Viscoplastic constitutive law LETK

Summarized:

Model L&K describes a behavior élasto-visco-plastic of the rocks. Elastoplasticity is characterized by a positive hardening in pre-peak and a lenitive behavior beyond strength. Viscoplasticity translates the effect of time on the behavior. It is described by a model in power of Perzyna.

The initiation of the phenomena elastoplastic or viscoplastic starts as of the crossing of the corresponding threshold The behavior related to each phase is described by the evolution of these various thresholds. This evolution is governed by functions of plastic or viscoplastic hardening.

For the elastoplastic mechanism, a surface of load evolves through various thresholds:

- A threshold of damage confused with the threshold of initial viscosity,
- a macroscopic threshold of peak, defined starting from the laboratory tests,
- an intermediate, qualified threshold of limit of cleavage, determined analytically,
- a definite threshold characteristic as the envelope of the threshold of damage and limit of cleavage, called also limiting of contractance/dilatancy (this limit is confused with the threshold of maximum viscoplasticity),
- a threshold of residual strength.

For the viscous mechanism, a viscoplastic surface evolves through:

An initial threshold confused with the threshold of damage
a maximum threshold of viscosity considered confused with the limit of contractane/dilatancy

Contents

1 Notations

1.1 General information

$\boldsymbol{\sigma}$ indicates the tensor of the effective stresses in small disturbances, noted in the shape of the following vector:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\,\sigma_{12} \\ \sqrt{2}\,\sigma_{13} \\ \sqrt{2}\,\sigma_{23} \end{pmatrix}$$

One notes:

$I_1=tr(\boldsymbol{\sigma})$	first tensor invariant of
$\boldsymbol{s}=\boldsymbol{\sigma}-\frac{I_1}{3}\boldsymbol{I}$	the stresses of the stresses déviatoires
$s_{II}=\sqrt{\boldsymbol{s}\,.\,\boldsymbol{s}}$	second invariant of the tensor of the stresses déviatoires
$\sigma_{\max}$	major principal stress
$\sigma_{\min}$	minor principal stress
$\tilde{\boldsymbol{\varepsilon}}=\boldsymbol{\varepsilon}-\frac{Tr(\boldsymbol{\varepsilon})}{3}\boldsymbol{I}$	deviator of the strains
$\varepsilon_V=tr(\boldsymbol{\varepsilon})$	voluminal strain
$\cos(3\theta)=2^{1/2}\;3^{3/2}\,\frac{\det(\boldsymbol{s})}{s_{II}^3}$	θ being the angle of Lode
$\dot{\gamma}_p=\sqrt{\frac{2}{3}\tilde{\dot{\varepsilon}}_{ij}^p\tilde{\dot{\varepsilon}}_{ij}^p}$	cumulated plastic deviatoric strains
$\dot{\gamma}_{vp}=\sqrt{\frac{2}{3}\tilde{\dot{\varepsilon}}_{ij}^{vp}\tilde{\dot{\varepsilon}}_{ij}^{vp}}$	cumulated viscoplastic deviatoric strains
ξ_p	plastic hardening parameter
ξ_{vp}	viscoplastic hardening parameter
$\boldsymbol{G}^{visc}$	function controlling the evolution of the viscous strains and describing the deviative flow direction
$\tilde{\boldsymbol{G}}=\boldsymbol{G}-\frac{Tr(\boldsymbol{G})}{3}\boldsymbol{I}$	of $\boldsymbol{G}$
$G=Tr(\boldsymbol{G})$	trace of $\boldsymbol{G}$
$\tilde{G}_{II}=\sqrt{\tilde{\boldsymbol{G}}\cdot\tilde{\boldsymbol{G}}}$	angle $\tilde{\boldsymbol{G}}$
ψ	of dilatancy normalizes
f^d	surfaces of elastoplastic load
f^{vp}	surfaces of viscoplastic load

1.2 Sign convention

•In Code_Aster, sign convention is that of the mechanics of the continuums:

In compression: $\sigma<0$; $\varepsilon=\frac{\partial u}{\partial x}<0$

In tension : $\sigma>0$; $\varepsilon=\frac{\partial u}{\partial x}>0$

•In the model LETK , sign convention is that of the soil mechanics:

In compression : $\sigma>0$
Contractance : $\varepsilon_v>0$
In tension : $\sigma<0$
Dilatancy : $\varepsilon_v<0$

Note:

To integrate this model in *Code_Aster* such as it is presented, it is necessary to change the sign of all the fields at the entrance of the routine corresponding to the constitutive law and its output.

At the entrance of the routine:

$\sigma^-_{\mathrm{L\&K}}=-\sigma^-$

$\varepsilon^-_{\mathrm{L\&K}}=-\varepsilon^-$

$\Delta\varepsilon_{\mathrm{L\&K}}=-\Delta\varepsilon$

At the output of the routine:

$\sigma=-\sigma_{\mathrm{L\&K}}$

$\varepsilon=-\varepsilon_{\mathrm{L\&K}}$

$\Delta\varepsilon=-\Delta\varepsilon_{\mathrm{L\&K}}$

1.3 Parameters of the model

Notation	Description
P_a	atmospheric pressure
σ_c	strength in simple compression, intervening in the statement of the parameter
H_0^{ext}	criteria controlling strength in extension, intervening in the statement of the criteria
σ_{point1}	$\sigma_{\min}$ of the intersection enters the thresholds of peak and intermediate
x_{ams}	non-zero parameter intervening the models of hardening pre-peak
η	non-zero parameter intervening in the models of hardening post-peak
a_0	value of a on the threshold of damage
m_0	value of m on the threshold of damage
s_0	value of S on the threshold of damage
a_{pic}	value of a on the threshold of peak
m_{pic}	value of m on the threshold of peak
ξ_{pic}	level of hardening necessary to ξ_p reach the threshold of peak
a_e	value of a on the threshold of cleavage
m_e	value of m on the threshold of cleavage
ξ_e	level of hardening necessary to ξ_p reach the threshold of cleavage
m_{ult}	value of m on the residual threshold
ξ_{ult}	level of hardening necessary to ξ_p reach the residual threshold
$m_{v-\max}$	value of m on the maximum viscoplastic threshold
$\xi_{v-\max}$	value of ξ_v for which the maximum viscoplastic criterion is reached
A_v	parameter characterizing the amplitude velocity of exposing
n_v	creep intervening in the formula controlling the kinetics of creep

$\mu_{0,v}$	parameter relating to dilatancy in pre-peak
$\xi_{0,v}$	parameter relating to dilatancy in pre-peak
μ_1	parameter relating to dilatancy in post peak
ξ_1	parameter relating to dilatancy in post-peak

2 Introduction

This document the model presents rheological L&K developed with the CIH by F. Laigle and A. Kleine. It is a model élasto-visco-plastic dedicated to the rocks. The specificity of elastoplasticity resides in the modelization of a nonlinear behavior in phase pre-peak and of a behavior post peak softening. Viscosity characterizes the effect of time on the behavior of the rock. The initiation of each one of these phenomena starts as of the crossing of a threshold. The behavior related to each phase is described by the evolution of these various thresholds governed by functions of plastic or viscoplastic hardening.

3 Equations of model L&K

3.1 Simplification of the model

In order to describe as well as possible and a concise way this version of the model, it is necessary the model to give an outline on original. The difference between the two versions will be perceived better.

3.1.1 Short outline on the thresholds of the original model

In the original version of model L&K such as it is developed under the Flac software with the CIH (cf R1 1) or the thesis of A. Kleine (cf R 4), there exist three distinct mechanisms:

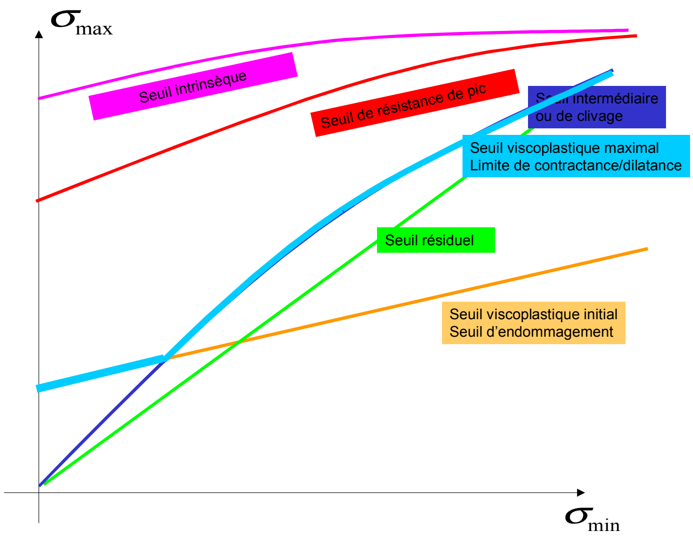
- An elastoplastic mechanism pre-peak, governed by a positive hardening,
- a viscoplastic mechanism governed also by a positive hardening,
- an elastoplastic mechanism post peak governed by a negative hardening describing the fracturing.

The characteristic of this original model lies in the fact that the coupling of the two mechanisms pre peak thus starts the fracturing the mechanism post-peak. Indeed, the cracks of extension induce a degradation of the mechanical properties of the materials with the increase in dilatancy.

For the elastoplastic mechanism, a surface of load evolves through various thresholds. For the viscous mechanism, a viscoplastic surface evolves of an initial threshold to a final threshold.

The various thresholds delimit fields associated with particular physical mechanisms:

- A threshold of damage confused with the threshold of initial viscosity,
- an intermediate threshold, qualified of limit of cleavage,
- a definite threshold characteristic as the envelope of the threshold of damage and limit of cleavage, called also limiting of contractance/dilatancy (this limit is confused with the threshold of maximum viscoplasticity),
- a macroscopic threshold of peak, defined starting from the laboratory tests,
- a definite purely conceptual intrinsic threshold like extrapolation of the threshold of peak, (this threshold is eliminated in the version simplified from the model),
- a threshold of residual strength.

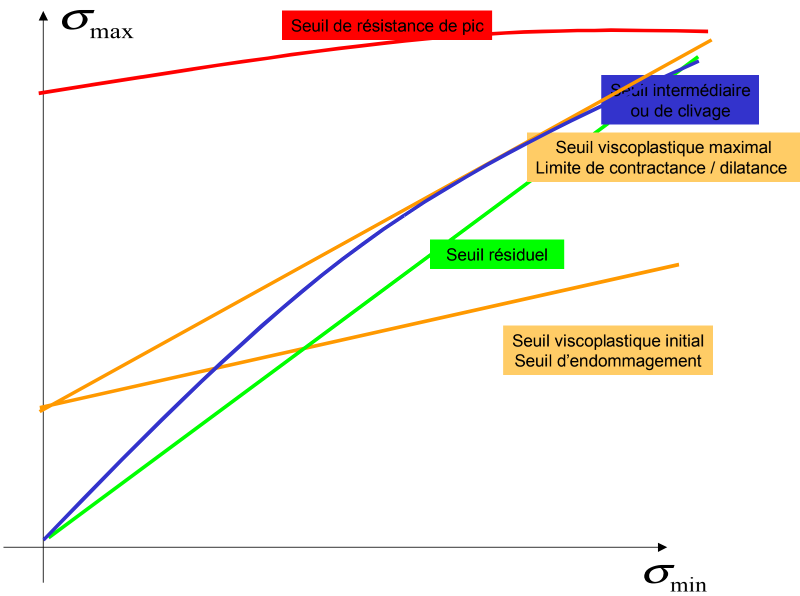


Appear 3.1.1-a. Thresholds of the original model presented in the Characteristics $(\sigma_{min}, \sigma_{max})$

3.1.2 plane of the model simplified

the simplified version proposed by the CIH (cf. R2 2) rests only on two mechanisms: an elastoplastic mechanism and a viscoplastic mechanism.

- The intrinsic threshold is eliminated in this version.
- The threshold characteristic delimiting the fields of contractance and dilatancy in phase pre-peak, is linearized to avoid any numerical problem. It is supposed to be confused with the threshold of maximum viscosity.



Appear 3.1.2-a. Thresholds of the model simplified in the Descriptive $(\sigma_{min}, \sigma_{max})$

3.2 plan of the mechanisms

3.2.1 the viscoplastic mechanism

This mechanism is activated as soon as the point of load exceeds the initial viscoelastic threshold (compare to the initial elastic limit). Unrecoverable deformations are generated. These irreversible strainrates are proportional to the distance from the point of load compared to the threshold of viscosity. Surface associated with the viscoplastic mechanism evolves of the initial elastic limit to the maximum viscoplastic threshold according to the generated unrecoverable deformations.

3.2.2 The elastoplastic mechanism

3.2.2.1 pre peak

the plastic mechanism élasto is activated at the same time as the viscoplastic mechanism. As soon as the point of load exceeds the initial elastic limit, the surface of load starts with écrouire positively.

3.2.2.2 post peak

In the simplified version of the model, this mechanism is governed by:
a negative hardening of the threshold of peak towards the intermediate threshold,
a negative hardening of the intermediate threshold towards the residual threshold.

3.2.3 The voluminal behavior

the voluminal behavior, during the phase pre-peak, can be contracting or dilating.
Below the initial elastic limit, the behavior is contracting.
Below the limit contractance/dilatancy, the voluminal behavior is contracting plastic.
Beyond this limit, the voluminal behavior is dilating.

N.B: In the simplified version of the model, the limit contractance/dilatancy is confused with the maximum viscoplastic threshold.

