Integration of the elastoplastic structural mechanics behaviors of Drucker-Prager, associated (DRUCK_PRAGER) and non-aligned (DRUCK_PRAG_N_A) and postprocessings

Summarized:

This document describes the principles of several developments concerning elastoplastic constitutive law of Drucker-Prager in associated version (DRUCK_PRAGER) and non-aligned (DRUCK_PRAG_N_A).

One is interested initially in integration itself of the model then, this model being lenitive, with an indicator of localization of Rice and finally with the sensitivity analysis by direct differentiation for this model.
For the integration of the model, one uses an implicit scheme.

Contents

Contents

1 Introduction

the model of Drucker-Prager makes it possible to model in an elementary way the elastoplastic behavior of the concrete or certain soils. Compared to the plasticity of Von-Put with isotropic hardening, the difference lies in the presence of a term in $Tr(\boldsymbol{\sigma})$ the formulation of the threshold and of a non-zero spherical component of the tensor of plastic strains.

In Code_Aster, the model exists in the associated version (`DRUCK_PRAGER)` and non-aligned (`DRUCK_PRAG_N_A)`, more adapted for certain soils because it makes it possible to better take into account dilatancy.

This note gathers the theoretical aspects several developments carried out in the code around this model: its integration according to an implicit scheme in time, an indicator of localization of Rice and the sensitivity analysis by direct differentiation. The isotropic material is supposed. The indicator of Rice and sensitivity analysis do not operate under the assumption of the plane stresses.

The theory and the developments were made for two types of function of hardening: linear and parabolic, this function being in all the cases constant beyond of a cumulated plastic strain “ultimate”.

2 Integration of the constitutive law of Drucker-Prager

2.1 Notations

the mechanical stresses are counted positive in tension, the positive strains in extension.

$\boldsymbol{u}$	displacements of the deviative squelette of u_x, u_y, u_z
$\boldsymbol{\varepsilon}=\frac{1}{2}(\nabla \boldsymbol{u}+\nabla^T \boldsymbol{u})$	components tensor of the linearized
$\boldsymbol{e}=\boldsymbol{\varepsilon}-\frac{Tr(\boldsymbol{\varepsilon})}{3}\boldsymbol{I}$	strains of the strains
$\varepsilon_v=Tr(\boldsymbol{\varepsilon})$	traces strains: variation of Tensor
$\boldsymbol{\varepsilon^p}$	volume of plastic strains,
$\varepsilon_v^p=Tr(\boldsymbol{\varepsilon^p})$	plastic variation of volume.
$\boldsymbol{e^p}$	tensor deviator of
p	plastic strains the cumulated
$\boldsymbol{\sigma}$	plastic strain of the stresses
$\boldsymbol{s}=\boldsymbol{\sigma}-\frac{Tr(\boldsymbol{\sigma})}{3}\boldsymbol{I}$	deviator of the stresses
$\sigma_{eq}=\sqrt{\frac{3}{2}\boldsymbol{s}:\boldsymbol{s}}$	Equivalent stress of Von Mises
$I_1=Tr(\boldsymbol{\sigma})$	traces stresses
E_0	Modulus Young
ν_0	Poisson's ratio
φ	Friction angle
c	Cohesion
ψ^0	initial Angle of dilatancy

One poses $2\mu=\frac{E_0}{1+\nu_0}$ and $K=\frac{E_0}{3(1-2\nu_0)}$

2.2 Formulation in version associated

2.2.1 Statement with the behavior

$\boldsymbol{\sigma}$ is the tensor of the stresses, which depends only on $\boldsymbol{\varepsilon}$ and its history. One considers the criterion of the Drücker-Prager type:

$$F(\boldsymbol{\sigma}, p) = \sigma_{eq} + A I_1 - R(p) \leq 0 \qquad (2.2.1\text{-}1)$$

where A is a given coefficient and R is a function of the cumulated plastic strain p (function of hardening), of type linear or parabolic:

- linear hardening

$$R(p) = \sigma_Y + h.p \quad \text{if} \quad p \in [0, p_{ultm}]$$
$$R(p) = \sigma_Y + h.p_{ultm} \quad p > p_{ultm}$$

the coefficients h , p_{ultm} and σ_Y are given.

(2.2.1-2)

- parabolic hardening

$$R(p) = \sigma_Y \left(1 - \left(1 - \sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right) \frac{p}{p_{ultm}}\right)^2 \quad \text{if} \quad p \in [0, p_{ultm}]$$
$$R(p) = \sigma_{Y_{ultm}} \quad p > p_{ultm}$$

the coefficients σ_{Yultm} , p_{ultm} and σ_Y are given.

