# Constitutive law of the porous environments: model of Barcelona

**Summarized:**

The model of BARCELONE [bib1] described the structural mechanics behavior of the unsaturated soils coupled with the hydraulic behavior (this model must thus be used in an environment THHM [bib7]). In the cas particulier of a soil completely saturated with water, it is reduced to the model of CAM_CLAY modified, also implemented in *Code_Aster* [bib5]. It is particularly adapted under investigation behavior of clays.

## Contents

# 1 Notations

$\boldsymbol{\sigma}^T$ indicates the tensor of the total stresses in small disturbances, noted in the shape of the following vector:

$$\begin{vmatrix} \sigma^{T}{}_{11} \\ \sigma^{T}{}_{22} \\ \sigma^{T}{}_{33} \\ \sqrt{2}\,\sigma^{T}{}_{12} \\ \sqrt{2}\,\sigma^{T}{}_{23} \\ \sqrt{2}\,\sigma^{T}{}_{31} \end{vmatrix}$$

The behavior is described in a space of stresses to two variables:

$\boldsymbol{\sigma}=\boldsymbol{\sigma}^T+p_{gz}\boldsymbol{I}$ and $p_c=p_{gz}-p_{lq}$ ,

with $p_{lq}, p_{gz}, p_c$ respectively pressure of fluid, gas pressure, capillary pressure (or suction)

One notes:

$\boldsymbol{I}$ the tensor unit of order 2 whose indicielle notation is $\delta_{ij}$

$\boldsymbol{I}_4$ the tensor unit of order 4 whose indicielle notation $\delta_{ijkl}$

| | |
|---|---|
| $\underline{\underline{\boldsymbol{P}=-\frac{1}{3}tr(\boldsymbol{\sigma})}}$ | is forced of deviative |
| $\boldsymbol{s}=\boldsymbol{\sigma}+\boldsymbol{PI}$ | containment of the stresses |
| $I_2=\frac{1}{2}tr(\boldsymbol{s}.\boldsymbol{s})$ | second invariant of the stresses |
| $Q=\sigma_{eq}=\sqrt{3I_2}$ | equivalent stress |
| $\varepsilon=\frac{1}{2}(\nabla u+\nabla^T u$ | total deflection |
| $\varepsilon=\varepsilon_e+\varepsilon_p+\varepsilon_{th}$ | partition of the strains (elastic, plastic, thermal) |
| $\underline{\underline{\varepsilon_v=-tr(\varepsilon)+3\alpha(T-T_0)}}$ | total deflection voluminal |
| $\varepsilon_v^p=-tr(\varepsilon^p)$ | voluminal plastic strain |
| $\tilde{\varepsilon}=\varepsilon+\frac{1}{3}\varepsilon_v I$ | deviator of the strains |
| $\tilde{\varepsilon}^e=\tilde{\varepsilon}-\tilde{\varepsilon}^p$ | elastic strain deviatoric |
| $\tilde{\varepsilon}^p=\varepsilon^p+\frac{1}{3}\varepsilon_v^p I$ | deviatoric plastic strain |

$\varepsilon_{eq}^{e}=\sqrt{\frac{2}{3}tr\left(\tilde{\varepsilon}^{e}.\tilde{\varepsilon}^{e}\right)}$ elastic strain equivalent

$\varepsilon_{eq}^{p}=\sqrt{\frac{2}{3}tr\left(\tilde{\varepsilon}^{p}.\tilde{\varepsilon}^{p}\right)}$ equivalent plastic strain

$e$ index of the vacuums of the material (ratio of the volume of the pores on the volume of the solid matter constituents)

$e_0$ initial index of the vacuums

$\varphi$ porosity (ratio of the volume of the vacuums on total volume (pores plus grains))

$\varphi_{lq}, \varphi_{lq}^{e}, \varphi_{lq}^{p}$ content of fluid total, elastic and plastic

$\kappa$ coefficient of swelling (elastic slope in a hydrostatic compression test)

$\kappa_s$ elastic coefficient of stiffness in a test of variation of suction

$k_0=\frac{(1+e_0)}{\kappa}$

$k_{0s}=\frac{(1+e_0)}{\kappa_s}$

$\lambda(p_c)$ coefficient of compressibility (plastic slope in a hydrostatic compression test)

$\lambda^{*}$ coefficient of compressibility in conditions of saturation

$\lambda_s$ coefficient of compressibility plastic in a test of variation of suction

$k=\frac{(1+e_0)}{(\lambda-\kappa)}$

$k_s=\frac{(1+e_0)}{(\lambda_s-\kappa_s)}$

$M$ slope of the right of critical condition

$\alpha$ coefficient of correction of normality of yielding

$P_{cons}(p_c)$ pressure of critical

$P_{cr}(p_c)$ consolidation pressure, local variable of the model, equalizes with half of the pressure of critical

$P_{cr}$ consolidation pressure in conditions of saturation

$P_s$ cohesion (hydrostatic tension limits to suction given)

$P_0$ confining pressure of reference generally equal to the atmospheric pressure $P_a$

$k_c$ slope of cohesion according to suction

$\beta$ parameter controlling the increase in $\lambda(p_c)$ with $p_c$

$r$ parameter defining the peak of $\lambda(p_c)$ with $p_c$

$\mu$ elastic coefficient of shears (coefficient of Lamé)

$f$ surfaces of load in space $(P, Q)$
$f_2$ surfaces of load in $p_c$
$p_{c0}$ threshold of irreversibility of plastic
$\Lambda$ suction multiplier
$S_{lq}$ water saturation, $S_{lq}=\frac{\varphi_{lq}}{\varphi}$
$\varepsilon_{vp}^p$ voluminal plastic strain due to a loading in hydrostatic pressure
$\varepsilon_{vs}^p$ voluminal plastic strain due to a loading in suction
$\tilde{\varepsilon}_p^p$ deviatoric plastic strain due to a loading in hydrostatic pressure
$b$ coefficient of Biot

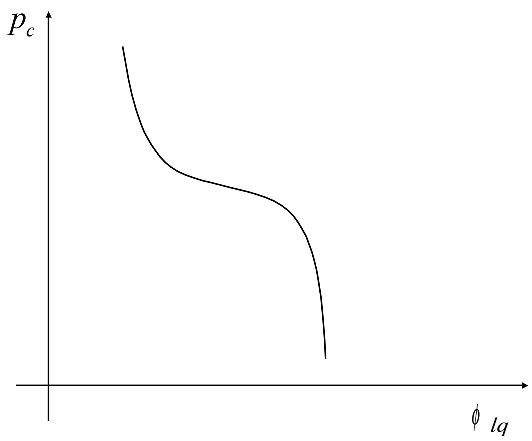
# 2 Introduction

the concepts of plasticity used for the water-logged soils are extended on the unsaturated ground. The model original of Barcelona is described using the variables $\boldsymbol{\sigma}, p_c$ , which distinguishes it from the models of mechanics coupled to a thermohydraulic behavior which are described using one only effective stress (forced of Bishop). One can notice that this model is rewritten in a frame poroplastic with the introduction of an additional poroplastic variable which is the water content [bib2], making it possible to collect the phenomena of hystereses which appear on the cycles of drying-damping. This phenomenon is not taken into account in the model original here exposed.

## 2.1 Phenomenology of the behavior of the soils unsaturated

### 2.1.1 Curve with Water retention

In addition to the principal common mechanical aspects with the water-logged soils [bib3], the porous environments comprising of the liquid and gas phases (soils unsaturated with water) have as a specific characteristic to be very sensitive to the phenomena of capillarity. The latter correspond to the localization of fluid meniscuses (increasingly small as soil désature) in which the water pressure is weaker than the air pressure (and all the more weak as the meniscus is small and thus désaturé soil). One thus sees appearing the notion of pressure capillary or suction $p_c=(p_{gz}-p_{lq})$ . While drying, a soil unsaturated has a water content $\varphi_{lq}$ lower what corresponds to a higher suction. The correspondence between these two quantities is the curve of Water retention (cf [Figure 2.1.1-a]). This one is obtained by drying of a soil initially saturated (suction is then null) and damping from the dry state.




**Appear 2.1.1-a: Curve of Water retention**

### 2.1.2 Extension of the definition of the effective stresses on the ground unsaturated

the structural mechanics behavior with the unsaturated soils is primarily observed in laboratory using devices with controlled suction (oedometers and triaxial). The modelization of this structural mechanics behavior was initially tried by extending the notion of effective stress to the unsaturated mediums. This one is a function of the total stress and intersticielle pressure: $\boldsymbol{\sigma}'=f(\boldsymbol{\sigma}^T, p_{lq})$ . In the saturated case, there is simply additivity of the pressure and the stress: $\boldsymbol{\sigma}'=\boldsymbol{\sigma}^T-p_{lq}\boldsymbol{I}$ because the pressure of water acts in the same way in water and solid in all the directions. The widening of this notion in the mediums unsaturated in the years 1950 (taking account of the pressures of the two liquid phases) brought the following form of the effective stress: $\boldsymbol{\sigma}'=(\boldsymbol{\sigma}^T-p_{gz}\boldsymbol{I})+\boldsymbol{g}\ (\boldsymbol{p_c})$ there remained the stress suggested in the form:

$$d\sigma'=d\ (\sigma^T-p_{gz})+bS_{lq}\ dp_c$$

where $S_{lq}$ is the degree of saturation out of water and B the coefficient of Biot [bib7].
As of the years 1960, the experimental observation clarifies certain limitations of the notion of effective stress extended on the unsaturated ground. In particular, the test of collapse to the oedometer puts at fault the stress of Bishop: This test consists in consolidating a sample unsaturated by maintaining suction constant, then to moisten again it with constant loading. A collapse of the soil then is observed. If the consolidation is continued, the curve corresponds to a standard test with of saturated. However, if one refers to the effective stress, this one decreases during the remoistening (since suction $p_c=(p_{gz}-p_{lq})$ is cancelled) and as it is supposed to control the strain, there should be swelling what is contradictory with the experimental observation. Most mechanics of the soils agree now on impossibility of completely describing the behavior of the soils unsaturated using one only stress and note the need for using two independent variables (stress and suction).

# 3 Description of the original model of Barcelona

In this model, the curve of Water retention does not have hysteresis, and it is not modified by the mechanical strain as it is the case in the presentation made by Dangla and saddle-point. [bib2]. There exists nevertheless a threshold in capillary pressure $p_{c0}$ with beyond which unrecoverable deformations appear. In this paragraph one distinguishes a mechanical part which treat induced mechanical strains by a mechanical loading and a hydro-mechanical part which treats effect of suction on the mechanics before writing the equations of the complete behavior.

## 3.1 Behavior purely mechanical

the assumption is made that suction $p_c$ remains constant during the mechanical transformation. The strains resulting from the variation of the stress are subscripted $_p$ .
One examines the behavior, under successively spherical and deviatoric loading, this behavior being considered isotropic.

### 3.1.1 Spherical loading

#### 3.1.1.1 Elasticity

the mechanical state of a soil unsaturated under hydrostatic request is determined by tests oedometric with controlled suction. As for the water-logged soils, the volume $v$ of the sample varies logarithmiquement with the load with a slope $\kappa$ of way reversible until a pressure of consolidation $P_{cons}(p_c)$ . One will choose $\kappa$ independent of $p_c$ , the experiment showing a weak dependence of the elastic slope with respect to $p_c$ .

The elastic component of the voluminal strain varies then like:

$$\dot{\varepsilon}^e_{vp}=\frac{\kappa}{1+e_0}\frac{\dot{P}}{P}\ \text{si}\ \ P<P_{cons}(p_c)$$ éq 3.1.1.1 -1

the preceding statement is in fact derived from a test oedometric with constant suction where one measures the variation of the index of the vacuums according to the loading, from where the following elastic model:

$$P=P_0\exp\left[k_0\left(\varepsilon_{vp}-\varepsilon^p_{vp}\right)\right]$$ éq 3.1.1.1 - 2

with $k_0=\frac{(1+e_0)}{\kappa}$, where $P_0$ is the value of reference corresponding to $\varepsilon_{vp}^e=0$ and $e=e_0$, initial index of the vacuums.

#### 3.1.1.2 Plasticity

Beyond the pressure of consolidation, the behavior of the soil is plastic and the slope $\lambda(p_c)$ is dependant on suction (cf [3.1.1.2 Figure - has]), this dependence being estimated way semi - empirical following:

$$\lambda(p_c)=\lambda(0)\left[(1-r)\exp(-\beta p_c)+r\right]$$

where $r=\frac{\lambda(p_c\to\infty)}{\lambda(0)}$ is a constant connected to the maximum of the stiffness of the soil and $\beta$ a parameter which controls the evolution of the stiffness according to suction.

Voluminal strain rate is then: $\dot{\varepsilon}_{vp}=\frac{\lambda(p_c)}{1+e_0}\frac{\dot{P}}{P}$ if $P>P_{cons}$,

from where the plastic component : $\dot{\varepsilon}_{vp}^p=\frac{(\lambda(p_c)-\kappa)}{1+e_0}\frac{\dot{P}}{P}$ .

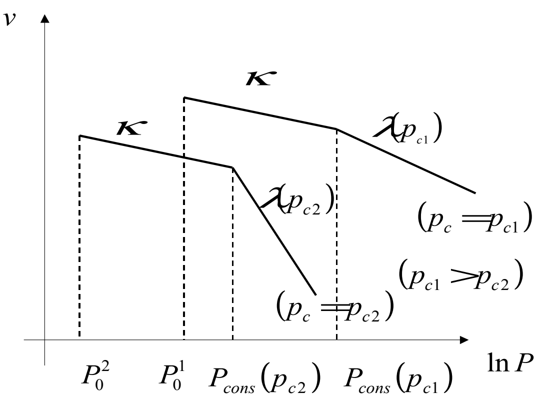
The statement of $P$ is thus written:

$$P=P_0\exp\left[k\left(\varepsilon_{vp}^p\right)\right] \quad \text{éq 3.1.1.2 - 1}$$

with $k=\frac{(1+e_0)}{\lambda\text{-}\kappa}$

**Note::**

*The two statements [éq 3.1.1.1 - 2] and [éq 3.1.1.2 - 1] are similar to those of the Camwood model - Clay [bib5] with the parameter $\lambda$ (or $k$ ) depend on the capillary pressure. The compressibility of the soil decreases with suction.*




**Appear 3.1.1.2 - has: Variation of specific volume under loading oedometric**

## 3.1.2 triaxial Loading

### 3.1.2.1 Elasticity

the elastic component of the deviatoric strain is proportional to the loading:

$$\tilde{\varepsilon}^e = \frac{s}{2\mu} \qquad \text{éq 3.1.2.1 - 1}$$

$\mu$ is independent of suction.