3.3 Decomposition of the strain tensor

the decomposition of the increment of total deflection is written:

$$\dot{\underline{\varepsilon}}=\dot{\underline{\varepsilon}}^{e}+\dot{\underline{\varepsilon}}^{p}+\dot{\underline{\varepsilon}}^{vp}$$

where $\dot{\underline{\varepsilon}}^{e}$, $\dot{\underline{\varepsilon}}^{p}$ and $\dot{\underline{\varepsilon}}^{vp}$ are the increments of the tensors elastic, irreversible instantaneous (plastic) and irreversible differed (viscoplastic).

3.3.1 Hypo-elasticity

the selected elastic model is a model hypo-elastic:

$$\dot{\varepsilon}_{ij}^{e}=\frac{1+\nu}{E}\dot{\sigma}_{ij}-\frac{3\nu}{E}\dot{p}\,\delta_{ij} \quad \text{or} \quad \dot{\varepsilon}_{ij}^{e}=\frac{1}{2G}\dot{s}_{ij}+\frac{1}{3K}\dot{p}\,\delta_{ij}$$

that one also notes: $\dot{\sigma}_{ij}=D^{e}\dot{\varepsilon}_{ij}^{e}$.

The shear moduli G and of compressibility K depend on the stress state: $K=K_{0}^{e}\left[\frac{I_{1}^{-}}{3P_{a}}\right]^{n_{elas}}$ and

$$G=G_{0}^{e}\left[\frac{I_{1}^{-}}{3P_{a}}\right]^{n_{elas}} \quad \text{with} \quad I_{1}^{-}=tr(\boldsymbol{\sigma}^{-}) .$$

$I_{1}^{-}=tr(\boldsymbol{\sigma}^{-})$ being the trace of the stresses at time $-$.

3.3.2 Plasticity

As in the simplified version of the model, only the mechanism deviatoric is taken into account, the instantaneous unrecoverable deformation is written:

$$\dot{\varepsilon}_{ij}^{p}=\dot{\lambda}\,G_{ij}$$

λ being the plastic multiplier and G is the flow function.

That is to say f^{d} the plasticity criterion:

So $f^{d}\leq 0$ then $\lambda=0$

So $f^{d}=0$ then $\lambda>0$

the statement of G rests on several works quoted in the note R2 2 and is form:

$$G_{ij}=\frac{\partial f^{d}}{\partial\sigma_{ij}}-\left(\frac{\partial f^{d}}{\partial\sigma_{kl}}n_{kl}\right)n_{ij}\qquad n_{ij}=\frac{\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}}{\sqrt{\beta'^{2}+3}}\ ,\ \beta'=-\frac{2\sqrt{6}\sin(\Psi)}{3-\sin(\Psi)}$$

the statements of $\sin(\Psi)$ are detailed in paragraph 3.6.1 and 3.6.2.

The computation in $\dot{\lambda}$ fact the object of the 4.2.1.2 4.2.1.2.

The computation of $\frac{\partial f}{\partial\sigma_{ij}}$ in paragraph 3.7.1 3.7.1

of elastoplasticity is detailed induces a plastic strain: ε_{p} connected through its deviatoric component $\tilde{\varepsilon}_{p}$ to the hardening parameter γ_{p} such as: $\gamma_{p}=\int\sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{p}\dot{\tilde{\varepsilon}}_{p}}dt$

from where the relation $\dot{\gamma}_{p}=\dot{\lambda}\sqrt{\frac{2}{3}\tilde{G}_{ij}\tilde{G}_{ij}}=\dot{\lambda}\sqrt{\frac{2}{3}}G_{II}$

3.3.3 the viscoplasticity

The computation of the differed unrecoverable deformations $\dot{\underline{\underline{\varepsilon}}}^{vp}$ is based on the theory of Perzyna.

$\dot{\underline{\underline{\varepsilon}}}^{vp}=\langle\Phi(f^{vp})\rangle G_{ij}^{visc}$ where $\Phi(f^{vp})$ and G^{visc} the amplitude and the direction velocity of the unrecoverable deformations characterize:

$$\Phi(f^{vp})=A_{v}\left(\frac{f^{vp}}{Pa}\right)^{n_{v}}\ \text{and}\ G_{ij}^{visc}=\frac{\partial f^{vp}}{\partial\sigma_{ij}}-\left(\frac{\partial f^{vp}}{\partial\sigma_{kl}}n_{kl}\right)n_{ij}$$

f^{vp} being the criterion of viscoplasticity, A_{v} and n_{v} are parameters of the model. P_{a} is the atmospheric pressure.

The evolution of viscoplasticity induces a viscous strain: connected through its deviatoric component $\tilde{\varepsilon}_{vp}$ to the hardening parameter γ_{vp} such as: $\gamma_{vp}=\int\sqrt{\frac{2}{3}\dot{\tilde{\varepsilon}}_{vp}\dot{\tilde{\varepsilon}}_{vp}}dt$.

Note:

That is to say S^{vp} the surface defined within the space of stresses by: $S^{vp}=\{\sigma,f^{vp}(\sigma,\xi^{vp})=0\}$

The velocity of creep for a stress state σ is proportional to the distance from σ to S^{vp} . That is to say P_{σ}^{vp} the projection of σ on S^{vp} and $d=\|\sigma-P_{\sigma}^{vp}\|$.

One can also write $d=\|\frac{\partial f^{vp}}{\partial\sigma}\left(P_{\sigma}^{vp}\right)\|C$. C being a constant which depends on the viscous parameters. At first approximation, one writes: $d=\|\frac{\partial f^{vp}}{\partial\sigma}(\sigma)\|C$. But this approximation poses a problem insofar as the function f^{vp} can not be defined for the value σ whereas it is it for P_{σ}^{vp} . For time, in *Code_Aster* one does not calculate this distance. If the situation arises, an alarm message warns the user.

3.4 Statements of the criteria

the statements of the two criteria viscoplastic f^{vp} and elastoplastic f^{d} depend on the stresses and the functions of hardening. In these statements, one finds I_1 the first invariant of the stresses and s_{II} the second invariant of the tensor of the stresses déviatoires. In the two criteria, the same definitions are adopted for:

$$H(\theta)=\frac{H_0^c+H_0^e}{2}+\left(\frac{H_0^c-H_0^e}{2}\right)\left(\frac{2\mathrm{h}(\theta)-\left(h_0^c+h_0^e\right)}{h_0^c-h_0^e}\right)$$

$$h(\theta)=(1-\gamma\cos 3\theta)^{\frac{1}{6}}\quad h_0^c=H_0^c=h(0°)=(1-\gamma)^{\frac{1}{6}}\quad h_0^e=h(60°)=(1+\gamma)^{\frac{1}{6}},$$

H_0^e is a parameter of the model. θ is the angle of Lode.

3.4.1 The viscoplastic criterion f^{vp}

$$f^{vp}(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^{vp}(\xi_{vp})s_{II}H(\theta)+B^{vp}(\xi_{vp})I_1+D^{vp}(\xi_{vp})\right]^{a^{vp}(\xi_{vp})}$$

with $A^{vp}(\xi_{vp})=-\frac{m^{vp}(\xi_{vp})k^{vp}(\xi_{vp})}{\sqrt{6}\sigma_c h_c^0}$ $B^{vp}(\xi_{vp})=\frac{m^{vp}(\xi_{vp})k^{vp}(\xi_{vp})}{3\sigma_c}$ $D^{vp}(\xi_{vp})=s^{vp}(\xi_{vp})k^{vp}(\xi_{vp})$,

$$k^{vp}(\xi_{vp})=\left(\frac{2}{3}\right)^{\frac{1}{2a^{vp}(\xi_{vp})}}$$

the functions of hardening $A^{vp}(\xi_{vp})$, $B^{vp}(\xi_{vp})$ and $D^{vp}(\xi_{vp})$ depend on the hardening parameters $a^{vp}(\xi_{vp})$, $m^{vp}(\xi_{vp})$ and $s^{vp}(\xi_{vp})$ whose statements evolve with the variables of hardening ξ_{vp} (see § 3.5). When ξ_{vp} reached certain particular values, surface f^{vp} reaches the corresponding thresholds.

Since the viscoplastic threshold is hammer-hardened only by viscosity, one always has $\dot{\xi}_{vp}=Min\left[\dot{\gamma}_{vp},\xi_{v-\max}-\xi_{vp}\right]$. $\xi_{vp-\max}$ corresponds to the maximum viscoplastic criterion and is a parameter of the model

3.4.2 the elastoplastic criterion f^{d}

$$f^{d}(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^{d}(\xi_p)s_{II}H(\theta)+B^{d}(\xi_p)I_1+D^{d}(\xi_p)\right]^{a^{d}(\xi_p)}$$

with $A^d(\xi_p)=-\frac{m^d(\xi_p)k^d(\xi_p)}{\sqrt{6}\sigma_c h_c^0}$ $B^d(\xi_p)=\frac{m^d(\xi_p)k^d(\xi_p)}{3\sigma_c}$ $D^d(\xi_p)=s^d(\xi_p)k^d(\xi_p)$,

$k^d(\xi_p)=\left(\frac{2}{3}\right)^{\frac{1}{2a^d(\xi_p)}}$

the functions of hardening $A^d(\xi_p)$, $B^d(\xi_p)$ and $D^d(\xi_p)$ depend on the hardening parameters $a^d(\xi_p)$, $m^d(\xi_p)$ and $s^d(\xi_p)$ whose statements evolve with the variables of hardening ξ_p (see § 3.5). Quandatteint ξ_p certain values particular, surface f^d reaches the corresponding thresholds.

Elastoplastic hardening depends on the position of the point of load compared to the limits contractance/dilatancy:

- if the point of load is below this limit $\dot{\xi}_p=\dot{\gamma}_p$,
- if the point of load is with the top of this limit $\dot{\xi}_p=\dot{\gamma}_p+\dot{\gamma}_{vp}$.

3.5 Functions of hardening

3.5.1 Functions of hardening of the viscous criterion

the viscoplastic criterion is governed by the following functions of hardening:

$$a(\xi_{vp})=a_0+(a_{v-\max}-a_0)\frac{\xi_{vp}}{\xi_{v-\max}} \quad \text{with } a_{v-\max}=1.$$

$$m(\xi_{vp})=m_0+(m_{v-\max}-m_0)\frac{\xi_{vp}}{\xi_{v-\max}}$$

$$s(\xi_{vp})=s_0+(s_{v-\max}-s_0)\frac{\xi_{vp}}{\xi_{v-\max}} \quad s_{v-\max}=s_0$$

3.5.2 Functions of hardening of the elastoplastic criterion and their derivatives

the statements of the functions of hardening which govern the elastoplastic criterion vary according to the value of the parameters ξ_p :

Evolution enters the threshold of damage and the threshold of peak : So $0\le\xi_p<\xi_{pic}$

$$a(\xi_p)=a_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{a_{pic}-a_0}{\ln(1+1/x_{ams})}\right) \qquad \frac{\partial a}{\partial\xi_p}=\left(\frac{a_{pic}-a_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

$$m(\xi_p)=m_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{m_{pic}-m_0}{\ln(1+1/x_{ams})}\right) \qquad \frac{\partial m}{\partial\xi_p}=\left(\frac{m_{pic}-m_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

$$s(\xi_p)=s_0+\ln\left(1+\frac{\xi_p}{x_{ams}\xi_{pic}}\right)\left(\frac{s_{pic}-s_0}{\ln(1+1/x_{ams})}\right)$$

$$\frac{\partial s}{\partial\xi_p}=\left(\frac{s_{pic}-s_0}{\ln(1+1/x_{ams})}\right)\left(\frac{1}{\xi_p+x_{ams}\xi_{pic}}\right)$$

with $s_{pic}=1$.

Evolution the threshold of peak and the intermediate threshold or the limit of cleavage enter: If $\xi_{pic} \leq \xi_p < \xi_e$

$$a(\xi_p) = a_{pic} + (a_e - a_{pic})\left(\frac{\xi_p - \xi_{pic}}{\xi_e - \xi_{pic}}\right) \qquad \frac{\partial a}{\partial \xi_p} = \frac{a_e - a_{pic}}{\xi_e - \xi_{pic}}$$

$$s(\xi_p) = 1 - \left(\frac{\xi_p - \xi_{pic}}{\xi_e - \xi_{pic}}\right) \qquad \frac{\partial s}{\partial \xi_p} = \frac{-1}{\xi_e - \xi_{pic}}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\partial m}{\partial a}\frac{\partial a}{\partial \xi_p} + \frac{\partial m}{\partial s}\frac{\partial s}{\partial \xi_p}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\sigma_c}{\sigma_{point1}}\left[\left(-\frac{a_{pic}}{a(\xi_p)^2}\right)\left(m_{pic}\frac{\sigma_{point1}}{\sigma_c} + s_{pic}\right)^{\frac{a_{pic}}{a(\xi_p)}} \ln\left(m_{pic}\frac{\sigma_{point1}}{\sigma_c} + s_{pic}\right)\frac{\partial a}{\partial \xi_p} - \frac{\partial s}{\partial \xi_p}\right]$$

Evolution enters the intermediate threshold and the residual threshold : So $\xi_e \leq \xi_p < \xi_{ult}$

$$a(\xi_p) = a_e + \ln\left(1 + \frac{1}{\eta}\frac{\xi_p - \xi_e}{\xi_{ult} - \xi_e}\right)\left(\frac{a_{ult} - a_e}{\ln(1 + 1/\eta)}\right) \qquad \frac{\partial a}{\partial \xi_p} = \left(\frac{a_{ult} - a_e}{\ln(1 + 1/\eta)}\right)\left(\frac{1}{\xi_p + \eta\xi_{ult} - (1+\eta)\xi_e}\right)$$

$$s(\xi_p) = 0 \qquad \frac{\partial s}{\partial \xi^p} = 0$$

$$m(\xi_p) = \frac{\sigma_c}{\sigma_{point2}}\left(m_e\frac{\sigma_{point2}}{\sigma_c}\right)^{\frac{a_e}{a(\xi_p)}}$$

$$\frac{\partial m}{\partial \xi_p} = \frac{\sigma_c}{\sigma_{point2}}\left[\left(-\frac{a_e}{a(\xi_p)^2}\right)\ln\left(m_e\frac{\sigma_{point2}}{\sigma_c}\right)\left(m_e\frac{\sigma_{point2}}{\sigma_c}\right)^{\frac{a_e}{a(\xi_p)}}\right]\frac{\partial a}{\partial \xi_p}$$

