(\frac{\theta}{2})\cos(\frac{3\theta}{2}) \\ \sigma_{22}^{I} = \frac{1}{(2\pi r)^{1/2}} \cos(\frac{\theta}{2})(1 + \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \end{cases}$$

$$\begin{cases} \sigma_{11}^{II} = -\frac{1}{(2\pi r)^{1/2}} \sin(\frac{\theta}{2})(2 + \cos(\frac{\theta}{2})\cos(\frac{3\theta}{2})) \\ \sigma_{12}^{II} = \frac{1}{(2\pi r)^{1/2}} \cos(\frac{\theta}{2})(1 - \sin(\frac{\theta}{2})\sin(\frac{3\theta}{2})) \\ \sigma_{22}^{II} = \frac{1}{(2\pi r)^{1/2}} \sin(\frac{\theta}{2})\cos(\frac{\theta}{2})\cos(\frac{3\theta}{2}) \end{cases}$$

Note: the statement of the fields of displacement and singular stresses established in 2D – plane strains can extend to the case 2D – axisymmetric [bib12]. The singular functions are however valid only asymptotically, in other words the distance *R* with the crack tip must remain small compared to the radius of the crack tip.

1.2 Extension to 3D case

In addition to the outline views of opening and plane sliding, a third mode of sliding antiplan (characterized by K_{III}) can be defined. The singular fields associated with the mode III are identified by solving the equilibrium equations of an infinite plane medium fissured for displacements only according to the axis x_3 :

$$\begin{cases} u_{S1}^m = 0 \\ u_{S2}^m = 0 \\ u_{S3}^m = \dfrac{4(1+\nu)}{E}\left(\dfrac{r}{2\pi}\right)^{1/2}\sin\left(\dfrac{\theta}{2}\right) \end{cases} \quad \text{and} \quad \begin{cases} \sigma_{31}^m = -\dfrac{1}{(2\pi r)^{1/2}}\sin\left(\dfrac{\theta}{2}\right) \\ \sigma_{32}^m = \dfrac{1}{(2\pi r)^{1/2}}\cos\left(\dfrac{\theta}{2}\right) \end{cases}$$

In the three-dimensional case, one can show that the asymptotic behavior of displacements and the stresses is the sum of the solutions correspondings to the modes I and II (in plane strains) and to the mode III (antiplan), and of four other particular solutions, but which are more regular than the preceding ones [bib9]. The principal term of the singular fields thus remains unchanged, and the field of displacement in 3D is expressed then in the following way:

$$u(s) = u_R + K_I(s)\,u_S^I + K_{II}(s)\,u_S^{II} + K_{III}(s)\,u_S^{III}$$

where s is the curvilinear abscisse along the crack tip and u_S^I, u_S^{II} and u_S^{III} are singular displacements defined in the preceding paragraph (expressed, for each point of the crack tip, in an adapted local coordinate system).

1.3 Formulate IRWIN and rates of energy restitution G

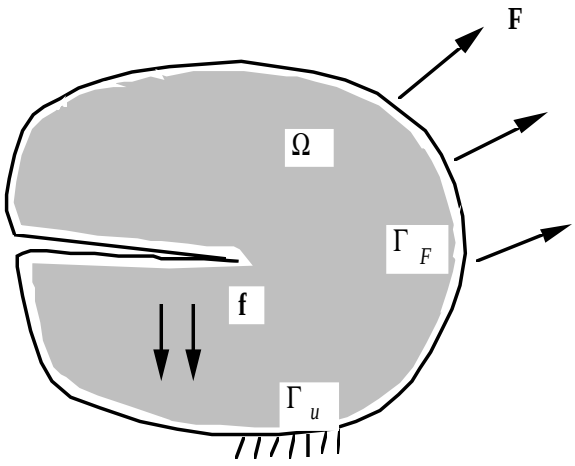
In linear elasticity, the stress intensity factors are connected to rate of energy restitution G by the formula of IRWIN:

$$G = \frac{1-\nu^2}{E}\left(K_I^2 + K_{II}^2\right) \text{ in plane strains}$$

$$G = \frac{1}{E}\left(K_I^2 + K_{II}^2\right) \text{ in plane stresses}$$

$$G(s) = \frac{1-\nu^2}{E}\left(K_I^2(s) + K_{II}^2(s)\right) + \frac{K_{III}^2(s)}{2\mu} \text{ , with } \mu = \frac{E}{2(1+\nu)} \text{ , in 3D}$$

rate of energy restitution G is defined by the opposite of derivative of potential energy compared to the propagation of the crack tip. In *Code_Aster*, G is calculated by the method theta [bib5 and R7.02.01 documentation] which is a Lagrangian method of derivative of potential energy.