### 3.1.2.2 Plasticity

In a triaxial compression test of revolution, one introduces the shearing stress $\boldsymbol{Q}=\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_3$ (one will be able to extend the formulation who follows to 3D by means of the norm within the meaning of von Mises of the stress). When suction becomes null (saturated medium), the model is supposed to be reduced to modified the Cam_Clay model [bib5]: the threshold of plasticity is then an ellipse of center $(P^*_{cr},0)$ which cuts the axis of the hydrostatic stresses into zero and in a value of pressure of consolidation $P^*_{cons}=2\mathrm{P}^*_{cr}$ . The surface of load associated with a non-zero $p_c$ suction is also an ellipse of center $(P_{cr}(p_c)-\frac{P_s}{2},0)$ (cf [3.1.2.2 Figure - has]) which cuts the hydrostatic axis in $P_{cons}(p_c)=2\mathrm{P}_{cr}(p_c)$ and $-P_s$ , $P_s$ representing a cohesion varying linearly with suction: $P_s=k_c p_c$ . Line representing the critical conditions (voluminal variation null) the same slope preserves $M$ as that in saturated but shifted condition $P_s$ . The equation of the surface of load in the diagram $(P,Q)$ for $p_c$ data is written:

$$Q^2-M^2(P+P_s)(2\mathrm{P}_{cr}-P)=0 \qquad \text{éq 3.1.2.2 - 1}$$

yielding in the plane $(P,Q)$ thus with $p_c$ constant is not associated with the surface of load. If it were the case, one would have:

$$\dot{\varepsilon}_{vp}^{p}=\dot{\Lambda}\frac{\partial f_1}{\partial P} \qquad \dot{\tilde{\varepsilon}}^{p}=\dot{\Lambda}\frac{\partial f_1}{\partial s}$$

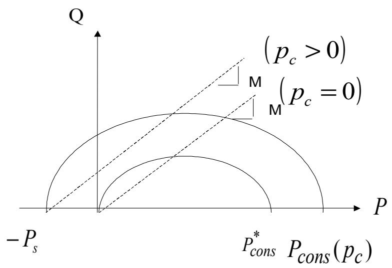
and the following ratio:

$$\frac{\dot{\tilde{\varepsilon}}_{eq^p}}{\dot{\varepsilon}_{vp}^{p}}=\frac{2Q}{M^2(2P+P_s-2P_{cr})}\ , \qquad \text{éq 3.1.2.2 - 2}$$

similar to the ratio obtained in the model from Camwood-Clay (with $P_s=0$ ). In fact in this model, one introduces a parameter of correction $\alpha$ which destroys the character of normality, so that: $\frac{\dot{\tilde{\varepsilon}}_{eq^p}}{\dot{\varepsilon}_{vp}^{p}}=\frac{2Q\alpha}{M^2(2P+P_s-2P_{cr})}$ . $\alpha$ is given by the authors of the model [bib1] as being:

$$\alpha=\frac{M(M-9)(M-3)}{9(6-M)}\left\{\frac{1}{\left[1-\frac{\kappa}{\lambda(0)}\right]}\right\} \qquad \text{éq 3.1.2.2 - 3}$$

This corrector allows to better take into account the experimental results, and in particular to better estimate the coefficient of thorough grounds.




**Appear 3.1.2.2 - has: Criterion in hydro-mechanical $(P,Q)$**

## 3.2 space Coupling or effect of suction on the mechanics

the variations of suction (with constant load) involve strains (those will be then subscripted by $_s$ ) reversible when $p_c<p_{c0}$ and irreversible when suction exceeds the threshold $p_{c0}$ .

### 3.2.1 Reversible part

the tests oedometric with constant stress and controlled suction give us the variation of the index of the vacuums according to suction [Figure 3.2.1-a] reversible below threshold in suction:

$e - e_0 = -\kappa_s \, Ln \dfrac{p_c}{p_{atm}}$ si $p_c < p_{c0}$ ,

with $\kappa_s$ independent of the state of containment.

Strain being able to be written: $\varepsilon_v - \varepsilon_{v0} = -\dfrac{e - e_0}{1 + e_0}$ , one a:

$$\dot{\varepsilon}^e_{vs} = \frac{\kappa_s}{1+e_0} \frac{\dot{p}_c}{p_c + p_{atm}} = \frac{1}{k_{0s}} \frac{\dot{p}_c}{\left(p_c + p_{atm}\right)} \qquad \text{éq 3.2.1-1}$$

the evolution of suction is written then:

$$p_c = p_{atm} \exp\left(k_{0s}\left(\varepsilon^e_{vs} - \varepsilon^e_{v0}\right)\right) \text{ , with } k_{0s} = \frac{1+e_0}{\kappa_s} \qquad \text{éq 3.2.1-2}$$

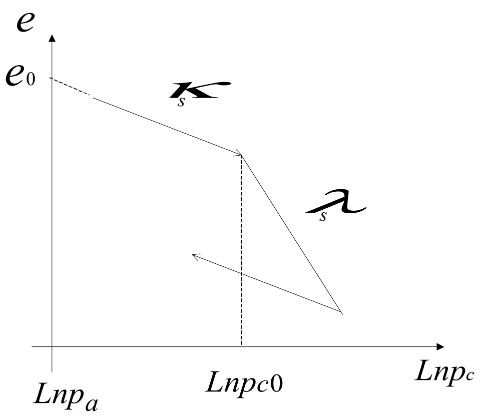



**Figure 3.2.1-a: Evolution of irreversible**

### 3.2.2 suction Left

Beyond the threshold $p_{c_0}$ , of the unrecoverable deformations appear, the slope in the test oedometric becoming $\lambda_s$. This slope can actually depend on the hydrostatic stress applied to the sample, but she is considered constant in the model original of Barcelona. As one can note it on [Figure 3.2.2], the pressure of consolidation increases with suction. [Figure the 3.2.2 (A)]) watch two compression tests in condition saturated $\left(p_c = 0\right)$ and unsaturated $\left(p_c > 0\right)$ . A relation between $P^*_{cons}$ (point 3) value of the preconsolidation with of saturated and $P_{cons}$ (not 2) the pressure with preconsolidation with unsaturated is established by comparing the specific volumes obtained on paths according to the items 1,2,3 [Figure 3.2.2 (a)] which describe a discharge of $P_{con.}$ with $P^*_{cons}$ constant suction followed by a remoistening of a value $p_c$ to 0 with constant pressure $P^*_{cons}$ , from where the following equation:

$$v_1 + \Delta v_{pression} + \Delta v_{succion} = v_3$$

The assumption is made that the reduction of suction $2 \rightarrow 3$ is accompanied by recoverable deformations. The elastic relation is the following one: $dv = -\kappa_s \frac{dp_c}{(p_c + p_{atm})}$ , where $p_{atm}$ is the atmospheric pressure. One writes for item 1 and 3 the statement of volume as follows:

$$v = N(P_0) - \lambda(p_c)\ln\frac{P}{P_0}$$

where $P_0$ is a pressure of reference corresponding to an initial volume $N(P_0)$ . One combines this statement and the elastic relations:

$$N(P_0) - \lambda(p_c)\ln\frac{P_{cons}}{P_0} + \kappa\ln\frac{P_{cons}}{P^*_{cons}} + \kappa_s\ln\frac{p_c + p_{atm}}{p_{atm}} = N(0) - \lambda(0)\ln\frac{P^*_{cons}}{P_0}$$

By eliminating initial volumes by the elastic relation:

$$\Delta v(P_0)\big|_{p_c}^{0} = N(0) - N(P_0) = \kappa_s \ln\frac{p_c + p_{atm}}{p_{atm}}$$

one then determines the following evolution of the threshold of consolidation in unsaturated condition:

$$\left(\frac{P_{cons}}{P_0}\right) = \left(\frac{P^*_{cons}}{P_0}\right)^{\left[\frac{\lambda(0)-\kappa}{\lambda(p_c)-\kappa}\right]}$$

Like $P_{cons} = 2\mathrm{P}_{cr}$,

One finds:

$$P_{cr} = \frac{P_0}{2}\left(\frac{2\mathrm{P}^*_{cr}}{P_0}\right)^{\left[\frac{\lambda(0)-\kappa}{\lambda(p_c)-\kappa}\right]} \qquad \text{éq 3.2.2-}$$

It [Figure 3.2.2] visualizes path 1-2-3 in the plane $(P, p_c)$ .

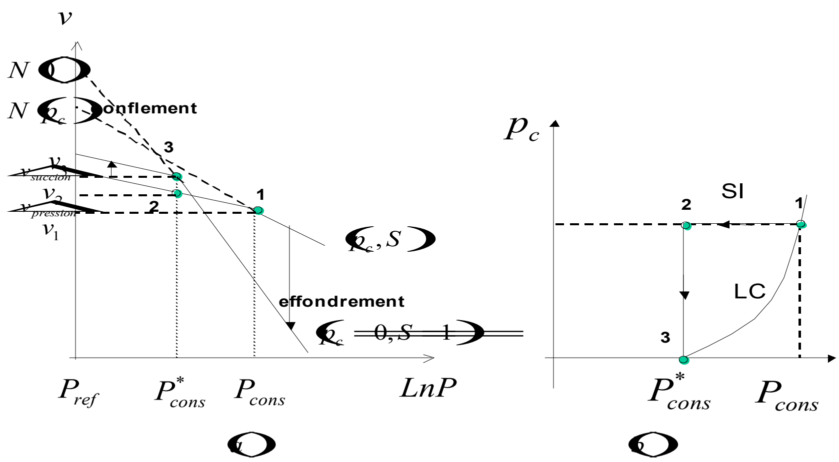



**Figure 3.2.2: (A) Curves of compression for water-logged soils and NON-saturated (b) criterion in the diagram $(P, p_c)$**

the total component with the voluminal strain due to the evolution with suction is:

$$\dot{\varepsilon}_{vs}=\frac{\lambda_s}{(1+e_0)}\frac{\dot{p}_c}{p_c+p_{atm}} \text{ si } p_c>p_{atm} \quad \text{éq 3.2.2-}$$

from where the plastic component which is written:

$$\dot{\varepsilon}^p_{vs}=\frac{(\lambda_s-\kappa_s)}{(1+e_0)}\frac{\dot{p}_c}{p_c+p_{atm}}=\frac{1}{k_s}\frac{\dot{p}_c}{p_c+p_{atm}} \quad \text{éq 3.3.2-}$$

**Note::**

*The variation of suction does not generate deviatoric strains.*

## 3.3 Behavior complete (mechanical and hydrous loading)

### 3.3.1 Behavior reversible

Under spherical loading, the evolution of the total voluminal elastic component is thus written:

$$\dot{\varepsilon}^e_v=\dot{\varepsilon}^e_{vp}+\dot{\varepsilon}^e_{vs}=\frac{1}{k_0}\frac{\dot{P}}{P}+\frac{1}{k_{0s}}\frac{\dot{p}_c}{(p_c+p_{atm})} \quad \text{éq the 3.3.1-1}$$

evolutions of the parts hydrostatics and deviatoric of the stress $\sigma$ are thus written:

$$\frac{\dot{P}}{P}=k_0\dot{\varepsilon}^e_v-\frac{k_0}{k_{0s}}\frac{\dot{p}_c}{(p_c+p_{atm})}, \quad \text{éq 3.3.1-2}$$

$$\dot{s}_{ij}=2\mu\dot{\tilde{\varepsilon}}^e_{ij}, \quad \text{éq 3.3.1-3}$$

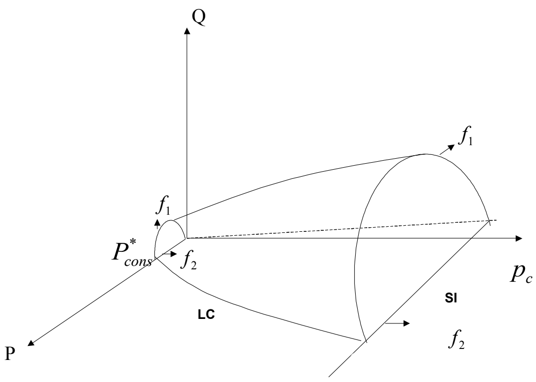
### 3.3.2 Thresholds of flow

the two thresholds of the reversible field are such as:

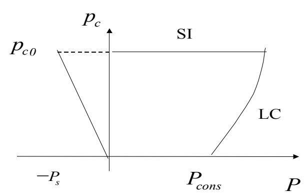
Mechanical criterion : $f_1(P, Q, P_{cr}(p_c), p_c) = Q^2 + M^2 (P + k_c p_c)(P - 2P_{cr}(p_c)) \leq 0$ éq 3.3.2-1

hydrous Criterion: $f_2(p_c, p_{c_0}) = p_c - p_{c_0} \leq 0$ éq 3.3.2-2

the three-dimensional field of reversibility in space $(P, Q, p_c)$ is represented on [Figure 3.3.2 - has]. These two criteria are reduced in the plane $(P, p_c)$ to curves called LC (loading collapse) and IF (suction increase) (cf [Figure 3.3.2-b]).




**Appear 3.3.2-a: Surface of load in space** $(P, Q, p_c)$




**Appears 3.3.2-b: Surface of load in space** $(P, p_c)$

### 3.3.3 Models of flow

yielding is controlled by the two criteria in pressure and suction:

$$\dot{\varepsilon}_v^p = \dot{\Lambda}\frac{\partial f_1}{\partial P} = \dot{\Lambda} M^2 \left( 2P - 2P_{cr} + k_c\, p_c \right) \quad \text{éq 3.3.3-1}$$

$$\dot{\tilde{\varepsilon}}^p = \alpha\, \dot{\Lambda}\frac{\partial f_1}{\partial s} = \alpha\, \dot{\Lambda}\frac{\partial f_1}{\partial Q}\frac{\partial Q}{\partial s} = 3\alpha\, \dot{\Lambda} s \quad \text{éq 3.3.3-2}$$

$$\text{or} \quad \dot{\varepsilon}_v^p = \frac{1}{k_s}\frac{\dot{p}_c}{p_c + p_{atm}} \;,\; \dot{\tilde{\varepsilon}}^p = 0 \quad \text{éq 3.3.3-3}$$

### 3.3.4 Models of hardening

the evolution of surfaces of load is controlled by the forces of hardening: $P_{cr}$ and $p_{c0}$.
The models of hardening of each surface are:

On $f_1$, $$\frac{\dot{P}_{cr}}{P_{cr}} = k\, \dot{\varepsilon}_v^p \quad \text{éq 3.3.4-1}$$

On $f_2$, $$\frac{\dot{p}_{c0}}{p_{c0} + p_{atm}} = k_s\, \dot{\varepsilon}_v^p \quad \text{éq 3.3.4-2}$$

### 3.3.5 Inventory of the configurations of mechanical and hydrous coupling

One examines the various configurations of loading in space $(P, p_c)$.

#### 3.3.5.1 Total reversibility

the loading represented by the point $M$ (cf [3.3.5.1 Figure - has]) is inside the field of reversibility: elasticity, and hydrous reversibility. That results in:

$f_1 < 0$, or ( $f_1 = 0,\ \dot{f}_1 < 0$), and $p_c < p_{c0}$, or ( $p_c = p_{c0}$ , $\dot{p}_c < 0$ ).