On the residual criterion : If $\xi_p \geq \xi_{ult}$

$$a(\xi_p) = a_{ult} = 1. \qquad \frac{\partial a}{\partial \xi_p} = 0$$

$$s(\xi_p) = 0 \qquad \frac{\partial s}{\partial \xi_p} = 0$$

$$m(\xi_p) = m_{ult} \qquad \frac{\partial m}{\partial \xi_p} = 0$$

3.6 Models of dilatancy

the elastoplastic and viscoplastic mechanisms are non-aligned. The laws of evolution of $\dot{\varepsilon}_{ij}^{p}$ and of $\dot{\varepsilon}_{ij}^{vp}$ are controls respectively by a function G and a function G^{visc} , such as:

$G_{ij} = \frac{\partial f^d}{\partial \sigma_{ij}} - \left(\frac{\partial f^d}{\partial \sigma_{kl}} n_{kl}\right) n_{ij}$ and $G_{ij}^{visc} = \frac{\partial f^{vp}}{\partial \sigma_{ij}} - \left(\frac{\partial f^{vp}}{\partial \sigma_{kl}} n_{kl}\right) n_{ij}$ with

$n_{ij} = \frac{\beta' \frac{s_{ij}}{s_{II}} - \delta_{ij}}{\sqrt{\beta'^2 + 3}}$ and $\beta' = -\frac{2\sqrt{6}\sin(\Psi)}{3 - \sin(\Psi)}$

The computation from the angle of dilatancy Ψ differs according to the viscous mechanisms or elastoplastic pre-peak and elastoplastic post-peak.

3.6.1 Elastoplastic pre-peak and viscoplastic angle of dilatancy of the mechanisms

$\sin(\Psi)=\mu_{0,v}\left(\frac{\sigma_{max}-\sigma_{lim}}{\xi_{0,v}\sigma_{max}+\sigma_{lim}}\right)$ with $\mu_{0,v}$ and $\xi_{0,v}$ of the parameters of the model.

where

$\sigma_{lim}=\sigma_{min}+\sigma_c\left(m_{v-max}\frac{\sigma_{min}}{\sigma_c}+s_{v-max}\right)^{a_{v-max}}$ with $s_{v-max}=s_0$ and $a_{v-max}=1$. σ_c and m_{v-max} are parameters of the model.

There exist conditions on the parameters $\mu_{0,v}$ and $\xi_{0,v}$ which are:

- $\mu_{0,v}<\xi_{0,v}$ or
- $\begin{cases}\mu_{0,v}>\xi_{0,v}\\ \frac{(s_{pic})^{a_{pic}}}{(s_0)^{a_0}}\le\frac{1+\mu_{0,v}}{\mu_{0,v}-\xi_{0,v}}\end{cases}$

3.6.2 Angle of dilatancy of the elastoplastic mechanism post-peak

$\sin(\Psi)=\mu_1\left(\frac{\alpha-\alpha_{res}}{\xi_1\alpha+\alpha_{res}}\right)$ with μ_1 and ξ_1 parameters of the model

where

$\alpha=\frac{\sigma_{max}+\tilde{\sigma}}{\sigma_{min}+\tilde{\sigma}}$ and $\alpha_{res}=\frac{\sigma_{max}}{\sigma_{min}}=1+m_{ult}$

$$\tilde{\sigma}=\begin{cases}\frac{\tilde{c}(\xi_p)}{\tan(\tilde{\varphi}(\xi_p))} & si \quad \xi_p\le\xi_e\\ 0 & si \quad \xi_p>\xi_e\end{cases}$$

with $\tilde{c}(\xi_p)=\frac{\sigma_c . s(\xi_p)^{a(\xi_p)}}{2\sqrt{1+a(\xi_p)m(\xi_p)s(\xi_p)^{a(\xi_p)-1}}}$ and

$\tilde{\varphi}(\xi_p)=2.\,arctg(\sqrt{1+a(\xi_p)m(\xi_p)s(\zeta_p)^{a(\xi_p)-1}})-\frac{\pi}{2}$

σ_{min} σ_{max} are calculated starting from the invariants of the stresses:

$$\sigma_{min}=\frac{1}{3}\left(I_1-\left(\frac{3}{2}-\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\sqrt{\frac{3}{2}}s_{II}\right)$$

$$\sigma_{\max}=\frac{1}{3}\left(I_1+\left(\frac{3}{2}+\frac{2\mathrm{H}(\theta)-\left(H_0^c+H_0^e\right)}{2\left(H_0^c-H_0^e\right)}\right)\sqrt{\frac{3}{2}}s_{II}\right)$$

3.7 Derived from the Computation

3.7.1 criterion from $\frac{\partial f}{\partial \sigma_{ij}}$

$$\frac{\partial I_1}{\partial \sigma_{ij}}=\frac{\partial tr\left(\sigma_{ij}\right)}{\partial \sigma_{ij}}=\delta_{ij}$$

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial \sigma_{ij}}=\frac{\partial\left(s_{II}H(\theta)\right)}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}=\left(\frac{\partial H(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\frac{\partial s_{II}}{\partial s_{kl}}\right)\frac{\partial s_{kl}}{\partial \sigma_{ij}}$$

$$\frac{\partial s_{II}}{\partial s_{kl}}=\frac{s_{kl}}{s_{II}} \;\; ; \;\; s_{II}=\sqrt{s_{kl}.s_{kl}}$$

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}}=\frac{\partial\left(\sigma_{kl}-\frac{1}{3}tr(\sigma)\delta_{kl}\right)}{\partial \sigma_{ij}}=\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}$$

Note: $s_{kl}.\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)=s_{ij}$

$$\frac{\partial H(\theta)}{\partial s_{kl}}=\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\frac{\partial h(\theta)}{\partial s_{kl}}$$

from where $\frac{\partial\left(s_{II}H(\theta)\right)}{\partial \sigma_{ij}}=\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\frac{\partial h(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$

There is the relation: $\cos(3\theta)=\sqrt{54}\frac{\det(\boldsymbol{s})}{s_{II}^3}$ (confer to the documentation R7.01.13-A: Model CJS in mechanics)

$$\frac{\partial h(\theta)}{\partial s_{kl}}=\frac{1}{6}\left(1-\gamma\cos(3\theta)\right)^{-\frac{5}{6}}\frac{\partial\left(1-\gamma\cos(3\theta)\right)}{\partial s_{kl}}=\frac{1}{6\mathrm{h}(\theta)^5}\frac{\partial}{\partial s_{kl}}\left(\frac{s_{II}^3-\gamma\sqrt{54}\det(s)}{s_{II}^3}\right)$$

$$\frac{\partial h(\theta)}{\partial s_{kl}}=\frac{1}{6\mathrm{h}(\theta)^5}\left\{\left[\frac{\partial s_{II}^3}{\partial s_{kl}}-\gamma\sqrt{54}\left(\frac{\partial\det(\underline{\underline{s}})}{\partial s_{kl}}\right)\right]\frac{s_{II}^3}{s_{II}^6}-\left(s_{II}^3-\gamma\sqrt{54}\det(\underline{\underline{s}})\right)\frac{3\mathrm{s}_{kl}s_{II}}{s_{II}^6}\right\}$$

$$=\frac{1}{6\mathrm{h}(\theta)^5}\left\{\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\gamma\sqrt{54}\left(\frac{\partial\det(\underline{\underline{s}})}{\partial s_{kl}}\right)\frac{1}{s_{II}^3}-\left(1-\gamma\cos(3\theta)\right)\frac{3\mathrm{s}_{kl}}{s_{II}^2}\right\}$$

$$=\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial\det(\underline{\underline{s}})}{\partial s_{kl}}\right)$$

One thus finds:

$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=$$
$$\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\left(\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial\det(\underline{s})}{\partial s_{kl}}\right)\right)s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$$

Finally:

For the elastoplastic criterion:

$$\frac{\partial f^d}{\partial\sigma_{ij}}=$$
$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}-a^d(\xi_p)\sigma_c H_0^c\left[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)\right]^{a^d(\xi_p)-1}$$
$$\left(A^d(\xi_p)\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d(\xi_p)I_d\right)$$

and for the viscous criterion:

$$\frac{\partial f^{vp}}{\partial\sigma_{ij}}=$$
$$\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}-a^{vp}(\xi_{vp})\sigma_c H_0^c\left[A^{vp}(\xi_{vp})s_{II}H(\theta)+B^{vp}(\xi_{vp})I_1+D^{vp}(\xi_{vp})\right]^{a^{vp}(\xi_{pp})-1}$$
$$\left(A^{vp}(\xi_{vp})\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^{vp}(\xi_{vp})I_d\right)$$

with $\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}=$

$$\left(\left(\frac{H_0^c-H_0^e}{h_0^c-h_0^e}\right)\left(\frac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\frac{3\mathrm{s}_{kl}}{s_{II}^2}-\frac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s_{II}^3}\left(\frac{\partial\det(\underline{\underline{s}})}{\partial s_{kl}}\right)\right)s_{II}+H(\theta)\frac{s_{kl}}{s_{II}}\right).\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$$

3.7.2 Computation of $\frac{\partial f^d}{\partial\xi_p}$

Statement of the threshold in stresses:

$$f^d(\sigma)=s_{II}H(\theta)-\sigma_c H_0^c\left[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)\right]^{a^d(\xi_p)}$$

with $A^d(\xi^p) = -\frac{m^d(\xi_p) k^d(\xi_p)}{\sqrt{6}\sigma_c h_c^0}$ $B^d(\xi_p) = \frac{m^d(\xi_p) k^d(\xi_p)}{3\sigma_c}$ $D^d(\xi_p) = s^d(\xi_p) k(\xi_p)$,

$$k^d(\xi_p) = \left(\frac{2}{3}\right)^{\frac{1}{2a^d(\xi_p)}}$$

$$\frac{\partial f^d}{\partial \xi_p} = \frac{\partial f^d}{\partial a^d} . \dot{a}^d(\xi_p) + \frac{\partial f}{\partial m^d} . \dot{m}^d(\xi_p) + \frac{\partial f}{\partial s^d} . \dot{s}^d(\xi_p)$$

$$\frac{\partial f^d}{\partial s^d} = -a^d k^d \sigma_c H_0^c \left[A^d s_{II} H(\theta) + B^d I_1 + D^d \right]^{a^d - 1}$$

$$\frac{\partial f^d}{\partial m^d} = -a^d \sigma_c H_0^c \left[\frac{A^d}{m^d} s_{II} H(\theta) + \frac{B^d}{m^d} I_1 \right] \left[A^d s_{II} H(\theta) + B^d I_1 + D^d \right]^{a^d - 1}$$

$$\frac{\partial f^d}{\partial a^d} =$$

$$\sigma_c H_0^c \dot{a}^d \left[A^d s_{II} H(\theta) + B^d I_1 + D^d \right]^{a^d} . \left[\ln\left[A^d s_{II} H(\theta) + B^d I_1 + D^d \right] - \frac{\frac{s^d}{2a^d} \ln\left(\frac{2}{3}\right) \left(\frac{2}{3}\right)^{\left(\frac{1}{2a}\right)}}{\left[A^d s_{II} H(\theta) + B^d I_1 + D^d \right]} \right]$$

4 in Code_Aster

the integration of model LETK `formulates` can be realized according to two distinct diagrams of integration. The first diagram of integration ("historical") is described like `SPECIFIQUE` and corresponds to a diagram of explicit integration. The second diagram is built on the basis of diagram of implicit integration. It is accessible under key word `NEWTON_PERT` .

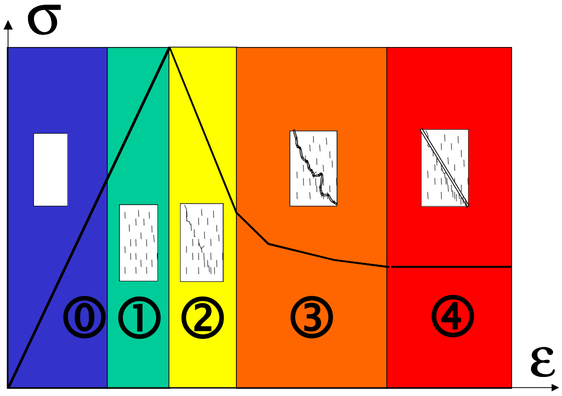
4.1 Local variables

V_1 : elastoplastic variable of hardening ξ_p
V_2 : plastic deviatoric strain γ_p
V_3 : viscoplastic variable of hardening ξ_{vp}
V_4 : viscoplastic deviatoric strain γ_{vp}
V_5 : 0 if contractance, 1 if dilatancy
V_6 : indicator of viscoplasticity
V_7 : indicator of plasticity
V_8 : The fields of behavior of the rock in plasticity

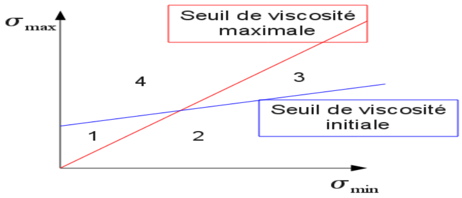
Five fields of behavior, numbered from 0 to 4 (cf appears), are identified to make it possible to have a relatively simple representation of the state of damage of the rock, since the intact rock to the rock in a residual state. These fields are function of the cumulated plastic strain déviatoire γ^p and the stress state. Each increment of number of field defines the transition in a field of higher damage.

