Rate of refund of energy in nonlinear thermoelasticity

Summarized:

One presents the computation of the rate of refund of energy by the method theta in 2D or 3D for a nonlinear thermoelastic problem. The nonlinear elastic behavior model is described in [R5.03.20].

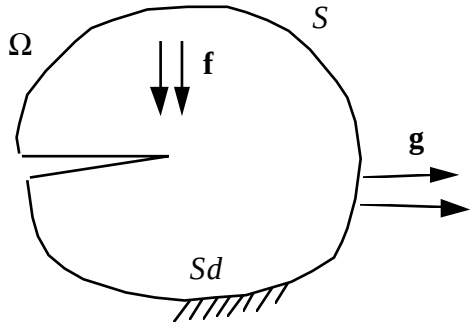
Contents

1 Computation of the rate of refund of energy by the method theta in nonlinear thermoelasticity

1.1 Behavior model

One considers a fissured solid occupying the field Ω of space R^2 or R^3. That is to say:

u the field of displacement,
T the field of temperature,
f the field of volume forces applied to Ω,
g the field of surface forces applied to part S of $\partial\Omega$,
U the field of displacements imposed on part S_d of $\partial\Omega$.



The behavior of solid is supposed to be elastic nonlinear such as the behavior model coincides with the elastoplastic model of Hencky-Von Put (isotropic hardening) in the case of a loading which induces a radial and monotonous evolution in any point. This model is selected in commands CALC_G via key word RELATION=' ELAS_VMIS_LINE' or "ELAS_VMIS_TRAC" or "ELAS_VMIS_PUIS" under factor key word the COMP_ELAS [R5.03.20].

One indicates by:

$\boldsymbol{\varepsilon}$ the strain tensor,
$\boldsymbol{\varepsilon}^\circ$ the tensor of the initial strains,
$\boldsymbol{\sigma}$ the tensor of the stresses,
$\boldsymbol{\sigma}^\circ$ the tensor of the initial stresses,
$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^\circ, \boldsymbol{\sigma}^\circ, T)$ density of free energy.

$\boldsymbol{\varepsilon}$ is connected to the field of displacement u by:

$$\boldsymbol{\varepsilon}(u)=\frac{1}{2}(\mathbf{u_{i,j}}+\mathbf{u_{j,i}})$$

The density of free energy $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^\circ, \boldsymbol{\sigma}^\circ, T)$ is a convex and differentiable, known function for a given state [R5.03.20 éq 3]. The behavior model of the material is written in the form:

$$\boldsymbol{\sigma}_{\mathbf{ij}}=\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}_{\mathbf{ij}}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^\circ, \boldsymbol{\sigma}^\circ, T)$$

It derives from the potential free energy. For this behavior model hyper elastic, one can give a meaning to rate of energy restitution in the frame of the comprehensive approach in fracture mechanics. It is not the case for a plastic behavior model.

1.2 Potential energy and relations of equilibrium

One defines spaces of the kinematically admissible fields V and V_0 .

$$V = \{\mathbf{v}\,\text{admissibles}, \mathbf{v}=\mathbf{U}\,\text{sur}\,S_d\}$$
$$V_0 = \{\mathbf{v}\,\text{admissibles}, \mathbf{v}=\mathbf{O}\,\text{sur}\,S_d\}$$

With the assumptions of the paragraph [§1.1] (with $\boldsymbol{\varepsilon}^\circ=\boldsymbol{\sigma}^\circ=0$), the relations of equilibrium in weak formulation are:

$$\begin{cases}\mathbf{u}\in V \\ \int_\Omega \boldsymbol{\sigma}_{\mathbf{ij}}\mathbf{v}_{\mathbf{i,j}}\,d\Omega=\int_\Omega \mathbf{f}_\mathbf{i}\mathbf{v}_\mathbf{i}\,d\Omega+\int_\Omega \mathbf{g}_\mathbf{i}\mathbf{v}_\mathbf{i}\,d\Gamma\,, \quad \forall\,\mathbf{v}\in V_0\end{cases}$$