The relations expressing this reversibility are:

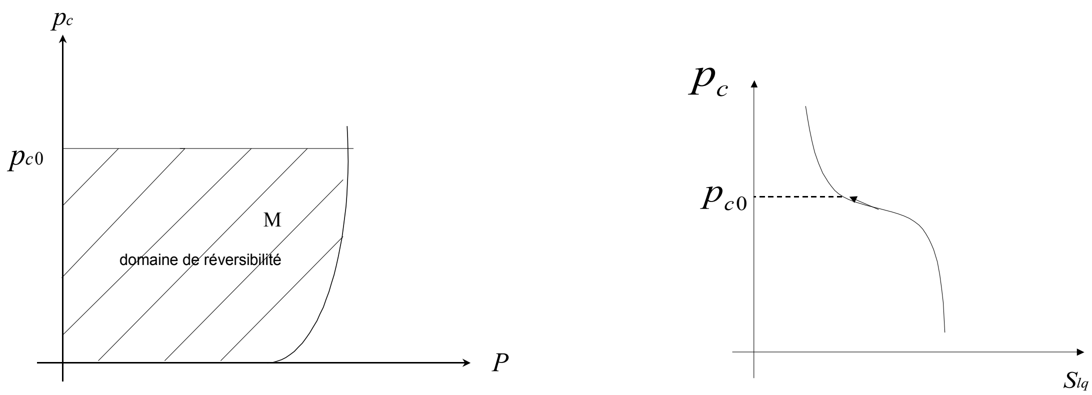
$$\frac{\dot{P}}{P} = k_0\, \dot{\varepsilon}_v - \frac{k_0}{k_{0s}}\frac{\dot{p}_c}{\left( p_c + p_{atm} \right)}$$

i.e.:

$$P = P_0 \frac{\exp\left( k_0 \left( \varepsilon_v - \varepsilon_{v0} \right) \right)}{\left( \dfrac{p_c + p_{atm}}{p_{atm}} \right)^{k_0 / k_{0s}}} \;, \quad \text{éq 3.3.5.1 - 1}$$

and

$$s=2\mu\,\tilde{\varepsilon}$$ éq 3.3.5.1 - 2




**3.3.5.1 Figure - has: Field of reversibility in the curved $(P, p_c)$ plan of Water retention**

### 3.3.5.2 Behavior elastoplastic

the point $M$ touches the criterion of mechanics alone (cf [3.3.5.2 Figure - has]):

$$f_1=0\ \ \dot{f}_1=0\text{, and } p_c<p_{c0}\ \ (\text{or } p_c=p_{c0} \text{ and } \dot{p}_c<0)$$

the elastic evolution is thus written:

$$\frac{\dot{P}}{P}=k_0\dot{\varepsilon}_v^e-\frac{k_0}{k_{0s}}\frac{\dot{p}_c}{(p_c+p_{atm})}\quad,$$

i.e.:

$$P=P_0\frac{\exp\left(k_0\left(\varepsilon_v^e-\varepsilon_{v0}^e\right)\right)}{\left(\frac{p_c+p_{atm}}{p_{atm}}\right)^{k_0/k_{0s}}}$$ éq 3.3.5.2 - 1

and

$$s=2\mu\,\tilde{\varepsilon}$$ éq 3.3.5.2 - 2

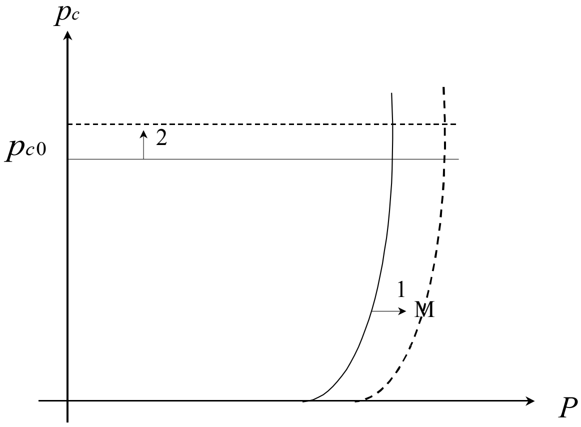
the evolution of the components of the plastic strain is:

$$\dot{\tilde{\varepsilon}}^p=3\alpha\Lambda s$$
$$\dot{\varepsilon}_v^p=\Lambda M^2\left[2P-2P_{cr}+k_c\,p_c\right]$$

The evolution of the mechanical threshold is written: $\dot{P}_{cr}=kP_{cr}\,\dot{\varepsilon}_{vp}^p=kP_{cr}\,\Lambda M^2\left[2P-2P_{cr}+k_c\,p_c\right]$ .

A specificity of the original model of Barcelona is the assumption that mechanical hardening is completely coupled with the hydrous hardening (cf [3.3.5.2 Figure - has]) from where the relation:

$$\frac{\dot{p}_{c0}}{p_{c0}+p_{atm}}=\frac{k_s}{k}\frac{\dot{P}_{cr}}{P_{cr}} \qquad \text{éq 3.3.5.2 - 3}$$




**3.3.5.2 Figure - has: Coupling of mechanical hardening to hydrous hardening**

### 3.3.5.3 Behavior hydrous generating unrecoverable deformations

the point M reached the threshold in suction (cf [3.3.5.2 Figure - has]):

$$p_c=p_{c0} \text{ and } \dot{p}_c>0$$

the structural mechanics behavior is elastic:

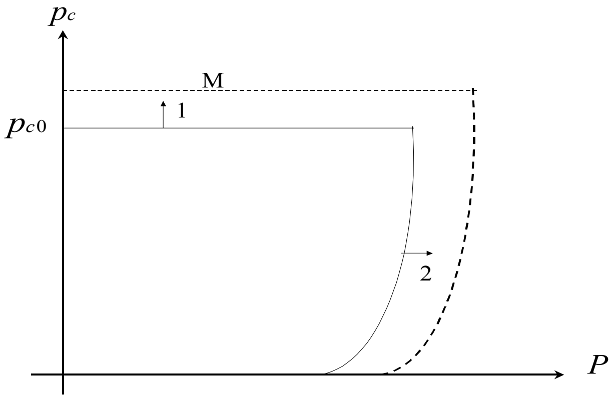
$$P=P_0\frac{\exp\left(k_0\left(\varepsilon_v^e-\varepsilon_{v0}^e\right)\right)}{\left(\frac{p_c+p_{atm}}{p_{atm}}\right)^{k_0/k_{0s}}} \quad , \quad \dot{s}=2\mu\,\dot{\tilde{\varepsilon}} \qquad \text{éq 3.3.5.3 - 1}$$

but as the mechanical threshold is coupled with that of suction, there is also mechanical hardening:

$$\frac{\dot{P}_{cr}}{P_{cr}}=\frac{k}{k_s}\frac{\dot{p}_{c0}}{p_{c0}+p_{atm}} \qquad \text{éq 3.3.5.3 - 2}$$

plastic strain rates are written:

$$\dot{\varepsilon}_v^p=\frac{1}{k_s}\frac{\dot{p}_c}{p_c+p_{atm}}$$
$$\dot{\tilde{\varepsilon}}^p=0$$




**Appear 3.3.5.3 - has: Coupling of hydrous hardening to mechanical hardening**

## 3.4 Data of the model of Barcelona

The model requires the following parameters:

1) Elastic parameters provided under key word ELAS :
The thermal coefficient of thermal expansion $\alpha$ , two elastic coefficients $E, \nu$ provided in data from which the coefficient of Lamé is calculated $\mu$.

2) Under key word CAM_CLAY :
   1) $P_0$ Initial hydrostatic pressure equal to the atmospheric pressure noted PA under key word CAM_CLAY
   2) instead of giving the initial index of the vacuums $e_0$ one gives the initial porosity which must be of value equal to that given under key word THM_INIT, noted PORO.
   3) Parameters associated with surface threshold LC (forced isotropic): $P_{cr}^*$ , equalizes with half of the pressure of preconsolidation $P_{cons}^*$ noted PRES_CRIT $\lambda^*$ , the coefficient of compressibility for a saturated state and $\kappa$ the elastic, noted coefficient of compressibility LAMBDA and KAPA.
   4) The critical slope $M$ ,

3) Under key word BARCELONE :
   1) $r$ and $\beta$ , coefficients allowing to calculate $\lambda(p_c)$ , noted R and BETA.
   2) parameters related to a variation of suction: $\lambda_s$, coefficient of compressibility related to a variation of suction in the plastic range, $\kappa_s$ coefficient associated with the change with suction in elastic domain, noted LABDAS and KAPAS.
   3) $k_c$ the parameter which controls the increase in cohesion with suction
   4) the initial threshold of suction $p_{c0}$, noted PC0_INIT
   5) $\alpha$ the coefficient of normality, noted ALPHAB.

Here a set of values of some of these parameters, resulting from [bib1]:

$\lambda(0)=0.2\,;\,\kappa=0.02\,;\,r=0.75\,;\,\beta=12.5\,MPa^{-1}\,;\,P_0=0.10\,MPa\,;$
$\lambda_s=0.08\,;\,\kappa_s=0.008\,;\,G=10\,MPa\,;\,M=1\,;\,k_c=0.6$

# 4 Numerical integration of the behavior models

## 4.1 Recall of the problem

the numerical integration of the model is similar to that carried out for the Camwood-Clay model [bib5], by operating a translation on the axis of the capillary pressures.
This model is obligatorily coupled with the hydraulic behavior, contrary to Camwood-Clay which can be used in a purely mechanical frame (one simulates a drained behavior then).
The model of Barcelona is thus usable only in the frame of behaviors THHM established in *Code_Aster* [bib7] and [bib8]. It will be more particularly employed with modelizations KIT_HHM and KIT_THHM (in this last case, there is not for the time of dependence of the mechanical characteristics specific to the model of Barcelona with the temperature).
The variables of entry of the model are $\Delta u$ or $\Delta\varepsilon$ and $\Delta P1$ $\Delta P2$ , $P1$ and $P2$ being equal in the quoted modelizations to $\Delta p_c$ $\Delta p_{gz}$ , $p_c$ and $p_{gz}$ that it is with hydrous modelizations LIQU_VAPE_GAZ or LIQU_GAZ.
The variables of output of the model are: $\sigma'$ $P_{cr}$ $p_{c0}$ $P_s$ .
The following notations are employed: $A^-, A, \Delta A$ respectively for the quantity evaluated at known time $t$ , with time $t+\Delta t$ and its increment. The equations are discretized in an implicit way, it is - with - to say expressed according to the unknown variables to time $t+\Delta t$ .

One will note:

$P_{cr}^-$ the quantity $P_{cr}^-(p_c^-)$,

$P_{cr}^-(p_c)$ the quantity $\frac{P_0}{2}\left(\frac{2P_{cr}^-}{P_0}\right)^{\left[\frac{\lambda(p_c^-)-\kappa}{\lambda(p_c)-\kappa}\right]}$ and

$P_{cr}(p_c)=P_{cr}^-(p_c)\exp\left(k\Delta\varepsilon_v^p\right)$

## 4.2 incremental Relations

the flow rules and the condition of consistency give the following flow relations:
If the threshold $f_1$ is reached, the voluminal increment of plastic strain is written:

$$\Delta\varepsilon_v^p=\frac{1}{k\left(P+k_c p_c\right)P_{cr}}\left[\frac{\left(2P-2P_{cr}+k_c p_c\right)}{2}\Delta P+\frac{Q}{M^2}\Delta Q-\frac{k_c}{2}(2P_{cr}-P)\,\Delta p_c\right] \quad \text{éq 4.2-1}$$

the increment of the norm of the equivalent plastic strain is then:

$$\Delta\varepsilon_{eqp}^p=\frac{\alpha}{kP_{cr}\left(P+k_c p_c\right)}\left[\frac{Q}{M^2}\Delta P+\frac{2Q^2}{M^4\left(2P-2P_{cr}+k_c p_c\right)}\Delta Q-\frac{k_c Q(2P_{cr}-P)}{M^2\left(2P-2P_{cr}+k_c p_c\right)}\Delta p_c\right] \quad \text{éq 4.2-2}$$

and the tensor deviatoric is written:

$$\Delta\tilde{\varepsilon}^p=\frac{3\alpha s}{M^2(2P-2P_{cr}+P_s)}\Delta\varepsilon_v^p \quad \text{éq 4.2-3}$$

If $f_2$ is reached, the voluminal increment of plastic strain is determined by:

$$\Delta\varepsilon_v^p=\frac{1}{k_s}Ln(\frac{p_{c0}+p_{atm}}{p_{c0}^-+p_{atm}})$$ éq 4.2-4

the plastic strain being purely voluminal ( $\Delta\tilde{\varepsilon}^p=0$ )

$\varepsilon_v^p$ will be the principal of the problem and determined unknown while solving $f_1(P^-,Q^-,P_{cr}^-(p_c),p_c,\Delta\varepsilon_v^p)=0$ , or $f_2(p_c,\Delta p_{c0})=0$ , the voluminal increment of plastic strain being obtained from $\Delta p_{c0}$. One from of then deduced the evolution from the stresses and the thresholds.

## 4.3 Computation of the stresses and the local variables

the elastic prediction of the deviatoric stress is written:

$$s^e=s^-+2\mu\Delta\tilde{\varepsilon}$$ éq 4.3-1

One chooses the elastic prediction $t+\Delta t$ in the following way:

$$P^e=P^-\frac{\exp(k_0\Delta\varepsilon_v)}{\left(\frac{p_c+_{atm}}{p_c^-+p_{atm}}\right)^{k_0/k_{0s}}}$$ **éq 4.3-2**

If $f_1<0$ and $f_2<0$, then $P=P^e, s=s^e, \Delta\varepsilon^p=0, P_{cr}=\frac{P_0}{2}\left(\frac{2P_{cr}^-}{P_0}\right)^{\left[\frac{\lambda(p_c^-)-\kappa}{\lambda(p_c)-\kappa}\right]}, \Delta p_{c0}=0$ ,

If not:

$$s=s^e-2\mu\Delta\tilde{\varepsilon}^p$$ éq 4.3-3

$$P=P^e\exp[-k_0\Delta\varepsilon_v^p]$$ éq 4.3-4

$$P_{cr}=\frac{P_0}{2}\left(\frac{2P_{cr}^-}{P_0}\right)^{\left[\frac{\lambda(p_c^-)-\kappa}{\lambda(p_c)-\kappa}\right]}\exp[k\Delta\varepsilon_v^p]$$ **éq 4.3-5**

$$(p_{c0}+p_{atm})=(p_{c0}^-+p_{atm})\exp[k_s\Delta\varepsilon_v^p]$$ éq 4.3-6

the principal unknown is thus $\Delta\varepsilon_v^p$.