- If the deviator is lower than 70% of the deviator of peak, then the material is in field 0;
- If not:
  - If $\gamma^p=0$ then the material is in field 1;
    1) If $0<\gamma^p<\gamma^e$ then the material is in field 2;
  - If $\gamma_e<\gamma^p<\gamma_{\text{ult}}$ then the material is in field 3;
  - $\gamma^p>\gamma_{\text{ult}}$ the material then is in field 4.

Domaine	Etat de la roche
0	Intacte
1	Endommagement pré-pic
2	Endommagement post-pic
3	Fissurée
4	Fracturée



V_9 : position of the stress state compared to the thresholds of viscosity.



Four fields of behavior, numbered from 1 to 4 (cf appears), are identified to make it possible to have a simple representation of the viscous behavior of the material, since the intact rock to the rock in a residual state. These fields are function of the cumulated viscoplastic strain déviatoire γ^{vp} and the stress state.

4.2 Diagram of integration clarifies (SPECIFIQUE)

4.2.1 Update of the stresses

One expresses the stresses brought up to date at time + compared to those calculated at time -:

$$\sigma=\sigma^{-}+D^{e}\Delta\varepsilon^{e}\ ;\ s=s^{-}+2\mathrm{G}\Delta\tilde{\varepsilon}^{e}\ ;\ I_1=I_1^{-}+3\mathrm{K}\Delta\varepsilon_v^{e}$$

$$\sigma_{ij}=s_{ij}+\frac{I_1}{3}\delta_{ij}\ ;\ \Delta\varepsilon_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{tr(\Delta\varepsilon)}{3}\delta_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{\Delta\varepsilon_v}{3}\delta_{ij}\ ;\ I_1=tr(\sigma)\ ;\ \varepsilon_v=tr(\Delta\varepsilon)$$

Elastic prediction:

$$\sigma^{e}=\sigma^{-}+D^{e}\Delta\varepsilon\ ;\ s^{e}=s^{-}+2\mathrm{G}\Delta\tilde{\varepsilon}\ ;\ I_1^{e}=I_1^{-}+3\mathrm{K}\Delta\varepsilon_v$$

$$K=K_0^{e}\left[\frac{I_1^{-}}{3\mathrm{P}_a}\right]^{n_{elas}}\text{ and }G=G_0^{e}\left[\frac{I_1^{-}}{3\mathrm{P}_a}\right]^{n_{elas}}$$

Note: The coefficient of compressibility K and the shear modulus G are considered at time **-**.

4.2.1.1 Hypoelasticity

$$\Delta\sigma_{ij}=\Delta s_{ij}+\frac{\Delta I_1}{3}\delta_{ij}\qquad \Delta\varepsilon_{ij}=\Delta\tilde{\varepsilon}_{ij}+\frac{\Delta\varepsilon_v}{3}\delta_{ij}\qquad \Delta\sigma_{ij}=2\mathrm{G}\Delta\varepsilon_{ij}+\left(K-\frac{2\mathrm{G}}{3}\right)tr(\Delta\varepsilon)\delta_{ij}$$

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \sqrt{2}\Delta\sigma_{12} \\ \sqrt{2}\Delta\sigma_{13} \\ \sqrt{2}\Delta\sigma_{23} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{4G}{3}+K & K-\frac{2G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & \frac{4G}{3}+K & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K-\frac{2G}{3} & \frac{4G}{3}+K & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix}}_{D^e} \cdot \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \sqrt{2}\Delta\varepsilon_{12} \\ \sqrt{2}\Delta\varepsilon_{13} \\ \sqrt{2}\Delta\varepsilon_{23} \end{Bmatrix}$$

4.2.1.2 Plasticity and viscoplasticity

One expresses the stress field at time $+$:

$$\sigma_{ij}=\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}^{e}=\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\varepsilon_{kl}^{p}-\Delta\varepsilon_{kl}^{vp}\right)$$

who is written by replacing the increase in plastic strains and viscous by their statements in the form:

$$\sigma_{ij}=\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\lambda G_{kl}(\sigma^{-},\xi_{p}^{-})-\langle\varphi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t\right)$$

The principal unknown is the plastic multiplier $\Delta\lambda$.

One seeks $\Delta\lambda / f^{d}\left(\sigma,\xi_{p}\right)=0$

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\left(\Delta\varepsilon_{kl}-\Delta\lambda G_{kl}(\sigma^{-},\xi_{p}^{-})-\langle\varphi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t\right),\xi_{p}^{-}+\Delta\xi_{p}\right)=0$$

with $\Delta\gamma_{p}=\Delta\lambda\sqrt{\frac{2}{3}\tilde{G}_{ij}\tilde{G}_{ij}}=\Delta\lambda\sqrt{\frac{2}{3}}G_{II}$

One chooses to make an explicit resolution with a development of Eulerian:

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}-D_{ijkl}^{e}\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right)-\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda+\frac{\partial f^{d}}{\partial\xi_{p}}\Delta\xi_{p}=0$$

The two cases are distinguished:

$\Delta\xi_{p}=\Delta\gamma_{p}+\Delta\gamma_{vp}$ (dilating case: the stress state exceeds the limit contractance/dilatancy)

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\varepsilon_{kl}-D_{ijkl}^{e}\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right) -\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda+\frac{\partial f^{d}}{\partial\xi_{p}}\left(\Delta\gamma_{p}+\Delta\gamma_{vp}\right)=0$$

$$f^{d}\left(\sigma_{ij}^{-}+D_{ijkl}^{e}\Delta\epsilon_{kl}-D_{ijkl}^{e}<\Phi>G_{kl}^{visc}\left(\sigma^{-},\xi_{v}^{-}\right)\Delta t,\xi_{p}^{-}\right)= \left(\frac{\partial f^{d}}{\partial\sigma_{ij}}D_{ijkl}^{e}G_{kl}(\sigma^{-},\xi_{p}^{-})-\frac{\partial f^{d}}{\partial\xi_{p}}\sqrt{\frac{2}{3}}G_{II}\left(\sigma^{-},\xi_{p}^{-}\right)\Delta\lambda-\frac{\partial f^{d}}{\partial\xi_{p}}\Delta\gamma_{vp}\right)$$

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\xi_p}\Delta\gamma_{vp}+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_{vp}^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

$\Delta\xi_p=\Delta\gamma_p$ (case contracting: the stress state is below the limit contractance/dilatancy)

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_{vp}^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

with $\Phi=A_v\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Pa}\right)^{n_v}$, A_v and n_v are parameters of the model.

4.2.2 Tangent operator

the stress with the state $+$:

$$\sigma_{ij}=\sigma_{ij}^-+D_{ijkl}^e\Delta\epsilon_{kl}^e=\sigma_{ij}^-+D_{ijkl}^e\left(\Delta\epsilon_{kl}-\Delta\epsilon_{kl}^p-\Delta\epsilon_{kl}^{vp}\right)=$$
$$\sigma_{ij}^-+D_{ijkl}^e\left(\Delta\epsilon_{kl}-\Delta\lambda\, G_{kl}(\sigma^-,\xi_p^-)-\langle\Phi\rangle\, G_{kl}^{visc}(\sigma^-,\xi_{vp}^-)\,\Delta t\right)$$

$$\frac{\partial\sigma_{ij}}{\partial\Delta\varepsilon_{kl}}=D_{ijkl}^e-D_{ijmn}^e G_{mn}^-\frac{\partial\Delta\lambda}{\partial\Delta\varepsilon_{kl}}-D_{ijmn}^e G_{imn}^{visc-}\frac{\partial\langle\varphi\rangle}{\partial\Delta\varepsilon_{kl}}\Delta t$$

The two cases are distinguished:

$\Delta\xi_p=\Delta\gamma_p$ (case contracting: the stress state is below characteristic threshold)

$$\Delta\lambda=\frac{f^d\left(\sigma_{ij}^-,\xi_p^-\right)+\frac{\partial f^d}{\partial\sigma_{ij}}\left[D_{ijkl}^e\Delta\varepsilon_{kl}-D_{ijkl}^e\langle\Phi\rangle G_{kl}^{visc}\left(\sigma^-,\xi_v^-\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijkl}^e G_{kl}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}\quad\text{and}\quad\Phi=A_v\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Pa}\right)^{n_v}$$

$$\frac{\partial\Delta\lambda}{\partial\Delta\varepsilon_{kl}}=\frac{\frac{\partial f^d}{\partial\sigma_{ij}}\cdot\left[D_{ijkl}^e-D_{ijmn}^e G_{mn}^{visc}\left(\sigma^-,\xi_v^-\right)\frac{\partial\Phi}{\partial\Delta\varepsilon_{kl}}\Delta t\right]}{\left(\frac{\partial f^d}{\partial\sigma_{ij}}D_{ijmn}^e G_{mn}\left(\sigma^-,\xi_p^-\right)-\frac{\partial f^d}{\partial\xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi_p^-\right)\right)}$$

$$\frac{\partial\Phi}{\partial\Delta\varepsilon_{kl}}=\frac{A_v.n_v}{Patm}\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial\sigma_{ij}^e}.\frac{\partial\sigma_{ij}^e}{\partial\Delta\varepsilon_{kl}}=\frac{A_v.n_v}{Patm}\left(\frac{f^{vp}\left(\sigma^e,\xi_{vp}^-\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial\sigma_{ij}^e}.D_{ijkl}^e$$

$$\frac{\partial \sigma_{ij}}{\partial \Delta\varepsilon_{kl}}=D^e_{ijkl}-D^e_{ijmn}.G^-_{mn}\frac{\frac{\partial f^d}{\partial \sigma_{ij}}.\left[D^e_{ijkl}-D^e_{ijmn}G^{visc}_{mn}\left(\sigma^-,\xi^-_p\right)\frac{\partial \Phi}{\partial \Delta\varepsilon_{kl}}\Delta t\right]}{\left(\frac{\partial f^d}{\partial \sigma_{ij}}D^e_{ijkl}G_{kl}\left(\sigma^-,\xi^-_p\right)-\frac{\partial f^d}{\partial \xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi^-_p\right)\right)}-$$

$$D^e_{ijmn}.G^{visc-}_{mn}\frac{A_v.n_v}{Patm}\left(\frac{f^{vp}\left(\sigma^e,\xi^-_{vp}\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial \sigma^e_{ij}}.D^e_{ijkl}\Delta t$$

$\Delta\xi_p=\Delta\gamma_p+\Delta\gamma_v$ (dilating case: the stress state exceeds the characteristic threshold)

$$\Delta\lambda=\frac{f^d\left(\sigma^-_{ij},\xi^-_p\right)+\frac{\partial f^d}{\partial \xi_p}\Delta\gamma_{vp}+\frac{\partial f^d}{\partial \sigma_{ij}}\left[D^e_{ijkl}\Delta\varepsilon_{kl}-D^e_{ijkl}\langle\Phi\rangle G^{visc}_{kl}\left(\sigma^-,\xi^-_{vp}\right)\Delta t\right]}{\left(\frac{\partial f^d}{\partial \sigma_{ij}}D^e_{ijkl}G_{kl}\left(\sigma^-,\xi^-_p\right)-\frac{\partial f^d}{\partial \xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi^-_p\right)\right)}$$

$$\frac{\partial \Delta\lambda}{\partial \Delta\varepsilon_{kl}}=\frac{\frac{\partial f^d}{\partial \sigma_{ij}}.\left[D^e_{ijkl}-D^e_{ijmn}G^{visc}_{mn}\left(\sigma^-,\xi^-_p\right)\frac{\partial \Phi}{\partial \Delta\varepsilon_{kl}}\Delta t\right]+\frac{\partial f^d}{\partial \xi_p}.\frac{\partial \Delta\gamma_{vp}}{\partial \Delta\varepsilon^{vp}_{kl}}.\frac{\partial \Delta\varepsilon^{vp}_{kl}}{\partial \Delta\sigma^e_{ij}}.D^e_{ijkl}}{\left(\frac{\partial f^d}{\partial \sigma_{ij}}D^e_{ijmn}G_{mn}\left(\sigma^-,\xi^-_p\right)-\frac{\partial f^d}{\partial \xi_p}\sqrt{\frac{2}{3}}\tilde{G}_{II}\left(\sigma^-,\xi^-_p\right)\right)}$$

where:

$$\frac{\partial \Delta\gamma_{vp}}{\partial \Delta\varepsilon^{vp}_{kl}}=\frac{1}{2}\left(\frac{2}{3}\Delta\tilde{\varepsilon}^{vp}_{ij}.\Delta\tilde{\varepsilon}^{vp}_{ij}\right)^{-\frac{1}{2}}.\frac{2}{3}.2.\Delta\tilde{\varepsilon}^{vp}_{ij}.\frac{\partial \Delta\tilde{\varepsilon}^{vp}_{ij}}{\partial \Delta\varepsilon^{vp}_{kl}}=\frac{2}{3}.\frac{\Delta\tilde{\varepsilon}^{vp}_{ij}}{\Delta\gamma_{vp}}.\left(\delta_{ik}.\delta_{jl}-\frac{1}{3}\delta_{ij}.\delta_{kl}\right)$$

$$\frac{\partial \Delta\varepsilon^{vp}_{kl}}{\partial \Delta\sigma^e_{ij}}=\frac{n_v.A_v}{Patm}.\left(\frac{f^{vp}\left(\sigma^e,\xi^-_v\right)}{Patm}\right)^{n_v-1}.\frac{\partial f^{vp}}{\partial \sigma^e_{ij}}.G^{visc-}_{kl}\left(\sigma^-,\xi^-_{vp}\right)\Delta t$$