Thus, one considers transformations $M \to M+\eta\theta(M)$ of the area of reference Ω_0 in a field Ω_η which correspond to propagations of crack. A these families of reference configuration thus defined Ω_η correspond of the families of deformed configurations whose crack was propagated. The potential energy defined on Ω_η is brought back on Ω_0.

One considers the voluminal forces $\boldsymbol{F}$ surfaciqueset $\boldsymbol{f}$ applied respectively to Γ_F and Ω_0. One notes $\Psi(\varepsilon(\boldsymbol{u}))$ the density of free energy, $\boldsymbol{u}$ the field of displacement, T the field of temperature and θ the field of vectors describing the direction of transport in $\eta=0$, then the general statement of rate of energy restitution G [bib5] is:

$$
\begin{aligned}
G= \quad & \int_\Omega \left[\sigma(\boldsymbol{u}):(\nabla \boldsymbol{u}.\nabla\theta)-\Psi(\varepsilon(\boldsymbol{u}))\operatorname{div}\theta\right] d\Omega && \leftarrow \textit{terme classique}\\
& -\int_\Omega \frac{\partial \Psi}{\partial T}(\nabla T.\theta)\, d\Omega && \leftarrow \textit{terme dû à la thermique}\\
& +\int_\Omega \left[(\nabla \boldsymbol{f}.\theta)\boldsymbol{u}+\boldsymbol{f}.\boldsymbol{u}\operatorname{div}\theta\right] d\Omega && \leftarrow \textit{terme dû aux forces volumiques } \boldsymbol{f} \textit{ sur } \Omega\\
& +\int_{\Gamma_F}\left[(\nabla \boldsymbol{F}.\theta)\boldsymbol{u}+\boldsymbol{F}.\boldsymbol{u}(\operatorname{div}\theta-\boldsymbol{n}.\frac{\partial\theta}{\partial \boldsymbol{n}})\right] d\Gamma && \leftarrow \textit{terme dû aux forces surfaciques } \boldsymbol{F} \textit{ sur } \Gamma_F\\
& -\int_{\Gamma_u}\left[(\sigma.\boldsymbol{n}).(\nabla \boldsymbol{u}.\theta)\right] d\Gamma && \leftarrow \textit{terme dû aux déplacements imposés sur } \Gamma_u
\end{aligned}
$$

In linear elasticity, G can be regarded as the symmetric bilinear shape of the field of displacement $\boldsymbol{u}$. The density of elastic strain energy $\Psi(\varepsilon(\boldsymbol{u}))$ is written:

$$\Psi(\varepsilon(\boldsymbol{u}))=\frac{1}{2}\varepsilon(\boldsymbol{u}):\Lambda:\varepsilon(\boldsymbol{u})=\frac{1}{2}B(\boldsymbol{u},\boldsymbol{u})$$

while noting:

Λ the elasticity tensor
B the symmetric bilinear form defined by: $B(\boldsymbol{u},\boldsymbol{v})=\varepsilon(\boldsymbol{u}):\Lambda:\varepsilon(\boldsymbol{u})$

and the bilinear form $g(.,.)$ associated with G is defined by:

$$
\begin{aligned}
g(\boldsymbol{u},\boldsymbol{v})= \quad & \frac{1}{2}\int_\Omega [\frac{\partial B}{\partial \nabla \boldsymbol{u}}.(\nabla \boldsymbol{v}.\nabla\theta)+\frac{\partial B}{\partial \nabla \boldsymbol{v}}.(\nabla \boldsymbol{u}.\nabla\theta)-B(\boldsymbol{u},\boldsymbol{v})\operatorname{div}\theta]\, d\Omega\\
& +\frac{1}{2}\int_\Omega [(\nabla f_u.\theta)\boldsymbol{v}+(\nabla f_v.\theta)\boldsymbol{u}+(f_u.\boldsymbol{v}+f_v.\boldsymbol{u})\operatorname{div}\theta]\, d\Omega
\end{aligned}
$$

while limiting itself under the terms classic and due to the volume forces $\boldsymbol{f}$.