If $f_1>0$, then

While replacing $\Delta\tilde{\varepsilon}^p$ by his statement according to $\Delta\varepsilon_v^p$ [éq 4.2-3] one obtains:

$$s=\frac{s^e}{1+\frac{6\alpha\mu\Delta\varepsilon_v^p}{M^2(2P-2P_{cr}+k_c p_c)}}$$ éq 4.3-7

and:

$$P=P^{e}\exp\left[-k_{0}\Delta\varepsilon_{v}^{p}\right]$$ éq 4.3-8

the unknown is given while solving $f_{1}(P,Q,P_{cr},p_{c})=f_{1}(P^{e},Q^{e},P_{cr}^{-}(p_{c}),p_{c},\Delta\varepsilon_{v}^{p})=0$ ,

I.e.:

$$Q^{e^{2}}=-M^{2}\left[1+\frac{6\alpha\mu\,\Delta\varepsilon_{v}^{p}}{M^{2}\left(2P-2P_{cr}+k_{c}p_{c}\right)}\right]^{2}\left(P+k_{c}p_{c}\right)\left(P-2P_{cr}\right) ,$$

or:

$$Q^{e^{2}}=-M^{2}\left[1+\frac{6\alpha\mu\,\Delta\varepsilon_{v}^{p}}{M^{2}\left(2P^{e}\exp\left(-k_{0}\Delta\varepsilon_{v}^{p}\right)-2P_{cr}^{-}(p_{c})\exp\left(k\Delta\varepsilon_{v}^{p}\right)+k_{c}p_{c}\right)}\right]^{2}\left[P^{e}\exp\left(-k_{0}\Delta\varepsilon_{v}^{p}\right)+k_{c}p_{c}\right]\left[P^{e}\exp\left(-k_{0}\Delta\varepsilon_{v}^{p}\right)-2P_{cr}^{-}(p_{c})\exp\left(-k\Delta\varepsilon_{v}^{p}\right)\right]$$ éq 4.3-9

If $f_{2}>0$ , then: $\Delta p_{c0}=\Delta p_{c}$ , the unknown is immediately given by:

$$\Delta\varepsilon_{v}^{p}=\frac{1}{k_{s}}Ln\left(\frac{p_{c0}+p_{atm}}{p^{-c0}+p_{atm}}\right) ,$$ éq 4.3-10

from where $s=s^{e}$ and $P=P^{e}\exp\left[-k_{0}\Delta\varepsilon_{v}^{p}\right]$ éq 4.3-11

One has moreover $P_{cr}=\frac{P_{0}}{2}\left(\frac{2P_{cr}^{-}}{P_{0}}\right)^{\left[\frac{\lambda(p_{c}^{-})-\kappa}{\lambda(p_{c})-\kappa}\right]}\exp\left[k\Delta\varepsilon_{v}^{p}\right]$ .éq 4.3-12

2

# 5 tangent Opérateur

If the option is: RIGI_MECA_TANG, option used at the time of the prediction, the tangent operator calculated in each Gauss point is known as of velocity:

$$\dot{\sigma}_{ij}=D_{ijkl}^{elp}\dot{\varepsilon}_{kl} ,$$

i.e. $D_{ijkl}^{elp}$ is calculated starting from the not discretized equations.

If the option is: FULL_MECA, option used when one and the reactualizes the tangent matrix by updating the stresses local variables:

$$d\sigma_{ij}=A_{ijkl}d\varepsilon_{kl}$$

In this case, $A_{ijkl}$ is calculated starting from the implicitly discretized equations.

The tangent operator of the generalized stresses is implemented in THHM under the name $D\Sigma\ DE$ and is partitionné in several blocks. The blocks concerned with the model are [DMECDE], [DMECP1] [bib8]. One calculates the contribution of the model to each one of these blocks for the tangent operator in elasticity, the operator of velocity and the coherent operator.

## 5.1 Nonlinear elastic tangent operator

the elastic relation of velocity of the model of Barcelona is written:

$$\dot{\sigma}_{ij}=-\dot{P}\delta_{ij}+\dot{s}_{ij}=k_0 Ptr\{\dot{\varepsilon}¿\,\delta_{ij}+2\mu\,\dot{\tilde{\varepsilon}}_{ij}+\frac{k_0}{k_{0s}}P\frac{\dot{p}_c}{p_c+p_{atm}}\delta_{ij}$$ éq 5.1-1

$$\dot{\sigma}_{ij}=(k_0 P-\frac{2}{3}\mu)tr\,\dot{\varepsilon}\,\delta_{ij}+2\mu\,\dot{\varepsilon}_{ij}+\frac{k_0}{k_{0s}}P\frac{\dot{p}_c}{p_c+p_{atm}}\delta_{ij}$$ éq 5.1-2

the tensor of the stresses used in the model of Barcelona (and the tests determining the data of the model) is function of the total stress and of the gas pressure and is written:

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^{T}+p_{gz}\boldsymbol{I}$$ éq 5.1-3

the tensor of the stresses of Bishop $\boldsymbol{\sigma}'$ used in *the Code_Aster* is such that: $\dot{\boldsymbol{\sigma}}_{T}=\dot{\boldsymbol{\sigma}}'+\dot{\sigma}_P\boldsymbol{I}$ with

$$\dot{\sigma}_P=-b\left(\dot{p}_{gz}-S_{lq}\dot{p}_c\right)$$ éq 5.1-4

From where the statement of the stress of Bishop according to the stress of the model of Barcelona:

$$\dot{\boldsymbol{\sigma}}'=\dot{\boldsymbol{\sigma}}+\left((b-1)\dot{p}_{gz}-bS_{lq}\dot{p}_c\right)\boldsymbol{I}$$ éq 5.1-5

**Note::**

> *The stress of Bishop is generally regarded as an effective stress (controlled only by the strain). It is not the case of the model of Barcelona where are needed two stresses ( $(\sigma, p_c)$ to describe the behavior. Consequently, in the tangent operator, the term $\frac{\partial\dot{\sigma}'}{\partial p_c}$ is not summarized with $-\frac{\partial\dot{\sigma}_p}{\partial p_c}$ .*

Part [DMECDE] of the matrix $D\Sigma\,DE$ corresponding to $\frac{\partial\sigma'}{\partial\varepsilon}$ is such as:

$$\begin{bmatrix}\dot{\sigma}'_{11}\\ \dot{\sigma}'_{22}\\ \dot{\sigma}'_{33}\\ \sqrt{2}\dot{\sigma}'_{12}\\ \sqrt{2}\dot{\sigma}'_{23}\\ \sqrt{2}\dot{\sigma}'_{31}\end{bmatrix}=\underbrace{\begin{bmatrix}k_0P+\frac{4}{3}\mu & k_0P-\frac{2}{3}\mu & k_0P-\frac{2}{3}\mu & 0 & 0 & 0\\ k_0P-\frac{2}{3}\mu & k_0P+\frac{4}{3}\mu & k_0P-\frac{2}{3}\mu & 0 & 0 & 0\\ k_0P-\frac{2}{3}\mu & k_0P-\frac{2}{3}\mu & k_0P+\frac{4}{3}\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix}}_{D^e}\begin{bmatrix}\dot{\varepsilon}_{11}\\ \dot{\varepsilon}_{22}\\ \dot{\varepsilon}_{33}\\ \sqrt{2}\dot{\varepsilon}_{12}\\ \sqrt{2}\dot{\varepsilon}_{23}\\ \sqrt{2}\dot{\varepsilon}_{31}\end{bmatrix}$$ éq 5.1-6

part [DMECP1] of the matrix $D\Sigma\,DE$ is reduced to $\frac{\partial\sigma'}{\partial p_1}$with $\left(p_1=p_c\right)$ which is such that:

$$\begin{Bmatrix} \dot{\sigma}'_{11} \\ \dot{\sigma}'_{22} \\ \dot{\sigma}'_{33} \\ \sqrt{2}\dot{\sigma}'_{12} \\ \sqrt{2}\dot{\sigma}'_{23} \\ \sqrt{2}\dot{\sigma}'_{31} \end{Bmatrix} = \left[ \left( \frac{k_0}{k_{0s}} \frac{P}{p_c + p_{atm}} - bS_{lq} \right) \quad \left( \frac{k_0}{k_{0s}} \frac{P}{p_c + p_{atm}} - bS_{lq} \right) \quad \left( \frac{k_0}{k_{0s}} \frac{P}{p_c + p_{atm}} - bS_{lq} \right) \quad 0 \quad 0 \quad 0 \right] \{ \dot{p}_1 \}$$

éq 5.1-7

## 5.2 plastic tangent Operator of velocity. Option RIGI_MECA_TANG

the total tangent operator is in this case obtained starting from the results known at time $t_{i-}$ (option RIGI_MECA_TANG called with the first iteration of a new increment of load).
So at $t_{i-}$ the border of the field of reversibility is reached, one writes the condition: $\dot{f}=0$ who must be checked jointly with the condition $f=0$ . So with $t_{i-}$ one is strictly inside the field $f<0$ , then the tangent operator is the operator of elasticity.

**If the mechanical criterion is reached:**

$\dot{f}_1=0$

$$\dot{f}_1 = \left( \frac{\partial f_1}{\partial \sigma} \right) \dot{\sigma} + \frac{\partial f_1}{\partial P_{cr}} \dot{P}_{cr} + \frac{\partial f_1}{\partial p_c} \dot{p}_c = 0 \quad \text{éq 5.2-1}$$

like $\dot{P}_{cr} = \frac{\partial P_{cr}}{\partial \varepsilon_v^p} \dot{\varepsilon}_v^p + \frac{\partial P_{cr}}{\partial p_c} \dot{p}_c$ , then:

$$\dot{f}_1 = \left( \frac{\partial f_1}{\partial \sigma} \right) \dot{\sigma} + \frac{\partial f_1}{\partial P_{cr}} \left( \frac{\partial P_{cr}}{\partial \varepsilon_v^p} \dot{\varepsilon}_v^p + \frac{\partial P_{cr}}{\partial p_c} \dot{p}_c \right) + \frac{\partial f_1}{\partial p_c} \dot{p}_c = 0 \quad \text{éq 5.2-2}$$

One has in addition: $\dot{\sigma}_{ij} = D^e_{ijkl} \dot{\varepsilon}^{e_{kl}} + k_0 P \frac{\dot{p}_c}{k_{0s}(p_c + p_{atm})} \delta_{ij}$ **éq 5.2-3**

i.e.:

$$\dot{\sigma}_{ij} = D^e_{ijkl} \dot{\varepsilon}_{kl} - \Lambda D^e_{ijkl} \left( \alpha \frac{\partial f_1}{\partial s_{kl}} - \frac{1}{3} \frac{\partial f_1}{\partial P} \delta_{kl} \right) + k_0 P \frac{\dot{p}_c}{k_{0s}(p_c + p_{atm})} \delta_{ij} \quad \text{éq 5.2-4}$$

By writing the plastic hardening modulus:

$$H_p = -\frac{\partial f_1}{\partial P_{cr}} \frac{\partial P_{cr}}{\partial \varepsilon_v^p} \frac{\partial f_1}{\partial P}, \quad \text{éq the 5.2-5}$$

equations [éq 5.2-2] and [éq 5.2-5] give:

$$\left( \frac{\partial f_1}{\partial \sigma} \right)_{ij} \dot{\sigma}_{ij} - \Lambda H_p + \left( \frac{\partial f_1}{\partial p_c} + \frac{\partial f_1}{\partial P_{cr}} \frac{\partial P_{cr}}{\partial p_c} \right) \dot{p}_c = 0 \quad \text{éq 5.2-6}$$

the multiplication of the equation [éq 5.2-4] by $\left( \frac{\partial f_1}{\partial \sigma} \right)_{ij}$ gives:

$$\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}\dot{\sigma}_{ij}=\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\dot{\varepsilon}_{kl}-\Lambda\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)+\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}k_0P\delta_{ij}\frac{\dot{p}_c}{k_{0s}(p_c+p_{atm})}$$ éq the 5.2-7

two preceding equations make it possible to find:

$$H_p\Lambda=\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\dot{\varepsilon}_{kl}-\Lambda\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)_{kl}+\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}k_0P\delta_{ij}\frac{\dot{p}_c}{k_{0s}(p_c+p_{atm})}+\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\dot{p}_c$$ **éq 5.2-8**

from where and to deduce the statement from it from the plastic multiplier:

$$\Lambda=\frac{\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\dot{\varepsilon}_{kl}+\left[\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}k_0P\delta_{ij}\frac{1}{k_{0s}(p_c+p_{atm})}+\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\right]\dot{p}_c}{\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)+H_p}$$ éq 5.2-9

Is $H$ the definite elastoplastic modulus like:

$$H=\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)+H_p$$ éq 5.2-10

the plastic multiplier is written:

$$\Lambda=\frac{\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}D^e_{ijkl}\dot{\varepsilon}_{kl}+\left[\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}k_0P\delta_{ij}\frac{1}{k_{0s}(p_c+p_{atm})}+\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\right]\dot{p}_c}{H}$$ éq 5.2-11

While replacing $\Lambda$ by its statement in the equation [éq 5.2-4], one obtains:

$$\dot{\sigma}_{ij}=D^e_{ijkl}\dot{\varepsilon}_{kl}-\frac{1}{H}\left[\left(\frac{\partial f_1}{\partial\sigma}\right)_{mn}D^e_{mnop}\dot{\varepsilon}_{op}\right].D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)-$$
$$\left[\frac{1}{H}\left[\left(\frac{\partial f_1}{\partial\sigma}\right)_{mn}k_0P\frac{1}{k_{0s}(p_c+p_{atm})}\delta_{mn}+\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\right]D^e_{ijkl}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)-\frac{k_0P}{k_{0s}(p_c+p_{atm})}\delta_{ij}\right]\dot{p}_c$$

éq 5.2-12

One from of thus deduced the elastoplastic operator $D^{elp}=D^e-D^p$ :

$$\dot{\sigma}_{ij}=\left[D^e_{ijkl}\underbrace{-\frac{1}{H}\left(\frac{\partial f_1}{\partial\sigma}\right)_{op}D^e_{ijop}D^e_{mnkl}\left(\alpha\frac{\partial f_1}{\partial s_{mn}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{mn}\right)}_{D^p_{ijkl}}\right]\dot{\varepsilon}_{kl}-$$
$$\underbrace{\left\{\frac{1}{H}\left(\alpha\frac{\partial f_1}{\partial s_{op}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{op}\right)D^e_{ijop}\left(\left(\frac{\partial f_1}{\partial\sigma}\right)_{mn}k_0P\delta_{mn}\frac{1}{k_{0s}(p_c+p_{atm})}-\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\right)-\frac{k_0P}{k_{0s}(p_c+p_{atm})}\delta_{ij}\right\}}_{D^{p_c}_{ij}}\dot{p}_c$$

éq 5.2-13

with,

$$D^p_{ijkl}=\frac{1}{H}\left(\frac{\partial f_1}{\partial\sigma}\right)_{op}D^e_{ijop}D^e_{mnkl}\left(\alpha\frac{\partial f_1}{\partial s_{mn}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{mn}\right)$$

and

$$D_{ij}^{p_c}=-\frac{1}{H}\left(\alpha\frac{\partial f_1}{\partial s_{op}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{op}\right)D_{ijop}^{e}\left(\left(\frac{\partial f_1}{\partial\sigma}\right)_{mn}k_0 P\delta_{mn}\frac{1}{k_{0s}(p_c+p_{atm})}+\left(\frac{\partial f_1}{\partial p_c}+\frac{\partial f_1}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\right)+\frac{k_0 P}{k_{0s}(p_c+p_{atm})}\delta_{ij}$$