4.2.3 Algorithm of resolution in Code_Aster

- Change of sign of the stresses to the state $-$ and the increase in strain:

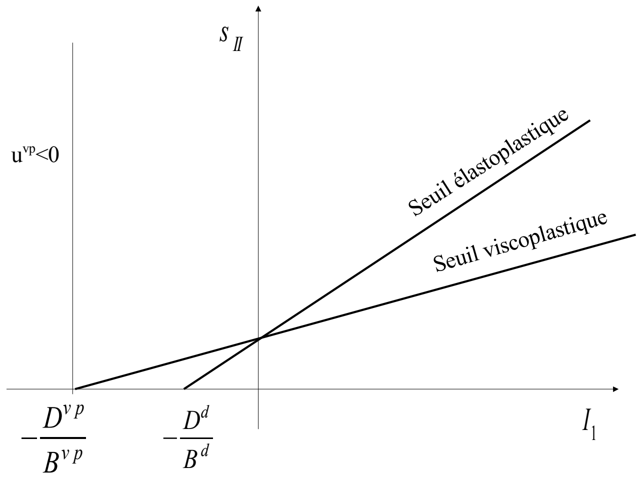
  $\sigma^-_{L\wedge K}=-\sigma^-$

  $\Delta\varepsilon_{L\wedge K}=-\Delta\varepsilon$
- Computation of the elastic stress of prediction σ^e : $\sigma^e=\sigma^-+D^e\Delta\varepsilon$
- Checking of the sign of the viscous criterion with the viscous variable max: $f^{vp}\left(\sigma^e,\xi_{vp-\max}\right)$
- If $f^{vp}\left(\sigma^e,\xi_{vp-\max}\right)>0$: dilating case and coupled hardening $V_5=1$. One regards as variable of hardening of the plastic criterion the office plurality between the plastic variable of hardening and the viscous variable: $\Delta\xi_p=\Delta\gamma_p+\Delta\gamma_{vp}$
- If $f^{vp}\left(\sigma^e,\xi_{vp-\max}\right)<0$: case contracting and hardening not coupled $V_5=0$. The variable of hardening of the plastic criterion is the plastic variable: $\Delta\xi_p=\Delta\gamma_p$

For the viscous criterion: $u^{vp}=A^{vp}\left(\xi_{vp}\right)s_{II}H\left(\theta\right)+B^{vp}\left(\xi_{vp}\right)I_1+D^{vp}\left(\xi_{vp}\right)$

If $u^{vp}\left(\sigma^e,\xi_{vp}^-\right)<0$:

•if $-\frac{D^{vp}}{B^{vp}}<-\frac{D^d}{B^d}$ (the Figure 4.2.3-a) then recutting of time step

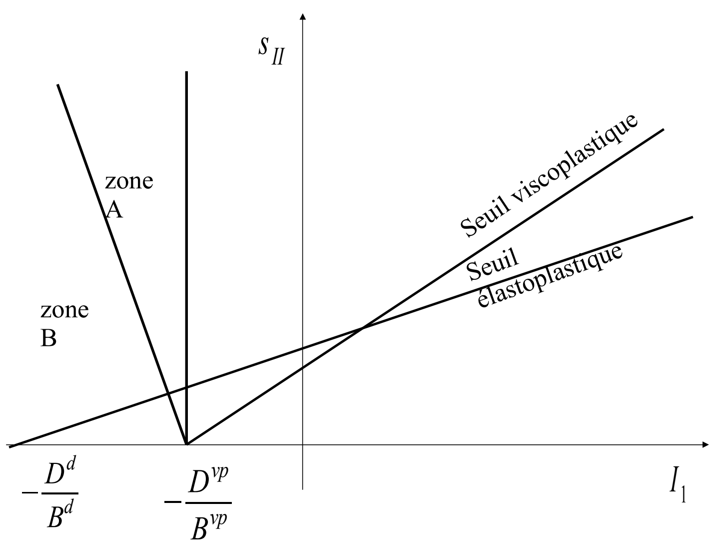


the Figure 4.2.3-a. Schematic representation in the case $u^{vp}\left(\sigma^e,\xi_{vp}^-\right)<0$ **,** $-\frac{D^{vp}}{B^{vp}}<-\frac{D^d}{B^d}$

If not (Appear 4.2.3-b) both cases arise:

- •if σ^e is in the zone A then $f^{vp}\left(\sigma^e\right)$ is not defined but one can make a geometrical projection (cf notices paragraph 3.3.3)
- •if σ^e in the zone then B it is necessary to make a projection at the top.

One is satisfied then with the message: “stop for coefficients non-cohesive materials of the model”.



Appear 4.2.3-b. Schematic representation in the case $u^{vp}\left(\sigma^{e},\xi_{vp}^{-}\right)<0$, $-\frac{D^{d}}{B^{d}}<-\frac{D^{vp}}{B^{vp}}$

For the elastoplastic criterion:

$$u^{d}=A^{d}\left(\xi_{p}\right)s_{II}\,H\left(\theta\right)+B^{d}\left(\xi_{p}\right)I_{1}+D^{d}\left(\xi_{p}\right)$$

So $u^{d}\left(\sigma^{e},\xi_{p}^{-}\right)<0$ then there is recutting of time step

For the viscous criterion:

So $f^{vp}\left(\sigma^{e},\xi_{vp}^{-}\right)<0$ then not of creep and $\Delta\varepsilon_{vp}=0$

- So $f^{vp}\left(\sigma^{e},\xi_{vp}^{-}\right)>0$ then creep develops according to the following form:

$$\Delta\varepsilon_{vp}=A\left[\frac{\langle f^{vp}\left(\sigma^{e},\xi_{vp}^{-}\right)\rangle}{Patm}\right]^{n_{v}}G^{visc}\left(\sigma^{-},\xi_{vp}^{-}\right)\Delta t \quad \text{with} \quad G^{visc}=\frac{\partial f^{vp}}{\partial\sigma}-\left(\frac{\partial f^{vp}}{\partial\sigma}n\right)n$$

$\langle\ \rangle$: The hooks of Macauley

where $\sin\left(\Psi\right)=\mu_{0,v}\left(\frac{\sigma_{\max}-\sigma_{\lim}}{\xi_{0,v}\sigma_{\max}+\sigma_{\lim}}\right)$ (see § 3.6.1), β' and n are deduced

one can deduce $\Delta\gamma_{vp}=\sqrt{\frac{2}{3}\Delta\tilde{\varepsilon}_{vp}.\Delta\tilde{\varepsilon}_{vp}}>0$ where $\Delta\tilde{\varepsilon}_{vp}=\Delta\varepsilon_{vp}-\frac{tr\left(\Delta\varepsilon_{vp}\right)}{3}I^{d}$

reactualization of the variable of hardening of the viscous criterion:

$\xi_{vp}=\xi_{vp}^{-}+\Delta\xi_{vp}$ with $\Delta\xi_{vp}=Min\left[\Delta\gamma_{vp},\xi_{v-\max}-\xi_{vp}^{-}\right]$

reactualization of the stresses: $\sigma=\sigma^{e}-D^{e}\Delta\varepsilon_{vp}$

reactualization of the local variables:

$$V_{1}=\xi_{p}=\xi_{p}^{-}+\Delta\gamma_{vp}$$
$$V_{2}=\gamma_{p}=\gamma_{p}^{-}$$
$$V_{3}=\xi_{vp}=\xi_{vp}^{-}+\Delta\xi_{vp}$$
$$V_{4}=\gamma_{vp}=\gamma_{vp}^{-}+\Delta\gamma_{vp}$$

For the elastoplastic criterion:

So $f^{d}\left(\sigma^{e}-D^{e}\Delta\xi_{vp},\xi_{p}^{-}+\Delta\gamma_{vp}\right)\leq0$ then: $\Delta\gamma=\Delta\gamma_{p}=\Delta\xi_{p}=0$

So $f^{d}\left(\sigma^{e}-D^{e}\Delta\varepsilon_{vp},\xi_{p}^{-}+\Delta\gamma_{vp}\right)>0$ then: $\Delta\tilde{\varepsilon}_{p}=\Delta\lambda G\left(\sigma^{-},\xi_{p}^{-}\right)$ with

$\sin(\Psi)=\mu_{0,v}\left(\frac{\sigma_{\max}-\sigma_{\lim}}{\xi_{0,v}\sigma_{\max}+\sigma_{\lim}}\right)$ (see § 3.6.1) if $0\le\xi_p^-<\xi_{pic}$

$\sin(\Psi)=\mu_1\left(\frac{\alpha-\alpha_{res}}{\xi_1\alpha-\alpha_{res}}\right)$ (see § 3.6.2) if $\xi_p^->\xi_{pic}$

$G=\frac{\partial f^d}{\partial\sigma}-\left(\frac{\partial f^d}{\partial\sigma}n\right)n$ β' and n are deduced

One seeks $\Delta\lambda>0$ such as $f^d\left(\sigma,\xi_p^-\right)=0$

One deduces $\Delta\gamma_p>0$

If hardening not coupled (contractance) $f^d\left(\sigma^e-D^e\Delta\xi_{vp},\xi_p^-+\Delta\gamma_{vp}\right)\le 0$ then: $\Delta\xi_p=\Delta\gamma_p$

if not coupled hardening (dilatancy) $f^d\left(\sigma^e-D^e\Delta\varepsilon_{vp},\xi_p^-+\Delta\gamma_{vp}\right)>0$ then: $\Delta\xi_p=\Delta\gamma_p+\Delta\gamma_{vp}$

The table of the local variables is supplemented:

$$V_1=\xi_p=\xi_p^-+\Delta\xi_p$$
$$V_2=\gamma_p=\gamma_p^-+\Delta\gamma_p$$

Update of the stresses:

$$\Delta\varepsilon^{irr}=\Delta\varepsilon^{vp}+\Delta\varepsilon^p$$
$$\Delta\varepsilon^e=\Delta\varepsilon-\Delta\varepsilon^{irr}$$
$$\Delta\sigma=D^e\Delta\varepsilon^e$$
$$\sigma=\sigma^-+\Delta\sigma$$

Summarized algorithm
• $\sigma^e=\sigma^-+D^e\Delta\varepsilon$ •so $f^{vp}\left(\sigma^e,\xi_{v\max}\right)<0$ then contractance ($VARV=0$) and the plastic variable is $\Delta\xi^p=\Delta\gamma^p$ •so $f^{vp}\left(\sigma^e,\xi_{v\max}\right)>0$ then dilatancy ($VARV=1$) and the plastic variable is $\Delta\xi^p=\Delta\gamma^p+\Delta\gamma^{vp}$ **Checking of creep:**   •computation of $f^{vp}\left(\sigma^e,\xi_{vp}^-\right)$ •if $f^{vp}\left(\sigma^e,\xi_{vp}^-\right)<0$ (**No creep**) $\Delta\xi_{vp}=\Delta\gamma_{vp}=0$ $\xi_{vp}=\xi_{vp}^-$ $\gamma_{vp}=\gamma_{vp}^-$

•if $f^{vp}\left(\sigma^e, \xi_{vp}^-\right) > 0$ (**creep)**

computation of $\Delta\varepsilon_{vp}$ and of $\Delta\gamma_{vp}$ according to σ^e and of ξ_{vp}^-

$$\Delta\xi_{vp} = \min\left(\Delta\gamma_{vp}, \xi_{v-\max} - \xi_{vp}^-\right)$$
$$\xi_{vp} = \xi_{vp}^- + \Delta\xi_{vp}$$
$$\gamma_{vp} = \gamma_{vp}^- + \Delta\gamma_{vp}$$

•Adjustment of the elastic prediction: $\sigma_n^e = \sigma^e - D^e \Delta\varepsilon_{vp}$

Checking of plasticity :

•computation of $f^d\left(\sigma_n^e, \xi_p^- + \Delta\gamma_{vp}\right)$
•if $f^d\left(\sigma_n^e, \xi_p^- + \Delta\gamma_{vp}\right) < 0$ (Elasticity**)**

$$\Delta\varepsilon_p = \Delta\gamma_p = 0$$
$$\gamma_p = \gamma_p^-$$
$\xi_p = \xi_p^- + \Delta\xi_p$ with
$$\Delta\xi_p = 0 \text{ if } VARV = 0$$
$$\Delta\xi_p = \Delta\gamma_{vp} \text{ if } VARV = 1$$

put formula of the stresses:
$$\sigma = \sigma^e - D^e \Delta\varepsilon_{vp}$$

•if $f^d\left(\sigma_n^e, \xi_p^- + \Delta\gamma_{vp}\right) > 0$ (**Plasticity)**

computation of $\Delta\lambda$, $\Delta\gamma_p$ and $\Delta\varepsilon_p$

$$\Delta\xi_p = \Delta\gamma_p \text{ if } VARV = 0$$
$$\Delta\varepsilon_p = \Delta\gamma_p + \Delta\gamma_{vp} \text{ if } VARV = 1$$
$$\xi_p = \xi_p^- + \Delta\xi_p$$

put formula of the stresses:
$$\sigma = \sigma^- + D^e(\Delta\varepsilon - \Delta\varepsilon_{vp} - \Delta\varepsilon_p)$$

table of the local variables:
$V1 = \xi_p$
$V2 = \gamma_p$
$V3 = \xi_{vp}$
$V4 = \gamma_{vp}$
$V5 = VARV$ (0 if contractance or 1 if dilatancy)
$V6 =$ of viscosity
$V7 =$ of plasticity

4.3 implicit Diagram of integration

the integration of model LETK according to the implicit diagram of integration is realized under environment PLASTI . The integration of model LETK under implicit scheme is currently available only by computation of a disturbed local jacobian matrix ("NEWTON_PERT").