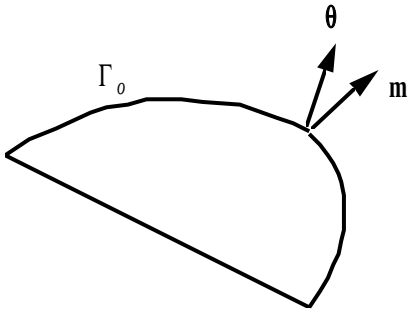
They are obtained by minimizing the total potential energy of the system:

$$\mathbf{W}(\mathbf{v})=\int_\Omega \boldsymbol{\Psi}(\boldsymbol{\varepsilon}(\mathbf{v}),T)\,d\Omega=\int_\Omega \mathbf{f}_\mathbf{i}\mathbf{v}_\mathbf{i}\,d\Omega+\int_\Omega \mathbf{g}_\mathbf{i}\mathbf{v}_\mathbf{i}\,d\Gamma$$

The demonstration is identical to that in linear elasticity [R7.02.01 §1.2].

1.3 Lagrangian statement of the rate of refund of energy

Is $\boldsymbol{m}$ the unit norm with Γ_0 located in the tangent plane in $\partial\Omega$ Ω .



That is to say the field $\boldsymbol{\theta}$ such as:

$$\boldsymbol{\theta}\in\Theta=\{\boldsymbol{\mu}\,\text{tels que}\,\boldsymbol{\mu}\,.\,\mathbf{n}=0\,\text{sur}\,\Omega\}$$

by noting $\mathbf{n}$ the norm with $\partial\Omega$.

Rate of energy restitution G is solution of the variational equation:

$$\int_{\Gamma_o} G\boldsymbol{\theta}\cdot\mathbf{m}\,\mathrm{d}S = G(\boldsymbol{\theta}),\ \forall\boldsymbol{\theta}\in\Theta$$

where $G(\boldsymbol{\theta})$ is defined by:

$$\begin{aligned} G(\theta) \;&= \int_\Omega \boldsymbol{\sigma}_{ij}\,\mathbf{u}_{\mathbf{i},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}} - \boldsymbol{\Psi}\,\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}} - \frac{\partial\boldsymbol{\Psi}}{\partial T}\mathbf{T}_{,\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}}\,d\,\Omega \\ &+ \int_\Omega \left(\boldsymbol{\sigma}_{ij} - \frac{1}{2}\boldsymbol{\sigma}^{\circ}_{ij}\right)\boldsymbol{\varepsilon}^{\circ}_{ij,\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}} - \left(\boldsymbol{\varepsilon}_{ij} - \boldsymbol{\varepsilon}^{th}_{ij} - \frac{1}{2}\boldsymbol{\varepsilon}^{\circ}_{ij}\right)\boldsymbol{\sigma}_{ij,\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}}\,d\,\Omega \\ &+ \int_\Omega \mathbf{f}_{\mathbf{i}}\,\mathbf{u}_{\mathbf{i}}\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}} + \mathbf{f}_{\mathbf{i},\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}}\,\mathbf{u}_{\mathbf{i}}\,d\,\Omega \\ &+ \int_S \mathbf{g}_{\mathbf{i},\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}}\,\mathbf{u}_{\mathbf{i}} + \mathbf{g}_{\mathbf{i}}\,\mathbf{u}_{\mathbf{i}}\left(\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}} - \frac{\partial\boldsymbol{\theta}}{\partial\,\mathbf{n}_{\mathbf{k}}}\mathbf{n}_{\mathbf{k}}\right)d\,\Gamma \\ &- \int_{S_d} \boldsymbol{\sigma}_{ij}\,\mathbf{n}_{\mathbf{j}}\mathbf{U}_{\mathbf{i},\mathbf{k}}\boldsymbol{\theta}_{\mathbf{k}}\,d\,\Gamma \end{aligned}$$

The demonstration is identical to that of the computation of G in linear elasticity [R7.02.01]. The statement is the same one, postprocessing is thus identical.