éq 5.2-14

Computation of $D_{ijkl}^{p}$:

$$\left(\frac{\partial f_1}{\partial\sigma}\right)_{ij}=-\frac{1}{3}M^2(2P-2P_{cr}+k_c p_c)\delta_{ij}+3s_{ij}\ , \qquad \text{éq 5.2-15}$$

which is written in vectorial notation:

$$\begin{bmatrix}-\frac{1}{3}M^2(2P-2P_{cr}+k_c p_c)+3s_{11}\\-\frac{1}{3}M^2(2P-2P_{cr}+k_c p_c)+3s_{22}\\-\frac{1}{3}M^2(2P-2P_{cr}+k_c p_c)+3s_{33}\\3\sqrt{2}s_{12}\\3\sqrt{2}s_{23}\\3\sqrt{2}s_{31}\end{bmatrix} \qquad \text{éq 5.2-16}$$

from where the statement of:

$$D_{ijkl}^{e}\left(\frac{\partial f}{\partial\sigma}\right)_{kl}\ :\ \begin{bmatrix}-k_0 M^2 P(2P-2P_{cr}+k_c p_c)+6\mu s_{11}\\-k_0 M^2 P(2P-2P_{cr}+k_c p_c)+6\mu s_{22}\\-k_0 M^2 P(2P-2P_{cr}+k_c p_c)+6\mu s_{33}\\6\mu\sqrt{2}s_{12}\\6\mu\sqrt{2}s_{23}\\6\mu\sqrt{2}s_{31}\end{bmatrix} \qquad \text{éq 5.2-17}$$

and

$$\left(\frac{\partial f}{\partial\sigma}\right)_{ij}D_{ijkl}^{e}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)=k_0 M^4 P(2P-2P_{cr}+k_c p_c)^2+12\alpha\mu Q^2 \qquad \text{éq 5.2-18}$$

Gold the plastic modulus $H$ is written in the form:

$$H=\left(\frac{\partial f}{\partial\sigma}\right)_{ij}D_{ijkl}^{e}\left(\alpha\frac{\partial f_1}{\partial s_{kl}}-\frac{1}{3}\frac{\partial f_1}{\partial P}\delta_{kl}\right)+H_p$$

$$H=M^4(2P-2P_{cr}+k_c p_c)\left[k_0 P(2P-2P_{cr}+k_c p_c)+2kP_{cr}(P+k_c p_c)\right]+12\alpha\mu Q^2 \qquad \text{éq 5.2-19}$$

While posing:

$A_{ij}=-k_0 M^2 P(2P-2P_{cr}+k_c p_c)\delta_{ij}+6\mu s_{ij}$, $A'_{ij}=-k_0 M^2 P(2P-2P_{cr}+k_c p_c)\delta_{ij}+6\alpha\mu s_{ij}$ , éq 5.2-20

with: $tr(A)=-3k_0 M^2 P(2P-2P_{cr}+k_c p_c)$

$$D^p=\frac{1}{H}\begin{bmatrix} A_{11}A'_{11} & A_{11}A'_{22} & A_{11}A'_{33} & 6\sqrt{2}\,\mu\alpha A_{11}s_{12} & 6\sqrt{2}\,\mu\alpha A_{11}s_{23} & 6\sqrt{2}\,\mu\alpha A_{11}s_{31} \\ A'_{11}A_{22} & A_{22}A'_{22} & A_{22}A'_{33} & 6\sqrt{2}\,\mu\alpha A_{22}s_{12} & 6\sqrt{2}\,\mu\alpha A_{22}s_{23} & 6\sqrt{2}\,\mu\alpha A_{22}s_{31} \\ A_{33}A'_{113} & A_{22}A'_{33} & A_{33}A'_{33} & 6\sqrt{2}\,\mu\alpha A_{33}s_{12} & 6\sqrt{2}\,\mu\alpha A_{33}s_{23} & 6\sqrt{2}\,\mu\alpha A_{33}s_{31} \\ 6\sqrt{2}\,\mu A'_{11}s_{12} & 6\sqrt{2}\,\mu A'_{22}s_{12} & 6\sqrt{2}\,\mu A'_{33}s_{12} & 36\,\alpha\mu^2 s_{12}^2 & 36\,\alpha\mu^2 s_{12}s_{23} & 36\,\alpha\mu^2 s_{12}s_{31} \\ 6\sqrt{2}\,\mu A'_{11}s_{23} & 6\sqrt{2}\,\mu A'_{22}s_{23} & 6\sqrt{2}\,\mu A'_{33}s_{23} & . & 36\,\alpha\mu^2 s_{23}^2 & 36\,\alpha\mu^2 s_{23}s_{31} \\ 6\sqrt{2}\,\mu A'_{11}s_{31} & 6\sqrt{2}\,\mu A'_{22}s_{31} & 6\sqrt{2}\,\mu A'_{33}s_{31} & . & . & 36\,\alpha\mu^2 s_{31}^2 \end{bmatrix}_{SYM}$$

éq. 5.2-21

One can write the components $\frac{\partial\sigma'}{\partial\varepsilon}$ of piece [DMECDE] of the matrix $D\Sigma\ DE$ which are those of the operator $D^{elp}=D^e - D^p$ .

According to the equation [éq 5.2.14]. The components $\frac{\partial\sigma'}{\partial p_1}$ with $(p_1=p_c)$ piece [ $DMECP1$ ] of the matrix $D\Sigma\ DE$ are:

$$\begin{bmatrix} \frac{-tr(A)}{3Hk_{0s}(p_c+p_{atm})}A'_{11}+\frac{M^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]}{H}A'_{11}+\frac{k_0P}{k_{0s}(p_c+p_{atm})}-bS_{lq} \\ \frac{-tr(A)}{3Hk_{0s}(p_c+p_{atm})}A'_{22}+\frac{M^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]}{H}A'_{22}+\frac{k_0P}{k_{0s}(p_c+p_{atm})}-bS_{lq} \\ \frac{-tr(A)}{3Hk_{0s}(p_c+p_{atm})}A'_{33}+\frac{M^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]}{H}A'_{33}+\frac{k_0P}{k_{0s}(p_c+p_{atm})}-bS_{lq} \\ \frac{-2\sqrt{2}\mu\,tr(A)}{Hk_{0s}(p_c+p_{atm})}alphas_{12}+\frac{6\sqrt{2}muM^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]}{H}alphas_{12} \\ \frac{-2\sqrt{2}\mu\,tr(A)}{Hk_{0s}(p_c+p_{atm})}alphas_{23}+\frac{6\sqrt{2}muM^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]}{H}alphas_{23} \\ \frac{-2\sqrt{2}\mu\,tr(A)}{Hk_{0s}(p_c+p_{atm})}alphas_{13}+6\sqrt{2}muM^2\left[k_c(2P_{cr}-P)-2P_{cr}(P+k_cp_c)Ln\left[\frac{2P^*_{cr}}{P_0}\right]\left[\frac{\lambda(0)-\kappa}{(\lambda(p_c)-\kappa)^2}\lambda'\right]\right]alphas_{13} \end{bmatrix}$$

éq 5.2-22

with $\lambda'=\frac{\partial\lambda}{\partial p_c}=-\beta\lambda(0)\left[(1-r)\exp(-\beta p_c)\right]$

**If the hydrous criterion is reached:**

One leaves again the equation [éq 5.2.3] with this time $\dot{\varepsilon}^p=\frac{\dot{p}_c}{k_s(p_c+p_{atm})}$ ,

One finds a direct relationship enters $\dot{\sigma}$ and $\dot{\varepsilon},\dot{p}_c$ of the form:

$$\dot{\boldsymbol{\sigma}}=\boldsymbol{D}^e\dot{\varepsilon}+k_0P\left(\frac{\dot{p}_c}{k_s(p_c+p_{atm})}+\frac{\dot{p}_c}{k_{0s}(p_c+p_{atm})}\right)\boldsymbol{I} \quad \text{éq 5.2-23}$$

One deduces then the stress from Bishop

$$\dot{\boldsymbol{\sigma}}'=\boldsymbol{D}^e\dot{\varepsilon}+\left(\left(\frac{k_0P}{k_s(p_c+p_{atm})}+\frac{k_0P}{k_{0s}(p_c+p_{atm})}\right)-bS_{lq}\right)\boldsymbol{I}\,\dot{p}_c \quad \text{éq the 5.2-24}$$

components $\frac{\partial\sigma'}{\partial\varepsilon}$ of the piece [ $DMECDE$ ] of the matrix $D\Sigma\ DE$ are not nothing other than those of the matrix $D^e$.

The only components of the piece [ $DMECP1$ ] of the matrix $D\Sigma\ DE$ are thus those of $\frac{\partial\sigma'}{\partial p_1}$ with $(p_1=p_c)$ :

$$\begin{bmatrix}\dot{\sigma}'_{11}\\ \dot{\sigma}'_{22}\\ \dot{\sigma}'_{33}\\ \sqrt{2}\dot{\sigma}'_{12}\\ \sqrt{2}\dot{\sigma}'_{23}\\ \sqrt{2}\dot{\sigma}'_{31}\end{bmatrix}=\begin{bmatrix}\frac{k_0P}{(p_c+p_{atm})}\left(\frac{1}{k_s}+\frac{1}{k_{0s}}\right)-bS_{lq} & \frac{k_0P}{(p_c+p_{atm})}\left(\frac{1}{k_s}+\frac{1}{k_{0s}}\right)-bS_{lq} & \frac{k_0P}{(p_c+p_{atm})}\left(\frac{1}{k_s}+\frac{1}{k_{0s}}\right)-bS_{lq} & 0 & 0 & 0\end{bmatrix}\left[\dot{p}_1\right] \quad \text{éq 5.2-25}$$

## 5.3 tangent Operator into implicit. Option `FULL_MECA`

To compute: the tangent operator into implicit, one chose as for the model Camwood Clay initially separating the processing from the deviatoric part of the hydrostatic part for then combining them in order to deduce the tangent operator connecting the disturbance from the total stress to the disturbance of the total deflection.

### 5.3.1 If the mechanical criterion is reached

#### 5.3.1.1 Processing of the deviatoric part

It is considered here that the variation of loading is purely deviatoric $(\delta P=0)$.
The increment of the deviatoric stress is written in the form:

$$\Delta s_{ij}=2\mu\left(\Delta\tilde{\varepsilon}_{ij}-\Delta\tilde{\varepsilon}_{ij}^p\right) \quad \text{éq 5.3.1.1 - 1}$$

Around the point of equilibrium $(\sigma^-+\Delta\sigma)$ , one considers a variation $\delta s$ of the deviatoric part of the stress:

$$\delta s_{kl}=2\mu\left(\delta\tilde{\varepsilon}_{kl}-\delta\tilde{\varepsilon}_{kl}^p\right) \quad \text{éq 5.3.1.1 - 2}$$

Computation of $\delta\tilde{\varepsilon}_{kl}^p$ :

It is known that:

$$\Delta\tilde{\varepsilon}_{kl}^p=3\Lambda\ \alpha s_{kl} \quad \text{éq 5.3.1.1 - 3}$$

By deriving this deviatoric equation compared to the forced, one obtains:

$$\delta\,\tilde{\varepsilon}^p_{kl} = 3\alpha\ \delta\ \Lambda\ s_{kl} + 3\ \alpha\ \Lambda\ \delta\ s_{kl}$$ éq 5.3.1.1 - 4

Computation of $\delta\ \Lambda$ :

One a:

$$\Lambda = \frac{1}{H_p}\left[\left(\frac{\partial f}{\partial \sigma}\right)_{mn}\Delta\sigma_{mn} + \left(\frac{\partial f}{\partial p_c} + \frac{\partial f}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\Delta p_c\right] = \frac{1}{H_p}\left[\left(\frac{\partial f}{\partial s}\right)_{mn}\Delta s_{mn} + \frac{\partial f}{\partial P}\Delta P + \left(\frac{\partial f}{\partial p_c} + \frac{\partial f}{\partial P_{cr}}\frac{\partial P_{cr}}{\partial p_c}\right)\Delta p_c\right]$$
$$= \frac{1}{H_p}\left[3 s_{mn}\Delta s_{mn} + M^2\left(2P - 2P_{cr} + k_c\, p_c\right)\Delta P - M^2\left[k_c\left(2P_{cr} - P\right) + 2\left(P + k_c\, p_c\right)P'_{cr}\right]\Delta p_c\right]$$

éq 5.3.1.1 - 5

If one considers only the evolution of the deviatoric part of $\sigma$ $(\delta P = 0)$, then:

$$\delta\left(\Lambda\ H_p\right) = \delta\ \Lambda H_p + \Lambda\delta H_p = \left[3\delta s_{mn}\Delta s_{mn} + 3 s_{mn}\delta s_{mn}\right] - 2M^2\Delta P\delta P_{cr} +$$
$$M^2 k_c\Delta P\delta p_c - 2M^2 k_c\delta P_{cr}\Delta p_c - M^2\left[k_c\left(2P_{cr} - P\right) + 2P'_{cr}\left(P + k_c\left(p_c + \Delta p_c\right)\right)\right]\delta p_c$$ éq. 5.3.1.1 - 6

with $P'_{cr} = \frac{\partial P_{cr}}{\partial p_c}$

Gold: $\delta\ P_{cr} = kP_{cr}\ \delta\ \varepsilon^P_v$ .