One has $G=g(\boldsymbol{u},\boldsymbol{u})$ if $\boldsymbol{u}$ is solution of the elastic problem.

1.4 Decoupling of the modes of fracture

to uncouple the three modes from fracture and to calculate the three factors of intensity of the stresses associated, one uses the symmetric bilinear form $g(.,.)$ and the decomposition of the field of displacement $\boldsymbol{u}$ in parts regular and singular [bib7]. To simplify the presentation, one places oneself initially in the plane case, but the properties spread without difficulty with 3D case.

$$\begin{cases} g(\boldsymbol{u},\boldsymbol{u}_S^I)=g(\boldsymbol{u}_R+K_I\boldsymbol{u}_S^I+K_{II}\boldsymbol{u}_S^{II},\boldsymbol{u}_S^I)=g(\boldsymbol{u}_R,\boldsymbol{u}_S^I)+K_I\,g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^I)+K_{II}\,g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^I) \\ g(\boldsymbol{u},\boldsymbol{u}_S^I)=g(\boldsymbol{u}_R+K_I\boldsymbol{u}_S^I+K_{II}\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II})=g(\boldsymbol{u}_R,\boldsymbol{u}_S^{II})+K_I\,g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^{II})+K_{II}\,g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II}) \end{cases}$$

One shows in plane linear thermoelasticity that $\boldsymbol{u}_S^I$ and $\boldsymbol{u}_S^{II}$ are orthogonal for the scalar product defined by the bilinear form $g(.,.)$ and that the terms utilizing the regular part are cancelled:

$$\begin{aligned} g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^{II}) &= g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^I) = 0 \\ g(\boldsymbol{u}^R,\boldsymbol{u}_S^I) &= g(\boldsymbol{u}^R,\boldsymbol{u}_S^{II}) = 0 \end{aligned}$$

One thus has finally:

$$\begin{aligned} g(\boldsymbol{u},\boldsymbol{u}_S^I) &= K_I\,g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^I) \\ g(\boldsymbol{u}^R,\boldsymbol{u}_S^{II}) &= K_{II}\,g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II}) \end{aligned}$$

Moreover, by writing rate of energy restitution in the form:

$$G=g(\boldsymbol{u},\boldsymbol{u})=g(\boldsymbol{u}_R+K_I\boldsymbol{u}_S^I+K_{II}\boldsymbol{u}_S^{II},\boldsymbol{u}_R+K_I\boldsymbol{u}_S^I+K_{II}\boldsymbol{u}_S^{II})$$

one finds the formula of IRWIN by re-using the properties of orthogonality:

$$g(\boldsymbol{u},\boldsymbol{u})=K_I^2\,g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^I)+K_{II}^2\,g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II})$$

with:

$$\begin{aligned} g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^I) &= g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II}) = \frac{1-\nu^2}{E} \quad \text{in C\_PLAN} \\ g(\boldsymbol{u}_S^I,\boldsymbol{u}_S^I) &= g(\boldsymbol{u}_S^{II},\boldsymbol{u}_S^{II}) = \frac{1}{E} \quad \text{in D\_PLAN} \end{aligned}$$

Finally, in a general way:

$$\begin{cases} K_I=E\,g(\boldsymbol{u},\boldsymbol{u}_S^I) \\ K_{II}=E\,g(\boldsymbol{u},\boldsymbol{u}_S^{II}) \end{cases} \quad \text{in C\_PLAN}$$

$$\begin{cases} K_I=\dfrac{E}{1-\nu^2}\,g(\boldsymbol{u},\boldsymbol{u}_S^I) \\ K_{II}=\dfrac{E}{1-\nu^2}\,g(\boldsymbol{u},\boldsymbol{u}_S^{II}) \end{cases} \quad \text{in D\_PLAN and 3D}$$

and $K_{III}=2\mu.g(\boldsymbol{u},\boldsymbol{u}_S^{III})$ 3D

the establishment of the computation of the stress intensity factors in linear thermoelasticity in *Code_Aster* is realized from the statement of rate of energy restitution G in linear elasticity, written in symmetric bilinear form, by introducing the known statements of singular displacements, and by means of the method theta.