The algorithm of resolution follows following logic. It uses an elastic prediction then iterations of correction if the viscous and/or plastic thresholds are requested. The purpose of the diagram is producing the variation of the stresses and variables of hardening under the effect of an increment of strain.

The local subdivision of the model is activable under this diagram of integration by the key word ITER_INTE_PAS of factor key word the COMP_INCR , cf [U4.51.11]).

4.3.1 Elastic phase of prediction

This phase is similar to that presented in section 19 .
The going beyond the thresholds of plasticity and viscosity is tested compared to this stress state. The statement of the thresholds tested is clarified in the § 9 .

- If none the thresholds is requested, the prediction is regarded as valid compared to the models. An update of the local variables is undertaken to display the state of activation of the various thresholds.

- If a threshold among both to consider (plasticity and/or viscosity) is requested, the resolution of a local system of nonlinear equations must be initiated. The defined mechanisms of dissipation as potentially active must lead to the going beyond the thresholds associated (plasticity and/or viscosity)

4.3.2 Phase with correction: nonlinear equations to solve

This stage consists in solving the system of equations local nonlinear established on the basis of viscous and/or plastic mechanism. After convergence, the stresses and local variables of the model are put up to date.

The nonlinear unknowns of the system of equations are the stresses σ_{n+1} , the plastic multiplier λ_{n+1}^{p} , the plastic variable of hardening ξ_{n+1}^{p} and the viscous variable of hardening ξ_{n+1}^{vp} .
The vector of the inconuu be thus comprises to the maximum for modelizations $3D$ 9 unknowns.

The nonlinear equations to solve are the following ones:

- The incremental equation of state $E1$:

$$\underline{\sigma}_{n+1}-\underline{\sigma}_{n}-C^{e}\left(\underline{\sigma}_{n+1}\right):\left(\Delta\underline{\epsilon}-\Delta\lambda\,\underline{G}^{p}-\Phi\left(f^{vp}\right)\cdot\underline{G}^{vp}\right)=0$$

- the condition of Kuhn-Tucker $E2$:

$$\begin{cases}\text{Si} \quad f^{d}\leqslant 0 \quad \text{alors} \quad \Delta\lambda=0\\ \text{Si} \quad f^{d}=0 \quad \text{alors} \quad \Delta\lambda>0\end{cases}$$

- the incremental evolution of the variable of hardening plastics $E3$:

$\xi_{n+1}^{p}-\xi_{n}^{p}-\Delta\xi^{p}=0$, with $\Delta\xi^{p}$ evolving according to the conditions specified with the § 11 .

- Incremental evolution of the viscoplastic variable of hardening $E4$:

$\xi_{n+1}^{vp}-\xi_{n}^{vp}-\Delta\xi^{vp}=0$, with $\Delta\xi^{vp}$ evolving according to the conditions specified with the § 11 .

These equations constitute a square system $R(\Delta Y)$, where the unknowns are $\Delta Y=(\Delta \underline{\underline{\sigma}}, \Delta \lambda, \Delta \xi^{p}, \Delta \xi^{vp})$. With the iteration j loop of on-the-spot correction of Newton formulates, one abstr. O C the following matric equation:

$$\frac{d R(\Delta Y^{j})}{d(\Delta Y^{j})} \cdot \delta(\Delta Y^{j+1})=-R(\Delta Y^{j})$$

the jacobian matrix $\frac{d R(\Delta Y^{j})}{d(\Delta Y^{j})}$, asymmetric, builds themselves in the following way:

$$\frac{d R(\Delta Y^{j})}{d(\Delta Y^{j})}=\begin{bmatrix} \frac{\partial E1}{\partial \underline{\underline{\sigma}}_{n+1}^{j}} & \frac{\partial E1}{\partial \Delta \lambda^{j}} & \frac{\partial E1}{\partial \xi_{p}^{j}} & \frac{\partial E1}{\partial \xi_{vp}^{j}} \\ \frac{E2}{\partial \underline{\underline{\sigma}}_{n+1}^{j}} & \frac{E2}{\partial \Delta \lambda^{j}} & \frac{E2}{\partial \xi_{p}^{j}} & \frac{E2}{\partial \xi_{vp}^{j}} \\ \frac{E3}{\partial \underline{\underline{\sigma}}_{n+1}^{j}} & \frac{E3}{\partial \Delta \lambda^{j}} & \frac{E3}{\partial \xi_{p}^{j}} & \frac{E3}{\partial \xi_{vp}^{j}} \\ \frac{E4}{\partial \underline{\underline{\sigma}}_{n+1}^{j}} & \frac{E4}{\partial \Delta \lambda^{j}} & \frac{E4}{\partial \xi_{p}^{j}} & \frac{E4}{\partial \xi_{vp}^{j}} \end{bmatrix}$$

This matrix is evaluated today analytically or by disturbance (`ALGO_INTE = "NEWTON"` or `ALGO_INTE = "NEWTON_PERT"`).

With an aim of standardizing the scales between the various equations to be solved, one makes the choice to put at the level of strains the equation E1 relating to the incremental equation of state. One applies for that the reverse of the nonlinear shear modulus of elasticity. This choice makes it possible to ensure a more uniform convergence on the group of the system.

Convergence famous is acquired since $\|R(\Delta Y^{j})\|<$ `RESI_INTE_RELA`. One also makes sure when the mechanism of plasticity is active that the plastic multiplier is strictly positive. If it is not the case, local integration is started again without taking account of the mechanism of plasticity. Only the mechanism of viscosity can then be considered.

4.3.2.1 Statement of the terms of the jacobian matrix

the derived terms associated with (R_{1}) are:

$$\frac{d(R_{1})_{ij}}{d((\Delta Y_{1})_{mn})}=I_{ilkl}-\frac{\partial C_{ijkl}^{e}}{\partial \sigma_{mn}}:[\Delta \epsilon_{kl}-\Delta \lambda \cdot G_{kl}^{p}-\Delta \epsilon_{kl}^{vp}]+\Delta \lambda C_{ijkl}^{e}:\frac{\partial G_{kl}^{p}}{\partial \sigma_{mn}}+$$
$$C_{ijkl}^{e}:G_{kl}^{vp} \otimes \frac{\partial(\langle \phi(f^{vp})\rangle^{+})}{\partial \sigma_{mn}} \Delta t+\langle \phi(f^{vp})\rangle^{+} \cdot \Delta t \cdot C_{ijkl}^{e}:\frac{\partial G_{kl}^{vp}}{\partial \sigma_{mn}}$$

$$\frac{d(R_{1})_{ij}}{d(\Delta Y_{2})}=C_{ijkl}^{e}:G_{kl}^{p}$$

$$\frac{d(R_{1})_{ij}}{d(\Delta Y_{3})}=\Delta \lambda \cdot C_{ijkl}^{e}:\frac{\partial G_{kl}^{p}}{\partial \xi^{p}}$$

$$\frac{d\left(R_1\right)_{ij}}{d\left(\Delta Y_4\right)}=C^e_{ijkl}:\left[\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^+}{\partial\xi^{vp}}G^{vp}_{kl}+\langle\phi\left(f^{vp}\right)\rangle^+\cdot\frac{\partial G^{vp}_{kl}}{\partial\xi^{vp}}\right]\cdot\Delta t$$

the derived terms associated with $\left(R_2\right)$ s.e distinguish according to the statement taken to satisfy the condition with Kuhn-Tucker:

If $\left(R_2\right)=\Delta\lambda$:

$$\frac{d\left(R_2\right)}{d\left(\left(\Delta Y_1\right)_{ij}\right)}=0_{ij}$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_2\right)}=1$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_3\right)}=0$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_4\right)}=0$$

S I $\left(R_2\right)=f^p$:

$$\frac{d\left(R_2\right)}{d\left(\left(\Delta Y_1\right)_{ij}\right)}=\frac{\partial f^p}{\partial\sigma_{ij}}$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_2\right)}=0$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_3\right)}=\frac{\partial f^p}{\partial\xi^p}$$

$$\frac{d\left(R_2\right)}{d\left(\Delta Y_4\right)}=0$$

the derived terms associated with $\left(R_3\right)$ are:

$$\frac{d\left(R_3\right)}{d\left(\Delta Y_1\right)_{ij}}=-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\sigma_{ij}}$$ (case contract ant: the stress state is below characteristic threshold)

$$\frac{d\left(R_3\right)}{d\left(\Delta Y_1\right)_{ij}}=-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\sigma_{ij}}-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left[\left(\tilde{G}^{vp}_{II}\cdot\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^+}{\partial\sigma_{ij}}+\langle\phi\left(f^{vp}\right)\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\partial\sigma_{ij}}\right)\right]$$ (dilating case: the stress state exceeds the characteristic threshold)

$$\frac{d\left(R_3\right)}{d\left(\Delta Y_2\right)}=-\sqrt{\frac{2}{3}}\cdot\tilde{G}^p_{II}$$

$$\frac{d\left(R_3\right)}{d\left(\Delta Y_3\right)}=1-\Delta\lambda\cdot\sqrt{\frac{2}{3}}\cdot\frac{\partial\tilde{G}^p_{II}}{\partial\xi^p}$$

$$\frac{d\left(R_3\right)}{d\left(\Delta Y_4\right)}=-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^+}{\xi^{vp}}\cdot\tilde{G}^{vp}_{II}+\langle\phi\left(f^{vp}\right)\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\xi^{vp}}\right)$$ (dilating case: the stress state exceeds the characteristic threshold)

the derived terms associated with $\left(R_4\right)$ are:

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_1\right)_{ij}}=-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^+}{\sigma_{ij}}\cdot\tilde{G}^{vp}_{II}+\langle\phi\left(f^{vp}\right)\rangle^+\cdot\frac{\partial\tilde{G}^{vp}_{II}}{\sigma_{ij}}\right)$$ if $\dot{\gamma}^{vp}\leqslant\xi^{vp}_{\max}-\xi^{vp}\left(t^-\right)$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_2\right)}=0$$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_3\right)}=0$$

$$\frac{d\left(R_4\right)}{d\left(\Delta Y_4\right)}=1-\sqrt{\frac{2}{3}}\cdot\Delta t\cdot\left(\frac{\partial\langle\phi\left(f^{vp}\right)\rangle^{+}}{\xi^{vp}}\cdot\tilde{G}_{II}^{vp}+\langle\phi\left(f^{vp}\right)\rangle^{+}\cdot\frac{\partial\tilde{G}_{II}^{vp}}{\xi^{vp}}\right)$$

the detailed statement of all the put ends concerned depends on following principal derivatives:

$$\frac{d\,C_{ijkl}^{e}}{d\,\sigma_{mn}}\;;\;\frac{d\,G_{ij}^{p}}{d\,\sigma_{kl}}\;;\;\frac{d\langle\phi\left(f^{vp}\right)\rangle^{+}}{d\,\sigma_{ij}}\;;\;\frac{d\,G_{ij}^{vp}}{d\,\sigma_{kl}}\;;\;\frac{d\,\tilde{G}_{II}^{p}}{d\,\sigma_{ij}}\;;\;\frac{d\,\tilde{G}_{II}^{vp}}{d\,\sigma_{ij}}\;;$$

$$\frac{d\,G_{ij}^{p}}{d\,\xi^{p}}\;;\;\frac{d\,\tilde{G}_{II}^{p}}{d\,\xi^{p}}\;;$$

$$\frac{d\langle\phi\left(f^{vp}\right)\rangle^{+}}{d\,\xi^{vp}}\;;\;\frac{d\,G_{ij}^{vp}}{d\,\xi^{vp}}\;;\;\frac{d\,\tilde{G}_{II}^{vp}}{d\,\xi^{vp}}\;.$$

The quantities mentioned above are presented in appendix of the document.

4.3.3 Phase of update

the update of the vector solution is carried out according to the following operation:

$$\Delta Y=\Delta Y^{j+1}=\Delta Y^{j}+\delta\,\Delta Y^{j+1}$$

This phase of update consists in deferring the evolution of the stresses, plastic strains, viscoplastic strains and hardening parameters plastic and viscoplastic.

4.3.4 Tangent operator of velocity

the tangent operator of velocity was introduced right now in the frame of the explicit diagram of integration. This operator of stiffness is used during the predictions on a total scale of Newton-Raphson out of tangent matrix (`PREDICTION=' TANGENTE'`). Sources FORTRAN associated with the onstruction with this operator are common to both diagrams of integration (`ALGO_INTE = ("NEWTON", "SPECIFIQUE")`).

4.3.5 Consistent tangent operator

On the basis of analytical development specified in the document [R5.03.12], it is possible to determine the tangent operator $\boldsymbol{M}_c=\frac{\partial\sigma}{\partial\epsilon}$ starting from the terms of the jacobian matrix defined above, § 27 ($J=\frac{d\,R}{d\,Y}$).

Indeed, the system $\Phi\left(\Delta Y\right)=0$ is checked at the end of the increment and for a small variation of Φ , by considering this time ϵ like a variable, the system remains with the equilibrium and thus one checks $d\,\Phi=0$.