Like $\Delta\varepsilon^p_v = \Lambda M^2\left(2P - 2P_{cr} + k_c\, p_c\right)$, one a:

$$\delta\varepsilon^p_v = \delta\Lambda M^2\left(2P - 2P_{cr} + k_c\, p_c\right) - 2M^2\Lambda\delta P_{cr} + k_c M^2\Lambda\delta p_c$$ éq. 5.3.1.1 - 7

From where:

$$\delta\ \Lambda M^2\left(2P - 2P_{cr} + k_c\, p_c\right) = \left[\frac{1}{kP_{cr}} + 2\Lambda M^2\right]\delta P_{cr} - k_c\Lambda M^2\delta p_c$$ éq 5.3.1.1 - 8

In addition,

$$H_p = 2\,kM^4 P_{cr}\left(P + k_c\, p_c\right)\left(2P - 2P_{cr} + k_c\, p_c\right)$$
et éq 5.3.1.1 - 9
$$\delta\ H_p = 2\,kM^4\left(P + k_c\, p_c\right)\left(2P - 4P_{cr} + k_c\, p_c\right)\delta\ P_{cr} + 2\,kP_{cr}M^4 k_c\left(3P - 2P_{cr} + 2k_c\, p_c\right)\delta p_c$$

By injecting this last equation in the equation [éq 5.3.1.1 - 6], one obtains:

$$\delta\ \Lambda H_p + \left[2\Lambda\, kM^4\left(P + k_c\, p_c\right)\left(2P - 4P_{cr} + k_c\, p_c\right) + 2M^2\Delta P + 2M^2 k_c\Delta p_c\right]\delta\ P_{cr} =$$
$$-\left[2\Lambda\, kP_{cr}M^4 k_c\left(3P + 2k_c\, p_c - 2P_{cr}\right) + M^2\left[k_c\left(2P_{cr} - \left(P + \Delta P\right)\right) + 2P'_{cr}\left(p_c + \Delta p_c\right)\right] + 2P'_{cr}P\right]\delta p_c +$$
$$\left[3\delta s_{mn}\Delta s_{mn} + 3 s_{mn}\delta s_{mn}\right]$$

éq 5.3.1.1 - 10

By means of the relation [éq 5.3.1.1 - 8], it comes then:

$$\delta\ \Lambda = \frac{\left[3\delta s_{mn}\Delta s_{mn} + 3 s_{mn}\delta s_{mn}\right]}{\left(H_p + A\right)} - \frac{Z\delta p_c}{\left(H_p + A\right)}$$ éq 5.3.1.1 - 11

$$\text{avec}\quad A = \left[k\ \Lambda M^4\left(P + k_c\, p_c\right)\left(2P - 4P_{cr} + k_c\, p_c\right) + M^2\Delta P + M^2 k_c\Delta p_c\right]\left[\frac{M^2\left(2P - 2P_{cr} + k_c\, p_c\right)}{\frac{1}{2\,kP_{cr}} + \Lambda M^2}\right]$$

$$Z = M^2 \Lambda \left( \frac{k k_c \Lambda M^2 (P + k_c p_c)(2P - 4P_{cr} + k_c p_c)}{\frac{1}{2kM^2 P_{cr}} + \Lambda} + 2kk_c P_{cr} M^2 (3P - 2P_{cr} + k_c p_c) \right)$$
$$- M^2 k_c \Delta P + \frac{\left( M^2 k_c \Delta P \Lambda + k_c^2 \Lambda M^2 \Delta p_c \right)}{\frac{1}{2kM^2 P_{cr}} + \Lambda} + M^2 k_c (2P_{cr} - P) + 2M^2 P'_{cr} (P + k_c (p_c + \Delta p_c))$$

One then obtains immediately the variation of the deviatoric part of the plastic strain:

$$\begin{aligned}\delta \tilde{\varepsilon}_{kl}^p = & \frac{9\alpha}{H_p + A} (\Delta s_{mn} \delta s_{mn} s_{kl} + s_{mn} \delta s_{mn} s_{kl}) + \frac{9\alpha}{H_p} s_{mn} \Delta s_{mn} \delta s_{kl} \\ & + \frac{3\alpha}{H_p} M^2 (2P - 2P_{cr} + k_c p_c) \Delta P \delta s_{kl} - \frac{3\alpha}{H_p} M^2 k_c (2P_{cr} - P) \Delta p_c \delta s_{kl} - \frac{3\alpha Z}{H_p + A} \delta p_c s_{kl} \\ & - \frac{6\alpha}{H_p} M^2 (P + k_c p_c) P'_{cr} \Delta p_c \delta s_{kl}\end{aligned}$$ éq 5.3.1.1 - 12

$\delta s_{ij}$ is written then:

$$\begin{aligned}\delta s_{ij} = & 2\mu \delta \{ \tilde{\varepsilon}_{ij} - \frac{18\mu\alpha}{(H_p + A)} [(\Delta s_{kl} s_{ij} \delta s_{kl} + s_{kl} s_{ij} \delta s_{kl})] - \frac{18\mu\alpha}{H_p} s_{kl} \Delta s_{kl} \delta s_{ij} \\ & - \frac{6\mu\alpha}{H_p} M^2 (2P - 2P_{cr} + k_c p_c) \Delta P \delta s_{ij} + \frac{6\mu\alpha}{H_p} M^2 k_c (2P_{cr} - P) \Delta p_c \delta s_{ij} \\ & + \frac{6\mu\alpha Z}{(H_p + A)} s_{ij} \delta p_c + \frac{12\mu\alpha}{H_p} M^2 (P + k_c p_c) P'_{cr} \Delta p_c \delta s_{ij}\end{aligned}$$ éq 5.3.1.1 - 13

i.e.:

$$\left( \begin{aligned} & \delta_{ijkl} + \delta_{ijkl} \frac{6\mu\alpha}{H_p} M^2 (2P - 2P_{cr} + k_c p_c) \Delta P + \frac{18\mu\alpha}{H_p + A} (\Delta s_{kl} s_{ij} + s_{kl} s_{ij}) + \frac{18\mu\alpha}{H_p} s_{mn} \Delta s_{mn} \delta_{ijkl} \\ & - \delta_{ijkl} \frac{6\mu\alpha}{H_p} k_c M^2 (2P_{cr} - 2P) \Delta p_c - \frac{12\mu\alpha}{H_p} M^2 (P + k_c p_c) P'_{cr} \Delta p_c \end{aligned} \right) \delta s_{kl} =$$
$$2\mu \delta \tilde{\varepsilon}_{ij} + \frac{6\mu\alpha Z}{H_p + A} s_{ij} \delta p_c$$

**éq 5.3.1.1 - 14**

or in tensorial writing:

$$\left(\begin{array}{l} I_4^d\left(1+\frac{6\,\mu\alpha}{H_p}M^2\left(2P-2P_{cr}+k_c\,p_c\right)\Delta P+\frac{18\,\mu\alpha}{H_p}\Delta s:s-\frac{6\,\mu\alpha}{H_p}M^2k_c\left(2P_{cr}-P\right)\Delta p_c\right) \\ +\frac{18\,\mu\alpha}{(H_p+A)}(s+\Delta s)\otimes s-\frac{12\,\mu\alpha}{H_p}M^2\left(P+k_c\,p_c\right)P'_{cr}\Delta p_c \end{array}\right)\delta s= \quad \text{éq 5.3.1.1 - 15}$$

$$2\,\mu\delta\,\{\,\tilde{\varepsilon}\,¿+\frac{6\,\mu\alpha Z}{(H_p+A)}s\delta p_c$$

which one can still write by symmetrizing the tensor $(s+\Delta s)\otimes s$ :

$$\left\{\begin{array}{l} I_4^d\left(1+\frac{6\,\mu\,\alpha}{H_p}M^2\left(2P-2P_{cr}+k_c\,p_c\right)\Delta P-\frac{6\,\mu\,\alpha}{H_p}k_c\,M^2\left(2P_{cr}-P\right)\Delta p_c \right.\\ \left. +\frac{18\,\mu\,\alpha}{H_p}\Delta s:s-\frac{12\,\mu\,\alpha}{H_p}M^2\left(P+k_c\,p_c\right)P'_{cr}\Delta p_c\right)+\frac{18\,\mu\,\alpha}{(H_p+A)}\aleph \end{array}\right\}\Delta s=2\,\mu\,\delta\,\tilde{\varepsilon}+\frac{6\,\mu\,\alpha Z}{(H_p+A)}s\,\delta\,p_c$$

éq 5.3.1.1 - 16

with: $\aleph=\frac{1}{2}\left[\left((s+\Delta s)\otimes s\right)+\left(s\otimes(s+\Delta s)\right)^T\right]$

Computation of $\aleph$, while posing: $T_{ij}=s_{ij}+\Delta s_{ij}$

$$T\otimes s=\begin{bmatrix} T_{11}s_{11} & T_{11}s_{22} & T_{11}s_{33} & \sqrt{2}\,T_{11}s_{12} & \sqrt{2}\,T_{11}s_{23} & \sqrt{2}\,T_{11}s_{31} \\ T_{22}s_{11} & T_{22}s_{22} & T_{22}s_{33} & \sqrt{2}\,T_{22}s_{12} & \sqrt{2}\,T_{22}s_{23} & \sqrt{2}\,T_{22}s_{31} \\ T_{33}s_{11} & T_{33}s_{22} & T_{33}s_{33} & \sqrt{2}\,T_{33}s_{12} & \sqrt{2}\,T_{33}s_{23} & \sqrt{2}\,T_{33}s_{31} \\ \sqrt{2}\,T_{12}s_{11} & \sqrt{2}\,T_{12}s_{22} & \sqrt{2}\,T_{12}s_{33} & 2T_{12}s_{12} & 2T_{12}s_{23} & 2T_{12}s_{31} \\ \sqrt{2}\,T_{23}s_{11} & \sqrt{2}\,T_{23}s_{22} & \sqrt{2}\,T_{23}s_{33} & 2T_{23}s_{12} & 2T_{23}s_{23} & 2T_{23}s_{31} \\ \sqrt{2}\,T_{31}s_{11} & \sqrt{2}\,T_{31}s_{22} & \sqrt{2}\,T_{31}s_{33} & T_{31}s_{12} & 2T_{31}s_{23} & 2T_{31}s_{31} \end{bmatrix}$$

$$\aleph=\frac{1}{2}\left[(T\otimes s)+(T\otimes s)^T\right]$$

That is to say:

$$C=I_4^d\left(\frac{1}{2\mu}+\frac{3\alpha}{H_p}M^2\left(2P-2P_{cr}+k_c\,p_c\right)\Delta P+\frac{9\alpha}{H_p}\Delta s:s-\frac{3\alpha k_c}{H_p}M^2\left(2P_{cr}-P\right)\Delta p_c-\frac{6\alpha}{H_p}M^2\left(P+k_c\,p_c\right)P'_{cr}\Delta\right.$$
$$+\frac{9\alpha}{(H_p+A)}\aleph$$

one poses:

$$c=\frac{9\alpha}{H_p}\left(\Delta s:s\right)$$

$$d=\frac{3\alpha}{H_p}M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c\,p_c\right)\Delta P$$
$$g=\frac{-3\alpha}{H_p}M^2k_c\left(2\mathrm{P}_{cr}-P\right)\Delta p_c$$
$$h=-\frac{6\alpha}{H_p}M^2\left(P+k_c\,p_c\right)P'_{cr}\Delta p_c$$

The symmetric matrix $C$ of dimensions (6,6) is too large to be presented whole, one breaks up it into 4 parts $C_1$ $C_2$, $C_3$ and $C_4$ :

$$C=\begin{bmatrix}C_1 & C_2\\ C_3 & C_4\end{bmatrix}$$

with

$$C_1=\begin{bmatrix}\frac{1}{2\mu}+c+d+g+h+\frac{9\alpha}{(H_p+A)}s_{11}T_{11} & \frac{9\alpha}{2(H_p+A)}(T_{11}s_{22}+T_{22}s_{11}) & \frac{9\alpha}{2(H_p+A)}(T_{11}s_{33}+T_{33}s_{11})\\ \frac{9\alpha}{2(H_p+A)}(T_{22}s_{11}+T_{11}s_{22}) & \frac{1}{2\mu}+c+d+g+h+\frac{9\alpha}{(H_p+A)}T_{22}s_{22} & \frac{9\alpha}{2(H_p+A)}(T_{22}s_{33}+T_{33}s_{22})\\ \frac{9\alpha}{2(H_p+A)}(T_{33}s_{11}+T_{11}s_{33}) & \frac{9\alpha}{2(H_p+A)}(T_{22}s_{33}+T_{33}s_{22}) & \frac{1}{2\mu}+c+d+g+h+\frac{9\alpha}{(H_p+A)}T_{33}s_{33}\end{bmatrix}$$

éq 5.3.1.1 - 17

$$C_2=\begin{bmatrix}\frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{11}s_{12}+s_{11}T_{12}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{11}s_{23}+s_{11}T_{23}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{11}s_{13}+s_{11}T_{13})\\ \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{22}s_{12}+s_{22}T_{12}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{22}s_{23}+s_{22}T_{23}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{22}s_{13}+s_{22}T_{13})\\ \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{33}s_{12}+s_{33}T_{12}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{33}s_{23}+s_{33}T_{23}) & \frac{9\alpha\sqrt{2}}{2(H_p+A)}(T_{33}s_{13}+s_{33}T_{13})\end{bmatrix}$$

éq 5.3.1.1 - 18

$$C_3=C_2$$ éq 5.3.1.1 - 19

$$C_4=\begin{bmatrix}\frac{1}{2\mu}+c+d+g+h+\frac{18\alpha}{(H_p+A)}s_{12}T_{12} & \frac{9\alpha}{(H_p+A)}(T_{12}s_{23}+T_{23}s_{12}) & \frac{9\alpha}{(H_p+A)}(T_{12}s_{23}+T_{23}s_{12})\\ \frac{9\alpha}{(H_p+A)}(T_{23}s_{12}+T_{12}s_{23}) & \frac{1}{2\mu}+c+d+g+h+\frac{18\alpha}{(H_p+A)}T_{23}s_{23} & \frac{9\alpha}{(H_p+A)}(T_{23}s_{13}+T_{13}s_{23})\\ \frac{9\alpha}{(H_p+A)}(T_{13}s_{12}+T_{12}s_{13}) & \frac{9\alpha}{(H_p+A)}(T_{13}s_{23}+T_{23}s_{13}) & \frac{1}{2\mu}+c+d+g+h+\frac{18\alpha}{(H_p+A)}T_{13}s_{13}\end{bmatrix}$$

éq 5.3.1.1 - 20

Computation of the rate of variation of volume:

$$\Delta\varepsilon_v^p=M^2\Lambda(2\mathrm{P}-2\mathrm{P}_{cr}+k_c\,p_c),$$

$$\begin{aligned}\delta\varepsilon_v^p&=M^2\delta\Lambda(2\mathrm{P}-2\mathrm{P}_{cr}+k_c\,p_c)-2\mathrm{M}^2\Lambda\delta\ P_{cr}+M^2\Lambda k_c\delta p_c\\&=B\delta\ \Lambda+D\delta p_c\\&=\frac{3\mathrm{B}}{(H_p+A)}(s+\Delta s).\delta\ s+(D-\frac{BZ}{(H_p+A)})\delta p_c\end{aligned}$$ éq 5.3.1.1 - 21

with: $B=M^2(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c)-M^2\Lambda\dfrac{M^2(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c)}{\dfrac{1}{2kP_{cr}}+M^2\Lambda}$.

and $D=k_c M^2\Lambda-M^2\Lambda\dfrac{k_c M^2\Lambda}{\dfrac{1}{2kP_{cr}}+M^2\Lambda}$.