1.4 Establishment of G in nonlinear thermoelasticity in *the Code_Aster*

element types and of loadings, the environment necessary are the same ones as for the establishment of G in linear thermoelasticity [R7.02.01 §2.4].

For the computation of the various terms of $G(\boldsymbol{\theta})$, in a given state, one recovers the density of free energy $\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, T)$, the strains $\boldsymbol{\varepsilon}$ and the forced $\boldsymbol{\sigma}$, calculated for the linear behavior model not - (routine NMELNL).

It is supposed that $\boldsymbol{\varepsilon}^{\circ} = \boldsymbol{\sigma}^{\circ} = 0$ (identical term in linear or nonlinear thermoelasticity). The density of free energy is written then [R5.03.20 §1.5]:

in linear thermoelasticity:

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, T) = \frac{1}{2}K\left(\boldsymbol{\varepsilon}_{kk} - 3\,\alpha\left(T - T_{réf}\right)\right)^2 + \frac{2\,\mu}{3}\boldsymbol{\varepsilon}^2_{eq}$$

with

$$\boldsymbol{\varepsilon}^2_{eq} = \frac{3}{2}\boldsymbol{\varepsilon}^{\mathbf{D}}_{ij}\boldsymbol{\varepsilon}^{\mathbf{D}}_{ij} = \frac{3}{2}\left(\boldsymbol{\varepsilon}_{ij} - \frac{1}{3}\boldsymbol{\varepsilon}_{kk}\boldsymbol{\delta}_{ij}\right)\left(\boldsymbol{\varepsilon}_{ij} - \frac{1}{3}\boldsymbol{\varepsilon}_{kk}\boldsymbol{\delta}_{ij}\right)$$

$$\boldsymbol{\varepsilon}^2_{eq} = \frac{3}{2}\left(\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{ij} - \frac{1}{3}\boldsymbol{\varepsilon}^2_{kk}\right)$$

in nonlinear thermoelasticity $\left(2\,\mu\,\boldsymbol{\varepsilon}_{eq} \geq \boldsymbol{\sigma}_{\mathbf{y}}\right)$:

$$\boldsymbol{\Psi}(\boldsymbol{\varepsilon}, T) = \frac{1}{2}K\left(\boldsymbol{\varepsilon}_{kk} - 3\,\alpha\left(T - T_{réf}\right)\right)^2 + \frac{1}{6\,\mu}\mathbf{R}\left(p\left(\boldsymbol{\varepsilon}_{eq}\right)\right)^2 + \int_0^{p\left(\boldsymbol{\varepsilon}_{eq}\right)} R(s)\,ds$$

with $\mathbf{R}\left(p\left(\boldsymbol{\varepsilon}_{eq}\right)\right)$: function of hardening.

For a linear isotropic hardening (RELATION= "ELAS_VMIS_LINE") one a:

$$\mathbf{R}(p)=\boldsymbol{\sigma}_{\mathrm{y}}+p\frac{E\,E_T}{E-E_T}=\boldsymbol{\sigma}_{\mathrm{y}}+a\,p$$
$$p=\frac{\boldsymbol{\sigma}_{eq}-\boldsymbol{\sigma}_{\mathrm{y}}}{3\,\mu+a}\quad \text{avec } a=\frac{E\,E_T}{E-E_T}$$
$$A(p)=\int_0^p \mathbf{R}(s)\,ds=\ \boldsymbol{\sigma}_{yp}+\frac{1}{2}a\,p^2=\frac{1}{2}\boldsymbol{\sigma}_{\mathrm{y}}\,p+\frac{p}{2}\left(\boldsymbol{\sigma}_{\mathrm{y}}+a\,p\right)$$
$$A(p)=\frac{p}{2}\left(\boldsymbol{\sigma}_{\mathrm{y}}+\mathbf{R}(p)\right)$$

postprocessing is then identical to the problem in linear elasticity except for the thermal term:

$$\mathrm{THER}=-\frac{\partial \boldsymbol{\Psi}}{\partial T}\mathbf{T}_{,\mathrm{k}}\boldsymbol{\theta}_{\mathrm{k}}$$