2 Establishment of KI, KII and KIII in linear thermoelasticity in *Code_Aster*

2.1 Element types and of loadings

To compute: the stress intensity factors K_I and K_{II} (and possibly K_{III}) in linear elasticity, it is necessary to use option CALC_K_G CALC_G of the command.

This option is valid as well for a crack with a grid (conventional finite elements) as for a crack nonwith a grid (finite elements nouveau riches: method X-FEM). It is available in 2D (modelizations "C_PLAN" and "D_PLAN" for a crack with a grid or not; modelization "AXIS" for a crack with a grid only) and in 3D, for of the finite elements linear or quadratic.

All the classical thermomechanical loadings are taken into account: thermal loadings, voluminal loadings (gravity, rotation,…), surface loadings (including on the lips of crack).

Note:

One does not take account of the term due to the displacements imposed on Γ_u , one thus should not impose conditions of DIRICHLET on the lips of crack.

2.2 Environment necessary for the computation of KI, KII and KIII

command CALC_G makes it possible to recover the model problem, the characteristics of the material, the field of displacement. The field theta is recovered or calculated by the command CALC_G.

Crack with a grid:

- In 2D, it is necessary to define the key word FOND_FISS, which makes it possible to recover a concept of the fond_fiss type (produced by the command DEFI_FOND_FISS) where the basic crack node and the norm with crack are defined.
- In 3D, it is necessary to define the key word FISSURES, which makes it possible to recover a concept of the fiss_xfem type (produced by the command DEFI_FISS_XFEM) where are defined the basic nodes of crack and the local bases along the crack tip.
- When that the crack is laid out along an axis of symmetry, one can be satisfied not to represent that half of the model, and to specify by the key word the symmetry of loading SYME (or if FOND_FISS is indicated the presence of symmetry is automatically detected). By default, one supposes that there is no symmetry. If one assigns the value "OUI' to key word SYME, that means that only the mode I of fracture acts (opening of the lips of crack) and one automatically affects the value zero to K_{II} (and possibly K_{III}).

Fissure nonwith a grid (method X-FEM):

- In 2D as in 3D, the crack must be defined, for mechanical computation and postprocessing, using command DEFI_FISS_XFEM. The key word CRACK must be indicated in CALC_G.
- If the crack is not with a grid, it is not possible to take into account possible symmetries of the model compared to the lips of crack.

Let us insist on the need for assigning to all the elements (including those of the lips) the values of the moduli of YOUNG E and the Poisson's ratio ν, because they are used in the computation of singular displacements. These values must be homogeneous on all the support of the field theta. In addition, if a loading is affected on meshes surface (in 3D) the or linear ones (in 2D) of the lips of crack, meshes of those must be correctly directed.

2.3 Bilinear form symmetric G (. .)

2.3.1 Elementary classical term

$$TCLA=\sigma(\boldsymbol{u}):(\nabla \boldsymbol{u}\nabla\theta)-\Psi(\varepsilon(\boldsymbol{u}))\operatorname{div}\theta$$

the density of elastic strain energy $\Psi(\varepsilon(\boldsymbol{u}))$ can be written in linear thermoelasticity in the following form:

in D_PLAN and C_PLAN:

$$2\Psi(\varepsilon(\boldsymbol{u}))=C_1(\varepsilon_{xx}^2+\varepsilon_{yy}^2)+2C_2\varepsilon_{xx}\varepsilon_{yy}+4C_3\varepsilon_{xy}^2-2\Psi_{th}$$

in 3D:

$$2\Psi(\varepsilon(\boldsymbol{u}))=C_1(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2)+2C_2(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{xx}\varepsilon_{zz}+\varepsilon_{yy}\varepsilon_{zz})+4C_3(\varepsilon_{xy}^2+\varepsilon_{xz}^2+\varepsilon_{yz}^2)-2\Psi_{th}$$

with Ψ_{th} = Density of energy due to the thermal:

$$\Psi_{th}=3K\alpha(T-T_{réf})tr\,\varepsilon$$

where:

$$\begin{aligned} 3K &= \frac{E}{1-2\nu} \\ \alpha &= \text{dilation thermique} \\ \varepsilon &= \text{tenseur de déformations} \\ T_{réf} &= \text{température de référence} \end{aligned}$$

and with:

$$\begin{cases} C_1=\dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}=\lambda+2\mu \\ C_2=\dfrac{\nu E}{(1+\nu)(1-2\nu)}=\lambda \\ C_3=\dfrac{E}{2(1+\nu)}=\mu \end{cases} \text{ in D_PLAN and 3D; } \begin{cases} C_1=\dfrac{E}{(1-\nu^2)} \\ C_2=\dfrac{\nu E}{(1-\nu^2)} \\ C_3=\dfrac{E}{(1+\nu)} \end{cases} \text{ in C_PLAN}$$

Note:

The exact statement of the density of energy due to the thermal Ψ_{th} is actually:
$\Psi_{th}=3K\alpha(T-T_{réf})tr\,\varepsilon-\frac{9}{2}K\alpha^2(T-T_{réf})^2$
One can however show that if one takes into account the term $-\frac{9}{2}K\alpha^2(T-T_{réf})^2$ in the classical term, and its derivative in the thermal term, then the two contributions are cancelled [bib10]. One thus does not take them into account.

While noting $\Psi(\varepsilon(\boldsymbol{u}))=\Psi(\boldsymbol{u},\boldsymbol{u})$, one has $2\Psi(\boldsymbol{u},\boldsymbol{v})=S1-S1TH$ with:

$$S1=C1\left(\frac{\partial u_k}{\partial x_k}\frac{\partial v_k}{\partial x_k}\right)+C2\left(\frac{\partial u_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}(1-\delta_{ij})\right)+C3\left(\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}(1-\delta_{ij})+\frac{\partial u_k}{\partial x_l}\frac{\partial v_l}{\partial x_k}(1-\delta_{kl})\right)$$

$$S1TH=3K\alpha\left(\left(T_u-T_{réf}\right)\ \mathrm{tr}\boldsymbol{\varepsilon}(v)+\left(T_v-T_{réf}\right)\ \mathrm{tr}\boldsymbol{\varepsilon}(u)\right)$$

where indices *I*, *J*, *K* and *L* correspond to a summation on the 2 coordinates of space (resp. 3 coordinates) in 2D (resp. 3D); $T_{\mathbf{u}}$ is the temperature associated with the field with displacement $\mathbf{u}$ by the relation:

$$\sigma=\Lambda(\varepsilon(\boldsymbol{u})-\varepsilon^{th})$$

where $\varepsilon_{ij}^{th}=\alpha(T-T_{réf})\delta_{ij}$ and σ the balance equations check.

In the same way, the term $\sigma(\boldsymbol{u}):(\nabla\boldsymbol{u}\nabla\theta)$ can be written:

$$\sigma(\boldsymbol{u}):(\nabla\boldsymbol{u}\nabla\theta)=S2-S2TH$$

with:

$$S2=C1\left[\frac{\partial u_k}{\partial x_k}\frac{\partial v_k}{\partial x_k}\frac{\partial\theta_k}{\partial x_k}+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_i}\frac{\partial v_i}{\partial x_j}+\frac{\partial v_i}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right)\left(1-\delta_{ij}\right)\frac{\partial\theta_j}{\partial x_i}\right]$$
$$+C2\left[\frac{1}{2}\frac{\partial\theta_i}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\frac{\partial v_j}{\partial x_i}+\frac{\partial v_k}{\partial x_k}\frac{\partial u_j}{\partial x_i}\right)\left(1-\delta_{jk}\right)\right]$$
$$+C3\left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}+\frac{\partial v_j}{\partial x_k}\right)+\frac{\partial v_k}{\partial x_i}\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right)\right)\left(1-\delta_{jk}\right)\frac{\partial\theta_i}{\partial x_j}\right]$$

and:

$$S2TH=\frac{TH1}{2}3K\alpha\left(T_u-T_{réf}\right)\left(\frac{\partial v_i}{\partial x_j}\frac{\partial\theta_j}{\partial x_i}\right)+\frac{THA}{2}3K\alpha\left(T_v-T_{réf}\right)\left(\frac{\partial u_i}{\partial x_j}\frac{\partial\theta_j}{\partial x_i}\right)$$

where $\mathrm{TH1}=1$ in `D_PLAN` and `3D` and $\mathrm{TH1}=\frac{1-2\nu}{1-\nu}$ `C_PLAN`.