By differentiation, one obtains:

$$\frac{\partial\Phi}{\partial\Delta\epsilon}d\left(\Delta\epsilon\right)+\frac{\partial\Phi}{\partial\Delta\sigma}d\left(\Delta\sigma\right)+\frac{\partial\Phi}{\partial\Delta\lambda}d\left(\Delta\lambda\right)+\frac{\partial\Phi}{\partial\Delta\xi_p}d\left(\Delta\xi_p\right)+\frac{\partial\Phi}{\partial\Delta\xi_{vp}}d\left(\Delta\xi_{vp}\right)=0$$

One rewrites the system by putting the ends in ϵ in the member of right:

$$\frac{\partial\Phi}{\partial\Delta\epsilon}d\left(\Delta\epsilon\right)+\frac{\partial\Phi}{\partial\Delta\sigma}d\left(\Delta\sigma\right)+\frac{\partial\Phi}{\partial\Delta\lambda}d\left(\Delta\lambda\right)+\frac{\partial\Phi}{\partial\Delta\xi_p}d\left(\Delta\xi_p\right)=-\frac{\partial\Phi}{\partial\Delta\xi_{vp}}d\left(\Delta\xi_{vp}\right)$$

This system can then be written in the following form:

$$J \cdot d(\Delta Y) = -\frac{\partial \Phi}{\partial(\Delta \epsilon)} d(\Delta \epsilon) \text{ with } \frac{\partial \Phi}{\partial(\Delta \epsilon)} = \{-C^e(\sigma), 0, 0, 0\}$$

Finally, one obtains: $J \cdot d(\Delta Y) = \{C^e(\sigma) : \Delta \epsilon, 0, 0, 0\}$

One writes then the system per blocks by separating $d(\Delta \sigma)$ from the other variables $Z = (\Delta \lambda, \Delta \xi_p, \Delta \xi_{vp})$, which gives:

$$\begin{bmatrix} J_{\sigma\sigma} & J_{\sigma Z} \\ J_{Z\sigma} & J_{ZZ} \end{bmatrix} . \begin{pmatrix} \Delta \sigma \\ Z \end{pmatrix} = \begin{pmatrix} C^e(\sigma) d(\Delta \epsilon) \\ 0 \end{pmatrix}$$

the statement of the tangent operator becomes:

$$\boldsymbol{M}_c = \frac{\partial \sigma}{\partial \epsilon} = \frac{d(\Delta \sigma)}{d(\Delta \epsilon)} = \left[J_{\sigma\sigma} - J_{\sigma Z} (J_{ZZ})^{-1} J_{Z\sigma} \right]^{-1} C^e(\sigma)$$

Note : The jacobian matrix not being symmetric, the tangent operator $\boldsymbol{M}_c$ is not it either.

5 References

R1 1 . The model “L&K” for Code_Aster. IH-HAVL-SIO-00015-A.

R2 2 . The model “L&K” for Code_Aster. IH-HAVL-SIO-00015-B.

R33. Réunion on the model viscoplastic L&K of the CIH simplified for Code_Aster. CR-AMA-2007-142.

R 4 . Numerical modelization of the behavior of the underground works by a viscoplastic approach. Thesis presented to the INPL by A. Kleine, November 2007.

6 Functionalities and checking

This document relates to the constitutive law `LETK` (key word `COMP_INCR` of `STAT_NON_LINE`) and associated material `LETK` (command `DEFI_MATERIAU`).

This behavior is checked by the two cases tests:
•SSNV206 - triaxial Compression test with the model LETK of the CIH - [V6.04.206]
•WTNV135 - triaxial Compression test drained with the model LETK of the CIH - [V7.31.135]

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
9.2	J.El-Gharib, C. Chavant, EDF-R&D/AMA F.Laigle, A.Kleine EDF-CIH	initial Text
11.2	A.Foucault	Algorithm of integration by implicit scheme
11.3	A.Foucault	analytical Development of Additional jacobian matrix

8 DR/DY: Terms of the jacobian matrix

Evaluating of the terms relative to $\dfrac{d(R_1)_{ij}}{d((\Delta Y_1)_{mn})}$:

$$\frac{d(R_1)_{ij}}{d((\Delta Y_1)_{mn})}=I_{ilkl}-\frac{\partial C^e_{ijkl}}{\partial\sigma_{mn}}:[\Delta\epsilon_{kl}-\Delta\lambda\cdot G^p_{kl}-\Delta\epsilon^{vp}_{kl}]+\Delta\lambda C^e_{ijkl}:\frac{\partial G^p_{kl}}{\partial\sigma_{mn}}+$$
$$C^e_{ijkl}:G^{vp}_{kl}\otimes\frac{\partial(\langle\phi(f^{vp})\rangle^+)}{\partial\sigma_{mn}}\Delta t+\langle\phi(f^{vp})\rangle^+\cdot\Delta t\cdot C^e_{ijkl}:\frac{\partial G^{vp}_{kl}}{\partial\sigma_{mn}}$$

$$\frac{\partial C^e_{ijkl}}{\partial\sigma_{mn}}=\frac{n_{elas}}{I_1}\cdot C^e_{ijkl}\otimes\delta_{mn}$$

$$\frac{\partial G^p_{ij}}{\partial\sigma_{kl}}=\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)-\left(\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right):n_{mn}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:\frac{\partial n_{mn}}{\partial\sigma_{kl}}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:n_{mn}\right)\cdot\frac{\partial n_{ij}}{\partial\sigma_{kl}}$$

$$\frac{\partial f^d}{\partial\sigma_{ij}}=$$

with $\dfrac{\partial(s_{II}H(\theta))}{\partial\sigma_{ij}}-a^d(\xi_p)\sigma_c H^c_0[A^d(\xi_p)s_{II}H(\theta)+B^d(\xi_p)I_1+D^d(\xi_p)]^{a^d(\xi_p)-1}$ either

$$\left(A^d(\xi_p)\frac{\partial(s_{II}H(\theta))}{\partial\sigma_{ij}}+B^d(\xi_p)I_d\right)$$

$$\frac{\partial}{\sigma_{kl}}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)=\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)-a^d(\xi^p)\sigma_c H^c_0[A^d(\xi^p)s_{II}H(\theta)+B^d(\xi^p)I_1+D^d(\xi^p)]^{a^d(\xi^p)-1}$$
$$\cdot A^d(\xi^p)\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)-a^d(\xi^p)(a^d(\xi^p)-1)\sigma_c H^c_0(A^d(\xi^p)s_{II}H(\theta)+B^d(\xi^p)I_1+D^d(\xi^p))^{a^d(\xi^p)-2}$$
$$\left(A^d(\xi^p)\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}+B^d(\xi^p)\delta_{ij}\right)\otimes\left(A^d(\xi^p)\frac{\partial s_{II}H(\theta)}{\partial\sigma_{kl}}+B^d(\xi^p)\delta_{kl}\right)$$

Gold $\dfrac{\partial(s_{II}H(\theta))}{\partial\sigma_{ij}}=\left(\left(\dfrac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\dfrac{\partial h(\theta)}{\partial s_{kl}}s_{II}+H(\theta)\dfrac{s_{kl}}{s_{II}}\right)\cdot\left(\delta_{ik}.\delta_{jl}-\dfrac{1}{3}\delta_{ij}.\delta_{kl}\right)$ from where

$$\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial s_{II}H(\theta)}{\partial\sigma_{ij}}\right)=\left(\frac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)s_{II}\frac{\partial s_{mn}}{\partial\sigma_{ij}}+\left(\frac{H^c_0-H^e_0}{h^c_0-h^e_0}\right)\frac{\partial h(\theta)}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}\frac{\partial s_{II}}{\partial s_{pq}}\frac{\partial s_{pq}}{\partial\sigma_{kl}}+$$
$$\frac{\partial H(\theta)}{\partial\sigma_{kl}}\frac{s_{mn}}{s_{II}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}+\frac{H(\theta)}{s_{II}}\frac{\partial s_{mn}}{\partial\sigma_{kl}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}-\frac{H(\theta)s_{mn}}{s^2_{II}}\frac{\partial s_{II}}{\partial s_{pq}}\frac{\partial s_{pq}}{\partial\sigma_{kl}}\frac{\partial s_{mn}}{\partial\sigma_{ij}}$$

One has moreover $\dfrac{\partial h(\theta)}{\partial s_{kl}}=\dfrac{\gamma\cos(3\theta)}{6\mathrm{h}(\theta)^5}\dfrac{3\mathrm{s}_{kl}}{s^2_{II}}-\dfrac{\gamma\sqrt{54}}{6\mathrm{h}(\theta)^5 s^3_{II}}\left(\dfrac{\partial\det(\underline{s})}{\partial s_{kl}}\right)$

$$\frac{\partial}{\partial\sigma_{kl}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)=\frac{\partial}{\partial s_{pq}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)\frac{\partial s_{pq}}{\partial\sigma_{kl}}$$

$$\frac{\partial}{\partial s_{pq}}\left(\frac{\partial h(\theta)}{\partial s_{mn}}\right)=\frac{\sqrt{54}\gamma}{2h^5(\theta)s_{II}^5}s_{mn}\otimes\frac{\partial det(s_{ij})}{\partial s_{pq}}-\frac{3\gamma\sqrt{54}\,det(s_{ij})}{2h^5(\theta)s_{II}^7}s_{mn}\otimes s_{pq}+\frac{\gamma\cos 3\theta}{2h^5(\theta)s_{II}^2}I_{mnpq}-$$
$$\frac{\gamma\cos 3\theta}{h^5(\theta)s_{II}^4}s_{mn}\otimes s_{pq}-\frac{5\gamma\cos 3\theta}{2h^6(\theta)s_{II}^2}s_{mn}\otimes\frac{\partial h(\theta)}{\partial s_{pq}}+\frac{5\gamma\sqrt{54}}{6s_{II}^3h^6(\theta)}\frac{\partial det(s_{ij})}{\partial s_{mn}}\otimes\frac{\partial h(\theta)}{\partial s_{pq}}+$$
$$\frac{\gamma\sqrt{54}}{2h^5(\theta)s_{II}^5}\frac{\partial det(s_{ij})}{\partial s_{mn}}\otimes s_{pq}-\frac{\gamma\sqrt{54}}{6h^5(\theta)s_{II}^3}\frac{\partial^2 det(s_{ij})}{\partial s_{mn}\partial s_{pq}}$$

with

$$\frac{\partial^2 det(s_{ij})}{\partial s_{mn}\partial s_{pq}}=\begin{bmatrix}0 & s_{33} & s_{22} & 0 & 0 & -\sqrt{2}s_{23}\\ s_{33} & 0 & s_{11} & 0 & -\sqrt{2}s_{13} & 0\\ s_{22} & s_{11} & 0 & -\sqrt{2}s_{12} & 0 & 0\\ 0 & 0 & -\sqrt{2}s_{12} & -s_{33} & s_{13} & s_{23}\\ 0 & -\sqrt{2}s_{13} & 0 & s_{13} & -s_{11} & s_{12}\\ -\sqrt{2}s_{23} & 0 & 0 & s_{23} & s_{12} & -s_{22}\end{bmatrix}$$

One recalls that $n_{ij}=\frac{\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}}{\sqrt{\beta'^2+3}}$ and $\beta'=-\frac{2\sqrt{6}\sin(\Psi)}{3-\sin(\Psi)}$

$$\frac{\partial n_{ij}}{\partial\sigma_{kl}}=\frac{\left[\frac{\partial\beta'}{\partial\sigma_{kl}}\frac{s_{ij}}{s_{II}}+\frac{\beta'}{s_{II}}\frac{\partial s_{ij}}{\partial\sigma_{kl}}-\frac{\beta' s_{ij}}{s_{II}^2}\frac{\partial s_{II}}{\partial\sigma_{kl}}\right](\beta'^2+3)-\beta'\left(\beta'\frac{s_{ij}}{s_{II}}-\delta_{ij}\right)\otimes\frac{\partial\beta'}{\partial\sigma_{kl}}}{(\beta'^2+3)\sqrt{\beta'^2+3}}$$

$$\frac{\partial\beta'}{\partial\sigma_{kl}}=\frac{\partial\beta'}{\partial s_{mn}}\frac{\partial s_{mn}}{\partial\sigma_{kl}}+\frac{\partial\beta'}{\partial I_1}\frac{\partial I_1}{\partial\sigma_{kl}}$$

$$\frac{\partial\beta'}{\partial s_{mn}}=\frac{-6\sqrt{6}}{(3-\sin\psi)^2}\frac{\partial\sin\psi}{\partial s_{mn}}\quad\text{and}\quad\frac{\partial\beta'}{\partial I_1}=\frac{-6\sqrt{6}}{(3-\sin\psi)^2}\frac{\partial\sin\psi}{\partial I_1}$$

Distinction of the statements between the behavior pre-peak or viscoplastic and post-peak

Form of derivatives in pre-peak or viscoplastic

$$\begin{aligned}\frac{\partial\sin\psi}{\partial s_{mn}} &= \frac{\partial\sin\psi}{\partial\sigma_{max}}\frac{\partial\sigma_{max}}{\partial s_{mn}}+\frac{\partial\sin\psi}{\partial\sigma_{\lim}}\frac{\partial\sigma_{\lim}}{\partial s_{mn}}\\ &= \mu_{0,v}\left(\frac{(1+\xi_{0,v})\sigma_{\lim}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\frac{\partial\sigma_{max}}{\partial s_{mn}}-\frac{(1+\xi_{0,v})\sigma_{\max}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\frac{\partial\sigma_{\lim}}{\partial s_{mn}}\right)\end{aligned}$$

and

$$\frac{\partial \sin\psi}{\partial I_1} = \frac{\partial \sin\psi}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial I_1}+\frac{\partial \sin\psi}{\partial \sigma_{\lim}}\frac{\partial \sigma_{\lim}}{\partial I_1}$$
$$= \frac{\mu_{0,v}}{3}\left(\frac{(1+\xi_{0,v})\sigma_{\lim}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}-\frac{(1+m_{v,\max})(1+\xi_{0,v})\sigma_{\max}}{(\xi_{0,v}\sigma_{max}+\sigma_{\lim})^2}\right)$$

$$\frac{\partial \sigma_{max}}{\partial s_{mn}}=\frac{1}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}+\left(\frac{3}{2}+\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{max}}{\partial I_1}=\frac{1}{3}$$

$$\frac{\partial \sigma_{\lim}}{\partial s_{mn}}=\frac{(1+m_{v,\max})}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}-\left(\frac{3}{2}-\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{\lim}}{\partial I_1}=\frac{(1+m_{v,\max})}{3}$$