If the coefficients of Lamé $\lambda(T)$ and $\mu(T)$ are independent of the temperature, this term is null. In the contrary case, it is necessary to calculate $\frac{\partial \boldsymbol{\Psi}}{\partial T}(\boldsymbol{\varepsilon},T)$ at a given time.

For a linear isotropic hardening, one a:

$$\frac{\partial \boldsymbol{\Psi}}{\partial T}\quad(\boldsymbol{\varepsilon},T)=\left[\frac{1}{2}\frac{dK(T)}{dT}\left(\boldsymbol{\varepsilon}_{\boldsymbol{kk}}-3\,\alpha\left(T-T_{réf}\right)\right)-3\mathrm{K}\left(\alpha+\frac{d\,\alpha(T)}{dT}\left(T-T_{réf}\right)\right)\right]\left(\boldsymbol{\varepsilon}_{\boldsymbol{kk}}-3\,\alpha\,T-T_{réf}\right)$$
$$+\frac{R(p)}{6\,\mu^2}\left[2\,\mu\frac{dR(P)}{dT}-\frac{d\,\mu(T)}{dT}R(p)\right]+\frac{dA(p)}{dT}$$

with

$$\frac{dR(p)}{dT}=\frac{d\,\boldsymbol{\sigma}_{\mathrm{y}}(T)}{dT}+\frac{d\,a(T)}{dT}\,p+a\frac{d\,p(T)}{dT}$$
$$\frac{d\,a(T)}{dT}=\frac{1}{\left(E-E_T\right)^2}\left(\frac{d\,E_T(T)}{dT}E^2-\frac{d\,E(T)}{dT}E_T^{\,2}\right)$$
$$\frac{dp(T)}{dT}=\frac{1}{\left(3\,\mu+a\right)^2}\left[\left(\boldsymbol{\sigma}_{\mathrm{y}}-\boldsymbol{\sigma}_{eq}\right)\left(3\frac{d\,\mu(T)}{dT}+\frac{dA(T)}{dT}\right)-\left(3\,\mu+a\right)\frac{d\,\boldsymbol{\sigma}_{\mathrm{y}}(T)}{dT}\right]$$
$$\frac{\mathrm{dA}(p)}{dT}=\frac{1}{2}\frac{dp(T)}{dT}\left(\boldsymbol{\sigma}_{\mathrm{y}}+R_p\right)+\frac{1}{2}p\left(\frac{d\,\boldsymbol{\sigma}_{\mathrm{y}}(T)}{dT}+\frac{\mathrm{d\,Rp}(T)}{dT}\right)$$

1.5 Warning

Attention! By definition, in the general case:

$$\Psi(\varepsilon, T) \neq \sigma : \varepsilon$$

Although it is possible to carry out a followed elastoplastic computation by the computation of G in nonlinear elasticity, it should well be known that does not have any thermodynamic meaning and that it is normal that result depends on the field θ.

2 Computation of the rate of refund of energy by the method theta in great transformations

One extends the behavior model of [§1] to large displacements and large rotations, insofar as it derives from a potential (model hyper elastic). This functionality is started by the key word DEFORMATION=' GROT_GDEP' in command CALC_G.

2.1 Behavior model

One indicates by:

$\mathbf{E}$ the strain tensor of Green-Lagrange,
$\mathbf{S}$ the tensor of the stresses of Piola-Lagrange still called second tensor of Piola - Kirchoff,
$\Psi(\mathbf{E})$ density of internal energy.