Finally the classical term is written:

$$TCLA=\left(S2-S2TH\right)-\frac{1}{2}\left(S1-S1TH\right)\mathrm{div}\,\theta$$

2.3.2 Volume force term

$$TFOR=\left(\nabla f.\theta\right)\boldsymbol{u}+f.\boldsymbol{u}\,\mathrm{div}\,\theta$$

In any rigor, the symmetric bilinear statement of $TFOR$ is written in $(\boldsymbol{u},\boldsymbol{v})$:

$$TFOR(\boldsymbol{u},\boldsymbol{v})=\frac{1}{2}\left[\left(\nabla f_u.\theta\right)\boldsymbol{v}+\left(\nabla f_v.\theta\right)\boldsymbol{u}+\left(f_u.\boldsymbol{v}+f_v.\boldsymbol{u}\right)\mathrm{div}\,\theta\right]$$

where f_u are the volume forces associated with the field with displacement $\boldsymbol{u}$ for the elastic problem.

As the statements which we are brought to calculate are of the type $TFOR(\boldsymbol{u},\boldsymbol{u})$ and $TFOR(\boldsymbol{u},\boldsymbol{u^S})$, where $\boldsymbol{u}$ and $\boldsymbol{u^S}$ are respectively the field of displacement and the singular field, and that:

$$f_{u^S} = \mathrm{div}(\sigma(\boldsymbol{u^S})) = 0 \text{ in ON position } \Omega$$

limits itself to write:

$$TFOR(\boldsymbol{u},\boldsymbol{v}) = CS[(\nabla F_u.\theta)\boldsymbol{v} + F_u.\boldsymbol{v}\,\mathrm{div}\,\theta] \text{ with } \begin{cases} CS = 0.5 & v = u^S \\ CS = 1 & v = u \end{cases}$$

the same remark is valid for the thermal classical term, the additional term due to the thermal and the terms due to the surface forces.

2.3.3 Thermal term

One makes the assumption that the characteristics of the material (E,ν,α) do not depend on the temperature.

in D_PLAN and C_PLAN : $TTHE = \frac{-\partial\Psi}{\partial T}(\nabla T.\theta) = \frac{1}{2}3K\alpha tr\varepsilon(\frac{\partial T}{\partial x}\theta_x + \frac{\partial T}{\partial y}\theta_y)$

in 3D : $TTHE = \frac{1}{2}3K\alpha tr\varepsilon(\frac{\partial T}{\partial x}\theta_x + \frac{\partial T}{\partial y}\theta_y + \frac{\partial T}{\partial z}\theta_z)$

2.3.4 Term forces surface

the term forces surface is written:

$$TSUR = (\nabla \boldsymbol{F}.\theta)\boldsymbol{u} + \boldsymbol{F}.\boldsymbol{u}(\mathrm{div}\,\theta - n.\frac{\partial\theta}{\partial n})$$

The term $n.\frac{\partial\theta}{\partial n}$ is null because the gradient of θ is orthogonal with n. As for the volume force term, the bilinear statement $TSUR(\mathrm{u},\mathrm{v})$ is obtained by considering that v is either equal to displacement u (and thus $\mathrm{F}_v = \mathrm{F}_u$), or at a singular field u^S (and thus $\mathrm{F}_v = 0$). One has then:

$$TSUR(u,v) = CS[(\nabla F_u.\theta)v + F_u.v\,\mathrm{div}\,\theta] \text{ with } \begin{cases} CS = 0.5 & v = u^S \\ CS = 1 & v = u \end{cases}$$