One thus has:

$$\delta\varepsilon_v^p=\frac{3\mathrm{B}}{(H_p+A)}(s+\Delta s).\delta\ s-(\frac{BZ}{(H_p+A)}-D)\delta p_c \quad \text{éq 5.3.1.1 - 22}$$

and finally:

$$\delta\ \varepsilon_{ij}=(C_{ijkl}-\frac{B}{(H_p+A)}(s+\Delta s)_{kl}\delta_{ij})\ \delta\ s_{kl}- \\ (-\frac{BZ}{3(H_p+A)}\delta_{ij}+\frac{D}{3}\delta_{ij}+\frac{\delta_{ij}}{3\mathrm{k}_{0\mathrm{s}}(p_c+p_{atm})}+\frac{3\mathrm{Z}}{(H_p+A)}s_{ij})\delta p_c \quad \text{éq 5.3.1.1 - 23}$$

#### 5.3.1.2 Processing of the hydrostatic part

One considers now that the variation of loading is purely spherical ( $\delta\ s=0$ ).
The increment of $P$ is written in the form:

$$\Delta P=P^-\left[\frac{\exp\left(k_0\Delta\varepsilon_v^e\right)}{\left(\dfrac{p_c+p_{atm}}{p^-_c+p_{atm}}\right)^{k_0/k_{0s}}}-1\right] \quad \text{éq 5.3.1.2 - 1}$$

the derivative of this equation gives:

$$\delta\ P=k_0 P\left(\delta\ \varepsilon_v-\delta\ \varepsilon_v^p\right)-\frac{k_0}{k_{0\mathrm{s}}}\frac{P}{p_c+p_{atm}}\ \delta\ p_c \quad \text{éq 5.3.1.2 - 2}$$

Computation of $\delta\varepsilon_v^p$ :
It is known that:

$$\Delta\varepsilon_v^p=\Lambda M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right) \quad \text{éq 5.3.1.2 - 3}$$

By differentiating this equation, one obtains:

$$\delta\varepsilon_v^p=M^2\left(\delta\ \Lambda\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)+\Lambda\left(2\delta\mathrm{P}-2\delta\mathrm{P}_{cr}+k_c\delta p_c\right)\right) \quad \text{éq 5.3.1.2 - 4}$$

One knows the statement of $\Lambda$ :

$$\Lambda=\frac{M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)\Delta P+3\mathrm{s}\Delta\mathrm{s}-M^2\left[k_c(2\mathrm{P}_{cr}-P)+2(P+k_c p_c)P'_{cr}\right]\Delta p_c}{H_p}=\frac{b}{H_p}$$ éq 5.3.1.2 - 5

while posing
$$b=M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)\Delta P+3\mathrm{s}\Delta\mathrm{s}-M^2\left[k_c(2\mathrm{P}_{cr}-P)+2(P+k_c p_c)P'_{cr}\right]\Delta p_c$$

While differentiating $\Delta\Lambda$, it comes:

$$\delta\Lambda=\frac{M^2}{H_p}\begin{bmatrix}\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)\delta P+\left(2\delta\mathrm{P}-2\delta\mathrm{P}_{cr}+k_c\delta p_c\right)\Delta P-k_c\left(2\mathrm{P}_{cr}-P\right)\delta p_c-k_c\left(2\delta\mathrm{P}_{cr}-\delta P\right)\Delta p_c\\-2(P+k_c p_c)P'_{cr}\delta p_c-2(\delta P+k_c\delta p_c)P'_{cr}\Delta p_c\end{bmatrix}$$
$$-\frac{2kM^4 b}{H_p^2}\begin{bmatrix}2\delta PP_{cr}\left(2\mathrm{P}-P_{cr}+\frac{3}{2}k_c p_c\right)+\delta P_{cr}\left(2\mathrm{P}^2-4PP_{cr}-4\mathrm{P}_{cr}k_c p_c+3Pk_c p_c+k_c^2 p_c^2\right)\\+k_c P_{cr}\left(3\mathrm{P}-2\mathrm{P}_{cr}+2\mathrm{k}_c p_c\right)\delta p_c\end{bmatrix}$$
éq 5.3.1.2 - 6

One seeks the statement of $\delta P_{cr}$ according to $\delta\Lambda$ :

One a:

$$\delta P_{cr}=kP_{cr}\delta\varepsilon_v^p$$ éq 5.3.1.2 - 7

One can write:

$$\frac{\delta P_{cr}}{kP_{cr}}=\delta\Lambda M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)+\Lambda M^2\left(2\delta\mathrm{P}-2\delta\mathrm{P}_{cr}+k_c\delta p_c\right)$$ éq 5.3.1.2 - 8

$$\delta P_{cr}\left(\frac{1+\Lambda 2M^2 kP_{cr}}{kP_{cr}}\right)=\delta\Lambda M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)+\Lambda 2M^2\delta P+\Lambda M^2 k_c\delta p_c$$

$$\delta P_{cr}=\left(\frac{M^2\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)kP_{cr}}{1+2kP_{cr}\Lambda M^2}\right)\delta\Lambda+\left(\frac{2\Lambda\mathrm{M}^2 kP_{cr}}{1+2kP_{cr}\Lambda M^2}\right)\delta P+\left(\frac{\Lambda M^2 k_c kP_{cr}}{1+2kP_{cr}\Lambda M^2}\right)\delta p_c$$ éq 5.3.1.2 - 9

One poses

$$c=\frac{M^2 kP_{cr}\left(2\mathrm{P}-2\mathrm{P}_{cr}+k_c p_c\right)}{\left[1+2\mathrm{M}^2 kP_{cr}\Lambda\right]}\qquad a=\frac{2\mathrm{M}^2 kP_{cr}\Lambda}{\left[1+2\mathrm{M}^2 kP_{cr}\Lambda\right]},\qquad d=\frac{k_c M^2 kP_{cr}\Lambda}{\left[1+2\mathrm{M}^2 kP_{cr}\Lambda\right]}$$

One has then:

$$\delta P_{cr}=a\delta P+c\delta\Lambda+d\delta p_c$$ éq 5.3.1.2 - 10

By replacing the statement of $\delta P_{cr}$ in $\delta\Lambda$ [éq 5.3.1.1 - 6], one finds:

$$\delta\Lambda=\begin{bmatrix}\left(2P-2P_{cr}+k_c p_c+2\Delta P+k_c\Delta p_c-2a\Delta P-2ak_c\Delta p_c-2P'_{cr}\Delta p_c\right)\delta P-2c\left(\Delta P+k_c\Delta p_c\right)\delta\Lambda\\+\left(k_c\left(P+\Delta P-2P_{cr}\right)-2d\left(\Delta P+k_c\Delta p_c\right)-2P'_{cr}\left(P+k_c\left(p_c+\Delta p_c\right)\right)\right)\delta p_c\end{bmatrix}\cdot\frac{M^2}{H_p}$$
$$-\frac{2kM^4b}{H_p^2}\left[P_{cr}\left(4P-2P_{cr}+3k_c p_c\right)+a\left(2P-4P_{cr}+k_c p_c\right)\left(P+k_c p_c\right)\right]\delta P$$
$$-\frac{2kM^4b}{H_p^2}\left[c\left(2P-4P_{cr}+k_c p_c\right)\left(P+k_c p_c\right)\right]\delta\Lambda$$
$$-\frac{2kM^4b}{H_p^2}\left[k_c P_{cr}\left(3P-2P_{cr}+2k_c p_c\right)+d\left(2P-4P_{cr}+k_c p_c\right)\left(P+k_c p_c\right)\right]\delta p_c$$

éq 5.3.1.2 - 11

By gathering the terms in $\delta\Lambda$ and those in $\delta P$, one finds:

$$\delta\Lambda=\frac{f}{e}\delta P+\frac{h}{e}\delta p_c$$ éq 5.3.1.2 - 12

with,

$$f=\frac{M^2}{H_p}\left[2P-2P_{cr}+k_c\left(p_c+(1-2a)\Delta p_c\right)+2\Delta P-2a\Delta P-2P'_{cr}\Delta p_c\right]$$
$$-\frac{2kM^4b}{H_p^2}\left[\left(4P-2P_{cr}+3k_c p_c\right)P_{cr}+a\left(2P^2-4PP_{cr}-4P_{cr}k_c p_c+3Pk_c p_c+k_c^2 p_c^2\right)\right]$$
$$h=\frac{M^2}{H_p}\left[-2d\Delta P-2dk_c\Delta p_c+k_c\Delta P-2k_c P_{cr}+k_c P-2P'_{cr}\left(P+k_c\left(p_c+\Delta p_c\right)\right)\right]$$
$$-\frac{2kM^4b}{H_p^2}\left[d\left(2P-4P_{cr}+k_c p_c\right)\left(P+k_c p_c\right)+k_c P_{cr}\left(3P-2P_{cr}+2k_c p_c\right)\right]$$

$$e=1+\frac{2cM^2\left(\Delta P+k_c p_c\right)}{H_p}+\frac{2bckM^4}{H_p^2}\left(2P^2-4PP_{cr}-4P_{cr}k_c p_c+3Pk_c p_c+k_c^2 p_c^2\right)$$

the statement of $\delta\varepsilon_v^p$ thus becomes:

$$\delta\varepsilon_v^p=X\delta P+Y\delta p_c$$ éq 5.3.1.2 - 13

with,

$$X=M^2\left(2\Lambda-2a\Lambda-2\Lambda c\frac{f}{e}+\frac{f}{e}\left(2P-2P_{cr}+k_c p_c\right)\right)$$
$$Y=M^2\left(\left(2P-2P_{cr}+k_c p_c\right)\frac{h}{e}-\left(2c\frac{h}{e}+2d-k_c\right)\Lambda\right)$$

from where the statement of $\delta P$ according to $\delta\varepsilon_v$ and $\delta p_c$:

$$\delta P\left(1+k_0 PX\right)=k_0 P\left(\delta\varepsilon_V-\left[Y+\frac{1}{k_{0s}\left(p_c+p_{atm}\right)}\right]\delta p_c\right)$$ éq 5.3.1.2 - 14

Calculus of the variation of deviatoric strain:

$$\delta\ \tilde{\varepsilon}_{ij} = \delta\ \tilde{\varepsilon}^{p} = 3\delta\ \Lambda\ s = 3\frac{f}{e}\ \delta\ Ps_{ij} + 3\frac{h}{e}\ \delta\ p_c s \qquad \text{éq 5.3.1.2 - 15}$$

One thus have finally:

$$\delta\varepsilon_{ij} = F_{ij}\,\delta P + K_{ij}\,\delta p_c \qquad \text{éq 5.3.1.2 - 16}$$

with

$$F = \frac{3\mathrm{f}}{e}s - \frac{1+k_0 PX}{3\mathrm{k}_0 P}1^d,\quad K = \frac{3\mathrm{h}}{e}s - \left(\frac{k_0 PY}{3} + \frac{k_0 P}{3\mathrm{k}_{0\mathrm{s}}(p_c + p_{atm})}\right)1^d \qquad \text{éq 5.3.1.2 - 17}$$

#### 5.3.1.3 tangent Operator

the tangent operator connects the variation of total stress to the variation of the strain and suction. Being given that the increment of the total deflection under loading deviatoric is written:

$$\delta\ \varepsilon_{ij} + H_{ij}\,\delta p_c = \left(C_{ijkl} - \frac{B}{(H_p + A)}(s + \Delta s)_{kl}\,\delta_{ij}\right) D^1_{klmn}\,\delta\ \sigma_{mn}\ , \qquad \text{éq 5.3.1.3 - 1}$$

with:

$$D^1 = \begin{bmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & -1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{éq 5.3.1.3 - 2}$$

projection in space deviatoric,

and that under spherical loading one a:

$$\delta\ \varepsilon_{ij} - K_{ij}\,\delta p_c = F_{ij}\ D^2_{kl}\ \delta\ \sigma_{kl} \qquad \text{éq 5.3.1.3 - 3}$$

with:

$$D^2 = \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{éq 5.3.1.3 - 4}$$

hydrostatic projection, one has then:

$$\delta\sigma_{ij} = A_{ijkl}\,\delta\varepsilon_{kl} + B_{ij}\,\delta p_c \qquad \text{éq 5.3.1.3 - 5}$$

with:

$$A_{ijkl}=\left[(C_{ijmn}-\frac{B}{(H_p+A)}(s+\Delta s)_{mn}\delta_{ij})\ D_{mnkl'}+F_{ij}D^2_{kl}\right]^{-1}$$ éq 5.3.1.3 - 6

$$B_{ij}=\left[(C_{ijmn}-\frac{B}{(H_p+A)}(s+\Delta s)_{mn}\delta_{ij})\ D_{mnkl'}+F_{ij}D^2_{kl}\right]^{-1}(H_{kl}-K_{kl})$$ éq 5.3.1.3 - 7

the stress of Bishop is thus written:

$$\delta\sigma'_{ij}=A_{ijkl}\,\delta\varepsilon_{kl}+\left(B_{ij}-bS_{lq}\right)\delta p_c$$

### 5.3.2 Tangent operator at the point criticizes

As for the model `CAM_CLAY` one writes a tangent operator specific to the critical point. As for the general case, one makes a processing of the deviatoric part and another for the hydrostatic part.

#### 5.3.2.1 Processing of the deviatoric part

According to the equation [éq 4.3.3] one finds:

$$s=s^e-2\mu\Delta\,\tilde{\varepsilon}^p=s^e-2\mu\Lambda\frac{\partial f}{\partial s}=s^e-6\,\mu\alpha\,\Lambda s$$ éq 5.3.2.1 - 1

the statements of the plastic multiplier $\Lambda$ and its derivative $\delta\Lambda$ are written in the following way:

$$\Lambda=\left(\frac{Q^e}{Q}-1\right)/6\,\mu\alpha \quad \text{and} \quad \delta\Lambda=\frac{\delta Q^e}{6\,\mu\alpha Q}-\frac{Q^e\,\delta Q}{6\,\mu\alpha Q^2}$$ éq 5.3.2.1 - 2

with,

$$\delta Q^e=\frac{3}{2}\frac{s^e\,\delta s^e}{Q^e} \quad \text{and} \quad \delta Q=\frac{3}{2}\frac{s\delta s}{Q}$$

from where the statement of $\delta\Lambda$ :

$$\delta\Lambda=\frac{1}{6\mu}\frac{3}{2\alpha}\left[\frac{s^e\,\delta s^e}{Q^e Q}-\frac{Q^e\,s\delta s}{Q^3}\right]$$ éq 5.3.2.1 - 3

Let us point out in the same way the statement of $\delta s$ :

$$\delta s_{ij}=2\mu\left(\delta\,\tilde{\varepsilon}_{ij}-3\delta\Lambda s_{ij}-3\Lambda\delta s_{ij}\right)$$

While replacing $\Lambda$ and $\delta\Lambda$ by their statements, one can write:

$$\delta s_{ij}=2\,\mu\delta\ \{\ \tilde{\varepsilon}¿_{ij}-\frac{3}{2\alpha}\frac{s^e_{kl}\delta s^e_{kl}}{Q^e Q}s_{ij}+\frac{3}{2\alpha}\frac{Q^e}{Q^3}s_{kl}\,\delta s_{kl}\,s_{ij}-\frac{1}{\alpha}\left(\frac{Q^e}{Q}-1\right)\delta s$$ éq 5.3.2.1 - 4

$$\delta s_{kl}\left[\delta_{ijkl}+\frac{1}{\alpha}\frac{Q^e}{Q}\delta_{ijkl}-\frac{1}{\alpha}\delta_{ijkl}-\frac{3}{2\alpha}\frac{Q^e}{Q^3}s_{kl}.s_{ij}\right]=2\mu\left[\delta_{ijkl}-\frac{3}{2\alpha}\frac{s_{kl}^e.s_{ij}}{Q^e Q}\right]\delta\tilde{\varepsilon}_{kl}$$ éq 5.3.2.1 - 5

or in tensorial writing:

$$\delta s\underbrace{\left[\frac{Q^e}{Q\alpha}I_4^d+I_4^d\left(1-\frac{1}{\alpha}\right)-\frac{3}{2\alpha}\frac{Q^e}{Q^3}s\otimes s\right]}_{G}=2\mu\underbrace{\left[I_4^d-\frac{3}{2\alpha}\frac{s^e\otimes s}{Q^e Q}\right]}_{H}\delta\tilde{\varepsilon}$$ éq 5.3.2.1 - 6

As $\delta s$ does not depend on $\delta\varepsilon_v$, one can confuse $\delta\tilde{\varepsilon}$ with $\delta\varepsilon$.