The behavior of solid is supposed hyper elastic, namely that:

$\mathbf{E}$ is connected to the field of displacement $\mathbf{u}$ measured compared to the reference configuration Ω_0 by:

$$\mathbf{E_{ij}}(\mathbf{u}) = \frac{1}{2}(\mathbf{u_{i,j}} + \mathbf{u_{j,i}} + \mathbf{u_{k,i}}\,\mathbf{u_{k,j}})$$

$\mathbf{S}$ is connected to the tensor of the stresses of cauchy T by:

$$\mathbf{S_{ij}} = \det(\mathbf{F})\,\mathbf{F_{ik}^{-1}}\,\mathbf{T_{kl}}\,\mathbf{F_{jl}^{-1}}$$

$\mathbf{F}$ being the gradient of the transformation which makes pass from the reference configuration Ω_0 to the present configuration Ω, connected to displacement by:

$$\mathbf{F_{ij}} = (\delta_{ij} + \mathbf{u_{i,j}})$$

The behavior model of a material hyper elastic is written in the form:

$$\mathbf{S_{ij}} = \frac{\partial \Psi}{\partial \mathbf{E_{ij}}} = \frac{\partial \Psi}{\partial \mathbf{E_{jj}}} = \mathbf{S_{ji}}$$

This relation describes one nonlinear elastic behavior, similar to that of [§1.1]. She gives the opportunity of dealing with the problems of fracture mechanics without integrating plasticity into it. And in the case of a monotonous radial loading, it makes it possible to obtain strains and stresses of structure similar to those which one would obtain if the material presented an isotropic hardening. The material hyper elastic has a reversible structural mechanics behavior, i.e. any cycle of loading does not generate any dissipation.

This model is selected in command CALC_G [U4.82.03] via the key word:

RELATION: "ELAS"
for an elastic relation "linear", i.e. the relation between the strains and the forced considered is linear,

RELATION: "ELAS_VMIS_LINE" or "ELAS_VMIS_TRAC" or "ELAS_VMIS_PUIS"
for an elastic behavior model "nonlinear" (model of HENCKY-VON PUT at isotropic hardening).

Such a behavior model makes it possible in any rigor to take into account large deformations and large rotations. However, one confines oneself with small strains to ensure the existence of a solution and to be identical to an elastoplastic behavior under a monotonous radial loading [R5.03.20 §2.1].

2.2 Potential energy and relations of equilibrium

the loading considered is reduced to one density surface nonfollowing $\mathbf{R}$ applied to part Γ_0 of edge of Ω_0 (assumption of the dead loads [R5.03.20 §2.2]).

A kinematically admissible space of the fields is defined V:

$$\mathrm{V} = \{ \mathbf{v} \text{ admissibles}, \mathbf{v} = 0 \text{ sur } \Gamma_0 \}$$

The relations of equilibrium in weak formulation are:

$$\int_{\Omega_0} \mathbf{F_{ik}} \mathbf{S_{kj}} \mathbf{v_{i,j}} d\Omega_0 = \int_{\Gamma} \mathbf{R_i} \mathbf{v_i} d\Gamma$$

They can be obtained by minimizing the total potential energy of the system:

$$W(\mathbf{v}) = \int_{\Omega_0} \Psi(\mathbf{E}(\mathbf{v})) d\Omega - \int_{\Gamma} \mathbf{R_i} \mathbf{v_i} d\Gamma$$

Indeed, if this functional calculus is minimal for the field of displacement $\mathbf{u}$, then:

$$\begin{aligned}
\delta W &= \int_{\Omega_0} \frac{\partial \Psi}{\partial \mathbf{E_{ij}}} \delta \mathbf{E_{ij}} d\Omega - \int_{\Gamma} \mathbf{R_i} \delta \mathbf{v_i} d\Gamma = 0, \qquad \forall \delta \mathbf{v} \in \mathrm{V} \\
&= \int_{\Omega_0} \mathbf{S_{ij}} \frac{\mathbf{1}}{\mathbf{2}} (\delta \mathbf{v_{i,j}} + \delta \mathbf{v_{j,i}} + \delta \mathbf{v_{p,i}} \mathbf{u_{p,j}} + \mathbf{u_{p,i}} \delta \mathbf{v_{p,j}}) d\Omega - \int_{\Gamma} \mathbf{R_i} \delta \mathbf{v_i} d\Gamma \\
&= \int_{\Omega_0} \mathbf{S_{ij}} (\boldsymbol{\delta_{ip}} + \mathbf{u_{p,i}}) \delta \mathbf{v_{p,i}} d\Omega - \int_{\Gamma} \mathbf{R_i} \delta \mathbf{v_i} d\Gamma \\
&= \int_{\Omega_0} \mathbf{F_{pi}} \mathbf{S_{ij}} \delta \mathbf{v_{p,j}} d\Omega - \int_{\Gamma} \mathbf{R_i} \delta \mathbf{v_i} d\Gamma \\
&= \int_{\Omega_0} \mathbf{F_{ik}} \mathbf{S_{kj}} \delta \mathbf{v_{i,j}} d\Omega - \int_{\Gamma} \mathbf{R_i} \delta \mathbf{v_i} d\Gamma = 0
\end{aligned}$$

We thus find the balance equations and the behavior model while having posed:

$$\mathbf{S}_{ij} = \frac{\partial \Psi}{\partial \mathbf{E}_{ij}}$$

2.3 Lagrangian statement of the rate of refund of energy in nonlinear thermoelasticity and great transformations

By definition, rate of energy restitution G is defined by the opposite of derivative of potential energy in the equilibrium compared to the field Ω [bib1]. It is calculated by the method theta, which is a Lagrangian method of derivative of potential energy [bib4] and [bib2]. One considers transformations $F^{\eta}: \mathbf{M} \to \mathbf{M} + \eta \boldsymbol{\theta}(\mathbf{M})$ of the field in Ω_0 a field Ω_{η} which correspond to propagations of crack. A these families of reference configuration thus defined Ω_{η} correspond of the families of deformed configurations where the crack was propagated. The rate of refund of energy G is then the opposite of derivative of potential energy $W(\mathbf{u}(\eta))$ to the equilibrium compared to the initial evolution of the crack tip: η

$$G = - \left(\frac{\mathrm{d}\, W(\mathbf{u}(\eta))}{\mathrm{d}\, \eta} \right)_{\eta = 0}$$

One notes as in [feeding-bottle 4] par. Lagrangian derivative in a virtual crack propagation velocity $\boldsymbol{\theta}$. That is to say $\boldsymbol{\varphi}(\eta, \mathbf{M})$ an unspecified field (η real positive and $\mathbf{M}$ belonging to field Ω_0), we will note:

$$\bar{\boldsymbol{\varphi}}(\eta, \mathbf{M}) = \bar{\boldsymbol{\varphi}}(\eta, F^{\eta}\mathbf{M}) \text{ et } \dot{\boldsymbol{\varphi}} = \left(\frac{\partial \bar{\boldsymbol{\varphi}}}{\partial \eta} \right)_{\eta = 0}$$

The potential energy defined on Ω_η is brought back on Ω_0, $\mathbf{R}$ is supposed to be independent of η, the derivative compared to the parameter of propagation η is then easy and the rate of refund of energy in this propagation is solution of the variational equation:

$$\int_{\Gamma_o} G\boldsymbol{\theta}\cdot\mathbf{m}\,\mathrm{d}S = G(\boldsymbol{\theta}) \qquad ,\forall\,\boldsymbol{\theta}\in\Theta$$

with:

$$-G(\boldsymbol{\theta}) = \int_{\Omega_o} \dot{\overbrace{\Psi(\mathbf{E},T)}}\,\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}}\,\mathrm{d}\,\Omega - \int_\Gamma \mathbf{R}_\mathbf{i}\dot{\mathbf{u}}_\mathbf{i} + \mathbf{R}_{\mathbf{i},\mathbf{k}}\boldsymbol{\theta}_\mathbf{k}\mathbf{u}_\mathbf{i} + \mathbf{R}_\mathbf{i}\mathbf{u}_\mathbf{i}\left(\boldsymbol{\theta}_{\mathbf{k},\mathbf{k}} - \frac{\partial\boldsymbol{\theta}}{\partial\mathbf{n}_\mathbf{k}}\mathbf{n}_\mathbf{k}\right)\mathrm{d}\,\Gamma$$