2.3.5 Term of modal dynamics

In the case as of dynamic problems, an additional term $TDYN$ appears in the decomposition of rate of energy restitution expressed in paragraph 1.3 :

$$TDYN = -\int_\Omega \frac{1}{2}\rho\dot{u}\ddot{u}.div\,\sigma.d\Omega - \int_\Omega \frac{1}{2}\rho\dot{u}\nabla\dot{u}.\theta.d\Omega + \int_\Omega \frac{1}{2}\rho\ddot{u}\nabla u.\theta.d\Omega$$

where $\dot{u}$ and $\ddot{u}$ are respectively the derivative first and second of the field of displacement u compared to time. For the harmonic problems, this term can be simplified [feeding-bottle 10]. By noting ω the pulsation of the studied eigen mode and ρ the density, the bilinear form of the dynamic term is written as follows:

$$TDYN(u,v) = -\frac{1}{2}(\rho\omega^2 u_k v_{k,j}\theta_j + \rho\omega^2 v_k u_{k,j}\theta_j) .$$

2.4 Fields of singular displacements and their derivatives

the singular fields u_S^I, u_S^{II} and u_S^{III}, respectively associated with the modes I, II and III, are known explicitly like their derivatives. They are written according to the polar coordinates in the reference related to crack.
The successive introduction of these singular fields allows, as indicated in the §1, the elementary computation of the stress intensity factors K_I, K_{II} and K_{III}.

2.5 Postprocessing of the results of KI and KII

2.5.1 For the problems 2D

Knowing the values of the stress intensity factors K_I and K_{II} for a given crack, the formulas of AMESTOY - BUI and DANG-VAN, allow the computation of the angle of propagation of crack according to 3 criteria (K_I maximum, K_{II} and G maximum) [bib6].

That is to say Ω_m^η a field identical to Ω except that the crack is prolonged in the direction of angle m of a line segment length η.

$$\Omega_m^0 \equiv \Omega$$

Are $K_I(\eta, m)$ $K_{II}(\eta, m)$, $G(\eta, m)$ the stress intensity factors and the rate of energy restitution of Ω_m^η subjected to the same loading as Ω.

One poses:

$$\begin{aligned} K_I^*(m) &= \lim_{\eta\to 0} K_I(\eta, m) \\ K_{II}^*(m) &= \lim_{\eta\to 0} K_{II}(\eta, m) \\ G^*(m) &= \lim_{\eta\to 0} G(\eta, m) \end{aligned}$$

Criteria quoted by AMESTOY - BUI and DANG-VAN [bib6] are:

- to choose m_0 such as $K_I^*(m_0)$ either maximum,
- to choose m_0 such as $K_{II}^*(m_0)$ or no one,
- to choose m_0 such as $G^*(m_0)$ or maximum.

These criteria give very nearby results [bib8].

The results are given in the form of a table of 4 coefficients K_{11} K_{21} K_{12} , K_{22} making it possible to calculate K_I^* and K_{II}^* in all the cases of loading:

$$\begin{pmatrix} K_I^* \\ K_{II}^* \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} K_I \\ K_{II} \end{pmatrix}$$

Angle m (°)	K_{11}	K_{21}	K_{12}	K_{22}
0	1	0	0	1
10	0,9886	0,0864	— 0,2597	0,9764
20	0,9552	0,1680	— 0,5068	0,9071
30	0,9018	0,2403	— 0,7298	0,7972
40	0,8314	0,2995	— 0,9189	0,6540
50	0,7479	0,3431	— 1,0665	0,4872
60	0,6559	0,3696	— 1,1681	0,3077
70	0,5598	0,3788	— 1,2220	0,1266
80	0,4640	0,3718	— 1,2293	— 0,0453
90	0,3722	0,3507	— 1,1936	— 0,1988

$$K_{11}(-m)=K_{11}(m),\ K_{21}(-m)=-K_{21}(m),\ K_{12}(-m)=-K_{12}(m),\ K_{22}(-m)=K_{22}(m)$$

the search of the angle m_0 in CALC_G is made of 10 degrees in 10 degrees. The angle β of propagation is not calculated and is printed out (in the message file) only if INFO is worth 2.

2.5.2 For the problems 3D

the direction of propagation of a crack in 3D can be given by the principle of *the Maximum Hoop Stress Criterion* (maximization of the circumferential stress) [bib11]. The angle of propagation is expressed then in the following way:

$$\beta = 2\arctan\left[\frac{1}{4}.\left(\frac{K_I}{K_{II}} - sign(K_{II}).\sqrt{\left(\frac{K_I}{K_{II}}\right)^2 + 8}\right)\right]$$

The angle β , calculated systematically, is indicated in the table result produced by the command CALC_G (column BETA).

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