By means of the tensor of projection within the space of deviatoric stresses $D^1$ [éq 5.3.1.3 - 2], one can write:

$$\delta\varepsilon=\frac{D^1.G.H^{-1}}{2\mu}.\delta\sigma$$ éq 5.3.2.1 - 7

#### 5.3.2.2 Processing of the hydrostatic part

In tensorial writing, one has according to the equation [éq 5.3.1.2 - 2] the following relation:

$$I^d\,\delta P=k_0 P\,\delta\varepsilon_v-\frac{k_0}{k_{0s}}\frac{P}{p_c+p_{atm}}I^d\,\delta p_c$$ éq 5.3.2.2 - 1

knowing that at the critical point $\delta\varepsilon_v^p=0$ .

As $\delta P$ does not depend on $\delta\tilde{\varepsilon}$ then one can confuse $\delta\varepsilon_v$ with $\delta\varepsilon$.

$$I^d\,\delta P=k_0 P\,\delta\varepsilon-\frac{k_0}{k_{0s}}\frac{P}{p_c+p_{atm}}I^d\,\delta p_c$$ **éq 5.3.2.2 - 2**

By means of the tensor of projection within the space of hydrostatic stresses $D^2$ [éq 5.3.1.3 - 3], one can write:

$$I^d D^2\,\delta\sigma=k_0 P\,\delta\varepsilon-\frac{k_0}{k_{0s}}\frac{P}{p_c+p_{atm}}I^d\,\delta p_c$$

from where

$$\delta\varepsilon=\frac{I^d D^2}{k_0 P}\delta\sigma+\frac{I^d}{k_{0s}(p_c+p_{atm})}\delta p_c$$ éq 5.3.2.2 - 3

#### 5.3.2.3 tangent Operator

By combining the contributions of the two parts deviatoric and hydrostatic, one finds the writing of the tangent operator who connects the variation of the total stress to the variation of the total deflection at the critical point:

$$\delta\varepsilon=\left[\frac{D^1.G.H^{-1}}{2\mu}+\frac{I^d D^2}{k_0 P}\right].\delta\sigma+\frac{I^d}{k_{0s}(p_c+p_{atm})}\delta p_c$$

$$\delta\sigma_{ij}=A_{ijkl}\,\delta\varepsilon_{kl}-B_{ij}\,\delta p_c$$ éq 5.3.2.3 - 1

with

$$A_{ijkl}=\left[\frac{D^1.G.H^{-1}}{2\mu}+\frac{I^d D^2}{k_0 P}\right]^{-}$$ éq 5.3.2.3 - 2

and

$$B_{ij}=-\frac{I^d}{k_{0s}(p_c+p_{atm})}$$ **éq 5.3.2.3 - 3**

As it is necessary to deduce the variation from the stress of Bishop, one finds:

$$B_{ij}=-\frac{I^d}{k_{0s}(p_c+p_{atm})}-bS_{lq}$$ **éq 5.3.2.3 - 4**

### 5.3.3 If the hydrous criterion is reached

the variation of the elastic strain is written in the form:

$$\delta\varepsilon^e_{kl}=\delta\tilde{\varepsilon}^e_{kl}-\frac{1}{3}\delta\varepsilon^e_v\delta_{kl}$$ éq 5.3.3-1

is:

$$\delta\varepsilon^e_{kl}=\frac{\delta s_{kl}}{2\mu}-\frac{\delta P}{3k_0 P}\delta_{kl}-\frac{\delta p_c}{3k_{0s}(p_c+p_{atm})}\delta_{kl}$$ éq 5.3.3-2

In this case the plastic deviatoric strain is null thus the plastic strain has the following statement:

$$\delta\varepsilon^p_{kl}=-\frac{1}{3}\delta\varepsilon^p_v\delta_{kl}$$ éq 5.3.3-3

is:

$$\delta\varepsilon^p_{kl}=-\frac{\delta p_c}{3k_s(p_{c0}+p_{atm})}\delta_{kl}$$ éq 5.3.3-4

By combining each one of the components elastic and plastic one finds:

$$\delta\varepsilon_{kl}=\delta\varepsilon^e_{kl}+\delta\varepsilon^p_{kl}=\frac{\delta s_{kl}}{2\mu}-\frac{\delta P}{3k_0 P}\delta_{kl}-\frac{1}{3}\left(\frac{1}{k_{0s}(p_c+p_{atm})}+\frac{1}{k_s(p_{c0}+p_{atm})}\right)\delta p_c\delta_{kl}$$ éq the 5.3.3-5

By means of matrixes of projection within the space of deviatoric and hydrostatic stresses one leads to the following statement:

$$\delta\varepsilon_{kl}=\left(\frac{D^1_{ijkl}}{2\mu}-\frac{D^2_{ij}\delta_{kl}}{3k_0 P}\right)\delta\sigma_{ij}-\frac{1}{3}\left(\frac{1}{k_{0s}(p_c+p_{atm})}+\frac{1}{k_s(p_{c0}+p_{atm})}\right)\delta p_c\delta_{kl}$$ éq 5.3.3-6

thus one can write:

$$\delta\sigma_{ij}=\left(\frac{D^1_{ijkl}}{2\mu}-\frac{D^2_{ij}\delta_{kl}}{3k_0 P}\right)^{-1}\delta\varepsilon_{kl}+\frac{1}{3}\left(\frac{1}{k_{0s}(p_c+p_{atm})}+\frac{1}{k_s(p_{c0}+p_{atm})}\right)\left(\frac{D^1_{ijkl}}{2\mu}-\frac{D^2_{ij}\delta_{kl}}{3k_0 P}\right)^{-1}\delta_{kl}\delta p_c \qquad \text{éq 5.3.3-7}$$

one poses $A_{ijkl}=\left(\frac{D^1_{ijkl}}{2\mu}-\frac{D^2_{ij}\delta_{kl}}{3k_0 P}\right)$

or

$$A_{ijkl}=\frac{1}{2\mu}\begin{pmatrix}\frac{2}{3}+\frac{2\mu}{9k_0P} & -\frac{1}{3}+\frac{2\mu}{9k_0P} & -\frac{1}{3}+\frac{2\mu}{9k_0P} & 0 & 0 & 0\\ -\frac{1}{3}+\frac{2\mu}{9k_0P} & \frac{2}{3}+\frac{2\mu}{9k_0P} & -\frac{1}{3}+\frac{2\mu}{9k_0P} & 0 & 0 & 0\\ -\frac{1}{3}+\frac{2\mu}{9k_0P} & -\frac{1}{3}+\frac{2\mu}{9k_0P} & \frac{2}{3}+\frac{2\mu}{9k_0P} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} \qquad \textbf{éq 5.3.3-8}$$

and by deducing the stress from Bishop, one finds:

$$\delta\sigma'_{ij}=A^{-1}_{ijkl}\delta\varepsilon_{kl}+\left[\frac{1}{3}\left(\frac{1}{k_{0s}(p_c+p_{atm})}+\frac{1}{k_s(p_{c0}+p_{atm})}\right)A^{-1}_{ijkl}\delta_{kl}-bS_{lq}\right]\delta p_c \qquad \text{éq 5.3.3-9}$$

# 6 Abstract of the model of Barcelona

Modelizations THHM:

KIT_HHM and KIT_THHM (in this last case, there is no dependence of the mechanical characteristics with the temperature).

**Variables of entry:**

$$\sigma^{r}, p_c^+, p_{gz}^+, P_{cr}^-, p_{c0}^-, \ \Delta\varepsilon, \ \Delta p_c \text{and } \Delta p_{gz}$$

**Variables of output:**

1) $\Delta\sigma'$, more tangent operators (necessary to operator STAT_NON_LINE).
2) Local variables $P_{cr}^+$, newer variables $p_{c0}^+$ : threshold in suction and $P_s^+$ : pressure of cohesion, and indicators of mechanical and $I_1$ hydrous hardening $I_2$.

Elastic prediction:

$$P^e = P^- \frac{\exp\left(k_0 \Delta\varepsilon_v\right)}{\left(\dfrac{p_c + p_{atm}}{p^{-c} + p_{atm}}\right)^{k_0/k_{s_0}}} , \ s^e = s^- + 2\mu\Delta\tilde{\varepsilon}$$

1) $f_1 < 0$ and $f_2 < 0$ $\left(p_c < p_{c0}\right)$ : reversible behavior

$$P = P^e , \ s = s^e , \varepsilon^p = 0 \quad , P_{cr} = P_{cr}^- , p_{c0} = p_{c0}^-$$

1) $f_1 > 0$ or $f_2 > 0$ plasticization and mechanical and hydrous hardening

$$P = P^e \exp\left[-k_0 \Delta\varepsilon_v^p\right]$$

$$s = \frac{s^e}{1 + \dfrac{6\alpha\mu\Delta\varepsilon_v^p}{M^2\left(2\mathrm{P} - 2\mathrm{P}_{cr} + k_c p_c\right)}}$$

$$P_{cr} = P_{cr}^- \exp\left[k \Delta\varepsilon_v^p\right] ,$$

$$p_{c0} + p_{atm} = \left(p_{c0} + p_{atm}\right) \exp\left[k_s \Delta\varepsilon_v^p\right]$$

the single unknown $\Delta\varepsilon_v^p$ is determined by $f_1 = 0$ (one has then: $\Delta\tilde{\varepsilon}^p = \dfrac{2\mathrm{Q}\alpha\Delta\varepsilon_v^p}{M^2\left(2\mathrm{P} + P_s - 2\mathrm{P}_{cr}\right)}$ )

or $f_2 = 0$ (and $\Delta\tilde{\varepsilon}^p = 0$ )

**Note:**

*The stress resulting from the data of the model of Barcelona east* $\sigma=\sigma_{tot}+p_{gz}1^d$ *, it will be thus the variable used in the routine describing the behavior, the stress of output provided to* `STAT_NON_LINE` *being the stress of Bishop:* $\sigma'=\sigma_{tot}-\sigma_p$ *.*

Tangent operators:

The tangent operator of the generalized stresses is implemented in THHM under the name $D\Sigma DE$ and is partitionné in several blocks. The components concerned with the model are $\frac{\partial\dot{\sigma}'}{\partial\varepsilon}$ and $\frac{\partial\dot{\sigma}'}{\partial\dot{p}_c}$ of the blocks [ $DMECDE$ ] and [ $DMECP1$ ] corresponding with: $\begin{bmatrix}\frac{\partial\dot{\sigma}'}{\partial\dot{\varepsilon}}\\ \frac{\partial\dot{\sigma}_p}{\partial\dot{\varepsilon}}\end{bmatrix},\begin{bmatrix}\frac{\partial\dot{\sigma}'}{\partial\dot{p}_c} & \frac{\partial\dot{\sigma}'}{\partial\nabla\dot{p}_c}\\ \frac{\partial\dot{\sigma}_{p'}}{\partial\dot{p}_c} & \frac{\partial\dot{\sigma}_{p'}}{\partial\nabla\dot{p}_c}\end{bmatrix}$.

# 7 Implemented of the model

## 7.1 Material characteristics

the use of the model of `BARCELONE` requires to enrich the data of the model by `CAM_CLAY` by additional data specific to the unsaturated soils. This is concretized by the simultaneous adoption of two key words `CAM_CLAY` and `BARCELONE` under command `DEFI_MATERIAU`.

## 7.2 Initialization of computation

It is necessary that the initial state of the material either plastically acceptable (the stress and the capillary pressure are thus such as the point of initial loading or inside the surface of load). It is necessary thus on the one hand that suction is lower than the hydrous threshold, and on the other hand that the stress is inside the ellipse defined in the plane of initial suction. In particular, if the initial mechanical loading is purely hydrostatic, it must be understood between the limits represented by cohesion ( $-k_c p_c$ ) and the pressure of consolidation ( $2P_{cr}$ ). The stress $\sigma$ used to describe the behavior (forced total plus gas pressure) is different from the stress to initialize in `ETAT_INIT` (forced of Bishop $\sigma'$). The relation between the two types of stress is:

$$\dot{\sigma}'=\dot{\sigma}+\left[(b-1)\dot{p}_{gz}I-bS_{lq}\dot{p}_c\right]$$

## 7.3 Local variables in output

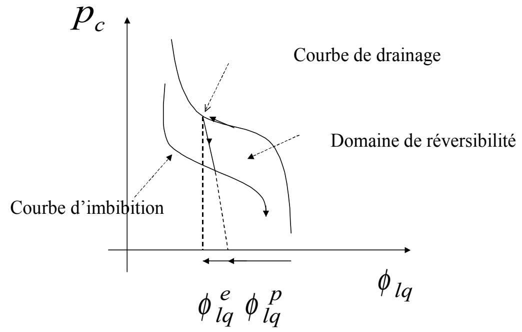
The model produced five local variables:

$V_1=P_{cr}$ : critical pressure
$V_2=I_1$ : mechanical indicator of irreversibility
$V_3=p_{c0}$ : hydrous threshold of irreversibility
$V_4=I_2$ : hydrous indicator of irreversibility
$V_5=P_S$ : pressure of cohesion

# 8 Developmental perspectives of the model

One of the phenomena not studied in the model original of Barcelona is it not reversibility of the capillary curve of pressure [Figure 8-a] and its dependence with the stress state. This is treated by Dangla and saddle-point. [bib2] while integrating the model of Barcelona in a frame poroplastic with the introduction of the water content like additional poroplastic variable, whose evolution is directly connected not only to the capillary variation of pressure via the curve of drainage-imbibition, but also to the mechanical evolution of the medium. It is necessary to distinguish two coupled distinct aspects there but nevertheless phenomenon. Nonthe reversibility of the curved drainage-imbibition is a phenomenon purely hydraulic and thus independent of the mechanical law adopted in a modelization THHM, but this curve thus depends on the index of the vacuums of the mechanical state of the medium. The partition of the water content partly elastic and plastic and of the thermodynamic considerations [bib2] makes it possible to deduce the evolution at the same time from the water content (and thus of the degree of saturation) and from the stress according to the strain and the capillary pressure. For example, the evolution in the field of reversibility is given by:

$$d\varphi_{lq}^{e}=-N(\varepsilon^{e},p_{c})\,dp_{c}+b(\varepsilon^{e},p_{c})\,dtr(\varepsilon^{e})$$
$$dP=b(\varepsilon^{e},p_{c})\,dp_{c}+K(\varepsilon^{e},p_{c})\,dtr(\varepsilon^{e})$$




**Appear 8-a**

Where $(N,b)$ are the generalized coefficients of Biot [bib6]. To enrich the model by Barcelona in this meaning thus implies two separate developments:

1) The introduction of a curve of drainage-imbibition into developments THHM.
2) The completeness of the model of Barcelona by the computation of the degree of saturation besides the stress.

# 9 Bibliography

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# 10 Checking

the constitutive law of BARCELONE is checked by the cases following tests:

| | | |
|---|---|---|
| WTNV123 | triaxial Compression test with built-in suction with the model of Barcelona | [V7.31.123] |
| WTNV124 | Test of desaturation-consolidation with the model of Barcelona | [V7.31.124] |
| WTNV126 | Response at mixed paths of saturation-consolidation with the model of Barcelona | [V7.31.126] |

# 11 Description of the versions of the document

| Version Aster | Author (S) Organization (S) | Description of the modifications |
|---|---|---|
| 7.4 | G.Debruyne, J.El-Gharib EDF-R&D/AMA | initial Text |