However:

$$\dot{\overbrace{\Psi(\mathbf{E},T)}} = \frac{\partial\Psi}{\partial\mathbf{E}_{ij}}\dot{\mathbf{E}}_{ij} + \frac{\partial\Psi}{\partial T}\dot{T}$$

Thereafter, we will consider only the term $\frac{\partial\Psi}{\partial\mathbf{E}_{ij}}\dot{\mathbf{E}}_{ij}$, the thermal term being treated in the same way that into small displacement [R7.02.01].

And according to proposal 2 of [bib4]:

$$\dot{\mathbf{E}}_{ij} = \frac{1}{2}\left(\dot{\mathbf{u}}_{\mathbf{i},\mathbf{j}} + \dot{\mathbf{u}}_{\mathbf{j},\mathbf{i}} + \dot{\mathbf{u}}_{\mathbf{k},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{j}} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\dot{\mathbf{u}}_{\mathbf{k},\mathbf{j}}\right)$$
$$-\frac{1}{2}\left(\mathbf{u}_{\mathbf{i},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{j}} + \mathbf{u}_{\mathbf{j},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{i}} + \mathbf{u}_{\mathbf{k},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{j}} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{j}}\right)$$

One can eliminate $\dot{\mathbf{u}}$ from the statement from G as in small strains by noticing that $\dot{u}$ is kinematically admissible (cf [bib3] for the problems of regularity) and by means of the balance equation:

$$\int_{\Omega_o} \dot{\overbrace{\Psi(\mathbf{E})}}\,\mathrm{d}\,\Omega - \int_\Gamma \mathbf{R}_\mathbf{i}\dot{\mathbf{u}}_\mathbf{i}\,\mathrm{d}\,\Gamma =$$
$$\int_{\Omega_o}\frac{\partial\Psi}{\partial\mathbf{E}_{ij}}\frac{1}{2}\left(\dot{\mathbf{u}}_{i,j} + \dot{\mathbf{u}}_{j,i} + \dot{\mathbf{u}}_{k,i}\mathbf{u}_{\mathbf{k},\mathbf{j}} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\dot{\mathbf{u}}_{k,j}\right)d\,\Omega$$
$$-\int_\Gamma \mathbf{R}_\mathbf{i}\dot{\mathbf{u}}_i\,d\,\Gamma - \int_{\Omega_o}\frac{\partial\Psi}{\partial\mathbf{E}_{ij}}\frac{1}{2}\left(\mathbf{u}_{\mathbf{i},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}} + \mathbf{u}_{\mathbf{j},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{i}} + \mathbf{u}_{\mathbf{k},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{j}} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}}\right)d\,\Omega$$
$$= -\int_{\Omega_o}\mathbf{S}_{ij}\left(\mathbf{u}_{\mathbf{i},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\mathbf{u}_{\mathbf{k},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}}\right)d\,\Omega$$
$$= -\int_{\Omega_o}\mathbf{S}_{ij}\left(\mathbf{d}_{ki} + \mathbf{u}_{\mathbf{k},\mathbf{i}}\right)\mathbf{u}_{\mathbf{k},\mathbf{p}}\boldsymbol{\theta}_{\mathbf{p},\mathbf{j}}\,d\,\Omega$$
$$= -\int_{\Omega_o}\mathbf{S}_{ij}\mathbf{F}_{ki}\mathbf{u}_{\mathbf{k},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{j}}\,d\,\Omega$$
$$= -\int_{\Omega_o}\mathbf{F}_{ik}\mathbf{S}_{kj}\mathbf{u}_{\mathbf{i},\mathbf{p}}\mathbf{q}_{\mathbf{p},\mathbf{j}}\,d\,\Omega$$