Form of derivatives in post-peak for the plastic model of dilatancy

$$\frac{\partial \sin\psi}{\partial s_{mn}}=\frac{\partial \sin\psi}{\partial \alpha}\left(\frac{\partial \alpha}{\partial \sigma_{min}}\frac{\partial \sigma_{min}}{\partial s_{mn}}+\frac{\partial \alpha}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial s_{mn}}\right)$$

and

$$\frac{\partial \sin\psi}{\partial I_1}=\frac{\partial \sin\psi}{\partial \alpha}\left(\frac{\partial \alpha}{\partial \sigma_{min}}\frac{\partial \sigma_{min}}{\partial I_1}+\frac{\partial \alpha}{\partial \sigma_{max}}\frac{\partial \sigma_{max}}{\partial I_1}\right)$$

$$\frac{\partial \sin\psi}{\partial s_{mn}}=\mu_1\frac{(1+\xi_1)\alpha_{res}}{(\xi_1\alpha+\alpha_{res})^2}\left(-\frac{\sigma_{max}+\tilde{\sigma}}{(\sigma_{min}+\tilde{\sigma})^2}\frac{\partial \sigma_{min}}{\partial s_{mn}}+\frac{1}{\sigma_{min}+\tilde{\sigma}}\frac{\partial \sigma_{max}}{\partial s_{mn}}\right)$$

and

$$\frac{\partial \sin\psi}{\partial I_1}=\frac{\mu_1\alpha_{res}(1+\xi_1)(\sigma_{min}-\sigma_{max})}{3(\xi_1\alpha+\alpha_{res})^2(\sigma_{min}+\tilde{\sigma})^2}$$

$$\frac{\partial \sigma_{min}}{\partial s_{mn}}=\frac{1}{\sqrt{6}}\left[\frac{s_{II}}{H_0^c-H_0^e}\frac{\partial H(\theta)}{\partial s_{mn}}-\left(\frac{3}{2}-\frac{2H(\theta)-(H_0^c+H_0^e)}{2(H_0^c-H_0^e)}\right)\frac{s_{mn}}{s_{II}}\right] \text{ and } \frac{\partial \sigma_{min}}{\partial I_1}=\frac{1}{3}$$

the evaluating of the formula $\frac{\partial G_{ij}^{vp}}{\partial \sigma_{kl}}$ is identical to that of $\frac{\partial G_{ij}^{p}}{\partial \sigma_{kl}}$ to the distinctions close already specified above.

One approaches now the evaluating of the term $\frac{\partial\left(\langle \phi(f^{vp})\rangle^+\right)}{\partial \sigma_{mn}}$ with $\Phi(f^{vp})=A_v\left(\frac{f^{vp}}{Pa}\right)^{n_v}$.

One obtains then:

$$\frac{\partial\left(\left\langle\phi\left(f^{vp}\right)\right\rangle^{+}\right)}{\partial\sigma_{mn}}=\frac{\partial f^{vp}}{\partial\sigma_{mn}}\cdot\frac{A_v n_v}{P_{\text{atm}}}\left(\frac{f^{vp}}{P_{\text{atm}}}\right)^{n_v-1}$$

Evaluating of the terms relative to $\frac{d\left(R_1\right)_{ij}}{d\left(\Delta Y_3\right)}=\Delta\lambda\cdot C^e_{ijkl}:\frac{\partial G^p_{kl}}{\partial\xi^p}$:

$$\frac{\partial G^p_{ij}}{\partial\xi^p}=\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)-\left(\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right):n_{mn}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:\frac{\partial n_{mn}}{\partial\xi^p}\right)\otimes n_{ij}-\left(\frac{\partial f^p}{\partial\sigma_{mn}}:n_{mn}\right)\cdot\frac{\partial n_{ij}}{\partial\xi^p}$$

with

$$\begin{aligned}\frac{\partial}{\xi^p}\left(\frac{\partial f^p}{\partial\sigma_{ij}}\right)=&\frac{\partial a^d}{\partial\xi^p}\sigma_c H^c_0\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)^{a^d-1}\cdot\left(A^d\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d\delta_{ij}\right)\\&-a^d\sigma_c H^c_0\left\{\frac{\partial a^d}{\xi^p}\ln\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)\ldots\right.\\&\left.\ldots+\frac{\left(a^d-1\right)}{A^d s_{II}H(\theta)+B^d I_1+D^d}\left(\frac{\partial A^d}{\partial\xi^p}s_{II}H(\theta)+\frac{\partial B^d}{\partial\xi^p}I_1+\frac{\partial D^d}{\partial\xi^p}\right)\right\}\cdot\\&\left(A^d s_{II}H(\theta)+B^d I_1+D^d\right)^{a^d-1}\left[A^d\frac{\partial\left(s_{II}H(\theta)\right)}{\partial\sigma_{ij}}+B^d\delta_{ij}\right]\\&-a^d\sigma_c H^c_0\left[A^d s_{II}H(\theta)+B^d I_1+D^d\right]^{a^d-1}\cdot\left(\frac{\partial A^d}{\partial\xi^p}\frac{\partial\left(s_{II}H(\theta)\right)}{\sigma_{ij}}+\frac{\partial B^d}{\partial\xi^p}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial n_{kl}}{\xi^p}&=\frac{\partial n_{kl}}{\partial\beta'}\frac{\partial\beta'}{\partial\xi^p}\\&=\frac{\left(\frac{s_{kl}}{s_{II}}\left(\beta'^2+3\right)-2\beta'^2\frac{s_{kl}}{s_{II}}+2\beta'\delta_{kl}\right)}{\left(\beta'^2+3\right)^{3/2}}\frac{\partial\beta'}{\partial\xi^p}\\&=\frac{\left(\frac{s_{kl}}{s_{II}}\left(\beta'^2+3\right)-2\beta'^2\frac{s_{kl}}{s_{II}}+2\beta'\delta_{kl}\right)}{\left(\beta'^2+3\right)^{3/2}}\frac{-6\sqrt{6}\sin\psi}{\left(3-\sin\psi\right)^2}\frac{\partial\sin\psi}{\partial\xi^p}\end{aligned}$$

if the model of followed dilatancy corresponds to the pre-peak field, $\frac{\partial\sin\psi}{\partial\xi^p}=0$

if the model of followed dilatancy corresponds to the post-peak field, the following operations are necessary:

$$\begin{aligned}\frac{\partial \sin\psi}{\partial \xi^p} &= \frac{\partial \sin\psi}{\partial \alpha}\frac{\partial \alpha}{\partial \xi^p}\\ &= \mu_1 \frac{(1+\xi_1)\alpha_{\text{res}}}{(\xi_1\alpha+\alpha_{\text{res}})^2}\frac{\partial \alpha}{\partial \xi^p}\\ &= \mu_1 \frac{(1+\xi_1)\alpha_{\text{res}}}{(\xi_1\alpha+\alpha_{\text{res}})^2}\frac{(\sigma_{\min}-\sigma_{\max})}{(\sigma_{\min}+\tilde{\sigma})^2}\frac{\partial \tilde{\sigma}}{\partial \xi^p}\end{aligned}$$

$$\begin{aligned}\frac{\partial \tilde{\sigma}}{\partial \xi^p} &= \frac{\partial \tilde{\sigma}}{\partial \tilde{c}}\frac{\partial \tilde{c}}{\partial \xi^p}+\frac{\partial \tilde{\sigma}}{\partial \tan(\phi)}\frac{\partial \tan(\phi)}{\partial \xi^p}\\ &= \frac{1}{\tan(\phi)}\frac{\partial \tilde{c}}{\partial \xi^p}-\frac{\tilde{c}}{\tan(\phi)^2}\frac{\partial \tan(\phi)}{\partial \xi^p}\end{aligned}$$

with

$$\frac{\partial \tilde{c}}{\partial \xi^p}=\frac{\left(\sigma_c (s^d)^{a^d}\left[\frac{\partial a^d}{\partial \xi^p}\ln(s^d)+\frac{a^d}{s^d}\frac{\partial s^d}{\partial \xi^p}\right]\right)}{2\sqrt{1+a^d m^d (s^d)^{a^d-1}}}+\ldots$$

$$\ldots\frac{\sigma_c (s^d)^{a^d}\left(\frac{\partial a^d}{\partial \xi^p}m^d (s^d)^{a^d-1}+a^d\frac{\partial m^d}{\partial \xi^p}(s^d)^{a^d-1}+a^d m^d\left(\frac{\partial a^d}{\partial \xi^p}\ln s^d+\frac{a^d-1}{s^d}\frac{\partial s^d}{\partial \xi^p}\right)(s^d)^{a^d-1}\right)}{4\left(1+a^d m^d (s^d)^{a^d-1}\right)^{3/2}}$$

$$\begin{aligned}\frac{\partial \tan\phi}{\partial \xi^p} &= (1+\tan^2\phi)\frac{\partial \phi}{\partial \xi^p}\\ &= \frac{(1+\tan^2\phi)\left[\frac{\partial a^d}{\partial \xi^p}m^d (s^d)^{a^d-1}+a^d\frac{\partial m^d}{\partial \xi^p}(s^d)^{a^d-1}+a^d m^d (s^d)^{a^d-1}\left(\frac{\partial a^d}{\partial \xi^p}\ln s^d+\frac{a^d-1}{s^d}\frac{\partial s^d}{\partial \xi^p}\right)\right]}{\left(2+a^d m^d (s^d)^{a^d-1}\right)\sqrt{\left(1+a^d m^d (s^d)^{a^d-1}\right)}}\end{aligned}$$

Computation of the terms relative to $\frac{d(R_1)_{ij}}{d(\Delta Y_4)}=C^e_{ijkl}:\left[\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial \xi^{vp}}G^{vp}_{kl}+\langle\phi(f^{vp})\rangle^+\cdot\frac{\partial G^{vp}_{kl}}{\partial \xi^{vp}}\right]\cdot\Delta t$:

The evaluating of the formula $\frac{\partial G^{vp}_{kl}}{\partial \xi^{vp}}$ is identical in its form to preceding computation for $\frac{\partial G^p_{kl}}{\partial \xi^p}$.

$$\frac{\partial\langle\phi(f^{vp})\rangle^+}{\partial \xi^{vp}}=\frac{A_v n_v}{P_{\text{atm}}}\left(\frac{f^{vp}}{P_{\text{atm}}}\right)^{n_v-1}\frac{\partial f^{vp}}{\partial \xi^{vp}}$$

the evaluating of the formula $\frac{\partial f^{vp}}{\partial \xi^{vp}}$ identical in its form at the end is previously calculated $\frac{\partial f^p}{\partial \xi^p}$.

Evaluating of the terms relative to $\frac{d\left(R_3\right)}{d\left(\Delta Y_1\right)_{ij}}$ and $\frac{d\left(R_4\right)}{d\left(\Delta Y_1\right)_{ij}}$

$$\begin{aligned}\frac{\partial \tilde{G}_{II}^p}{\partial \sigma_{ij}} &= \frac{\partial \tilde{G}_{II}^p}{\partial \tilde{G}_{kl}^p}\frac{\partial \tilde{G}_{kl}^p}{\partial G_{mn}^p}\frac{\partial G_{mn}^p}{\partial \sigma_{ij}}\\ &= \frac{\tilde{G}_{kl}^p}{\tilde{G}_{II}^p}\left(\delta_{mk}\delta_{nl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)\frac{\partial G_{mn}^p}{\partial \sigma_{ij}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{G}_{II}^{vp}}{\partial \sigma_{ij}} &= \frac{\partial \tilde{G}_{II}^{vp}}{\partial \tilde{G}_{kl}^{vp}}\frac{\partial \tilde{G}_{kl}^{vp}}{\partial G_{mn}^{vp}}\frac{\partial G_{mn}^{vp}}{\partial \sigma_{ij}}\\ &= \frac{\tilde{G}_{kl}^{vp}}{\tilde{G}_{II}^{vp}}\left(\delta_{mk}\delta_{nl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)\frac{\partial G_{mn}^{vp}}{\partial \sigma_{ij}}\end{aligned}$$

Evaluating of the terms relative to $\frac{d\left(R_3\right)}{d\left(\Delta Y_3\right)}$

$$\frac{\partial \tilde{G}_{II}^p}{\partial \xi^p}=\frac{\tilde{G}_{kl}^p}{\tilde{G}_{II}^p}\left(\delta_{mk}\delta_{nl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)\frac{\partial G_{mn}^p}{\partial \xi^p}$$

Evaluating of the terms relative to $\frac{d\left(R_3\right)}{d\left(\Delta Y_4\right)}$ and $\frac{d\left(R_4\right)}{d\left(\Delta Y_4\right)}$

$$\frac{\partial \tilde{G}_{II}^{vp}}{\partial \xi^{vp}}=\frac{\tilde{G}_{kl}^{vp}}{\tilde{G}_{II}^{vp}}\left(\delta_{mk}\delta_{nl}-\frac{1}{3}\delta_{mn}\delta_{kl}\right)\frac{\partial G_{mn}^{vp}}{\partial \xi^{vp}}$$