Finally, one obtains:

$$G(\theta)=\int_{\Omega_0} \mathbf{F_{ik}}\mathbf{S_{kj}}(\mathbf{u_{i,p}}\theta_{\mathbf{p,j}})-\Psi(\mathbf{E})\theta_{\mathbf{k,k}}\,\mathrm{d}\,\Omega+\int_{\Gamma}\mathbf{R_{i,k}}\theta_{\mathbf{k}}\mathbf{u_i}+\mathbf{R_i}\mathbf{u_i}(\theta_{\mathbf{k,k}}-\frac{\partial\theta}{\partial\mathbf{n_k}}\mathbf{n_k})\,\mathrm{d}\,\Gamma$$

The statement supplements for the following loadings:

nonfollowing surface density $\mathbf{R}$ applied to part Γ of edge of Ω_0 ,
nonfollowing voluminal density $\mathbf{f}$ applied to the field Ω ,

and by taking account of the thermal:

$$\begin{aligned}G(\theta)= & \int_{\Omega_0}\mathbf{F_{ik}}\mathbf{S_{kj}}(\mathbf{u_{i,p}}\theta_{\mathbf{p,j}})-\Psi(\mathbf{E})\theta_{\mathbf{k,k}}-\frac{\partial\Psi}{\partial\mathbf{T}}\mathbf{T_{,k}}\theta_{\mathbf{k}}\,\mathrm{d}\,\Omega\\ &+\int_{\Omega_0}\mathbf{f_i}\mathbf{u_i}\theta_{\mathbf{k,k}}+\mathbf{f_{i,k}}\theta_{\mathbf{k}}\mathbf{u_i}\,\mathrm{d}\,\Omega\\ &+\int_{\Gamma}\mathbf{R_{i,k}}\theta_{\mathbf{k}}\mathbf{u_i}+\mathbf{R_i}\mathbf{u_i}(\theta_{\mathbf{k,k}}-\frac{\partial\theta}{\partial\mathbf{n_k}}\mathbf{n_k})\,\mathrm{d}\,\Gamma\end{aligned}$$

2.4 Establishment in *the Code_Aster*

the comparison of the formulas of $G(\theta)$ [§1.3] and [§1.4] watch which the terms of $G(\theta)$ are very close. The introduction of the great transformations requires little modification in postprocessing.

The presence of key word DEFORMATION=' GROT_GDEP' under factor key word COMP_ELAS of the command CALC_G indicates that it is necessary to recover the tensor of the stresses of Piola-Lagrange $\boldsymbol{S}$ and the gradient of the transformation $\boldsymbol{F}$ (routines NMGEOM and NMELNL).

Element types finished are the same ones as in linear elasticity [R7.02.01 §2.4]. They are the isoparametric elements 2D and 3D.

The supported loadings are those supported in linear elasticity provided that they are dead loads: typically an imposed force is a dead load while the pressure is a following loading since it depends on the directional sense of surface, therefore of the transformation.

2.5 Restriction

With the behavior model specified on the §2, one has a formulation of G valid for large deformations for materials very-elastics, but… if one wishes a coherence with the actual material which, let us recall it, is elastoplastic, it is imperative to confine oneself with small strains, displacements and the rotations being able to be tall.

The conditions of loadings proportional and monotonous, essential to ensure the coherence of the model with the actual material, lead to important restrictions of the field of with the capable problems being dealt by this method (thermal in particular can lead it to local discharges). It can thus be a question only of one palliative solution before being able to give a meaning to rate of energy restitution in the frame of plastic behaviors.

3 Bibliography

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4 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of modifications
4	4E.VISSE EDF-R&D/MMN	initial Text
10,1,1	J.M.Proix, R.Bargellini EDF/R & D/ AMA Change	of key word GREEN in GROT_GDEP