(2.2.1-3)

Remark 1:

One can be given instead of A and σ_Y the binding fraction c and the friction angle φ :

$$A = \frac{2 \sin \varphi}{3 - \sin \varphi}$$
$$\sigma_Y = \frac{6 c \cos \varphi}{3 - \sin \varphi}$$

Notice 2:

One chose in this document to privilege the variable p . The cumulated plastic strain of shears $\gamma^p = p\sqrt{3/2}$ also is very much used in soil mechanics.

By considering an associated version one supposes that the potential of dissipation follows the same statement as that of the surface of load F . Yielding is summarized then with:

$$d\boldsymbol{\varepsilon}^{\boldsymbol{p}} = d\lambda \frac{\partial F(\boldsymbol{\sigma}, p)}{\partial \boldsymbol{\sigma}} \qquad (2.2.1\text{-}4)$$

with:

$$d\lambda \geq 0 \quad ; \quad F\cdot d\lambda = 0 \quad ; \quad F \leq 0 \tag{2.2.1-5}$$

the model of normality compared to the generalized force R gives the equality between the increment of cumulated plastic strain and the increment of the multiplier λ :

$$dp = -d\lambda \frac{\partial F(\boldsymbol{\sigma}, p)}{\partial R} = d\lambda \tag{2.2.1-6}$$

2.2.2 analytical Resolution of the mechanical formulation

One is placed in this chapter in the frame of finished increase. The integration of the model follows a pure implicit scheme, and the resolution is analytical. The finished increment of strain $\Delta\boldsymbol{\varepsilon}$ known and is provided by the iteration of total Newton. One uses by convention the following notations: an index – to indicate a component at the beginning of step of loading, any index for a component at the end of the step of loading, and the operator Δ to indicate the increase in a component. The equations translating the elastic behavior are written then:

$$\boldsymbol{s} = \boldsymbol{s}^- + 2\mu(\Delta\boldsymbol{e} - \Delta\boldsymbol{e}^p) = \boldsymbol{s}^e - 2\mu\Delta\boldsymbol{e}^p \tag{2.2.2-1}$$

$$I_1 = I_1^- + 3\mathrm{K}(\Delta\varepsilon_v - \Delta\varepsilon_v^p) = I_1^e - 3\mathrm{K}\,\Delta\varepsilon_v^p \tag{2.2.2-2}$$

the equations (2.2.1-4) and (2.2.1-6), taking into account (2.2.1-1), give:

$$\Delta\boldsymbol{\varepsilon}^p = \Delta p\left(\frac{\partial\sigma_{eq}}{\partial\boldsymbol{\sigma}} + A\frac{\partial I_1}{\partial\boldsymbol{\sigma}}\right) = \Delta p\left(\frac{3}{2}\frac{\boldsymbol{s}}{\sigma_{eq}} + A\boldsymbol{I}\right) \tag{2.2.2-3}$$

From where:

$$\Delta\varepsilon_v^p = 3A\Delta p \tag{2.2.2-4}$$

$$\Delta\boldsymbol{e}^p = \frac{3}{2}\frac{\mathbf{s}}{\sigma_{eq}}\Delta p \tag{2.2.2-5}$$

If the increment $\Delta\boldsymbol{e}^p$ is non-zero, the increment of cumulated plastic strain can be also written:

$$\Delta p = \sqrt{\frac{2}{3}\Delta\boldsymbol{e}^p : \Delta\boldsymbol{e}^p} \tag{2.2.2-6}$$

By combining the equations (2.2.2-1) and (2.2.2-5) one finds:

$$\mathbf{s}\left(1 + \frac{3\mu\Delta p}{\sigma_{eq}}\right) = \mathbf{s}^{\mathrm{e}} \tag{2.2.2-7}$$

from where:

$$\sigma_{eq} + 3\mu\cdot\Delta p = \sigma_{eq}^e \tag{2.2.2-8}$$

what leads to:

$$\mathbf{s}\frac{\sigma_{eq}^{e}}{\sigma_{eq}}=\mathbf{s}^{\mathbf{e}} \quad (2.2.2\text{-}9)$$

By respectively combining the equations (2.2.2-7) and (2.2.2-8), and the equations (2.2.2-2) and (2.2.2-4), one obtains:

$$\begin{cases}\mathbf{s}=\mathbf{s}^{\mathbf{e}}\left(1-\frac{3\mu}{\sigma_{eq}^{e}}\Delta p\right)\\ I_1=I_1^{e}-9KA\Delta p\end{cases} \quad (2.2.2\text{-}10)$$

By reinjecting the equation on I_1 and the relation $\sigma_{eq}=\sigma_{eq}^{e}-3\mu\cdot\Delta p$ in the formulation of the threshold, one obtains the scalar equation in Δp :

$$\sigma_{eq}^{e}+AI_1^{e}-\Delta p(3\mu+9KA^2)-R(p^-+\Delta p)=0 \quad (2.2.2\text{-}11)$$

It is supposed that: $F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)>0$.

To continue the resolution, one must now distinguish several cases:

1) **Case where** $p^->p_{ultm}$
One a: $R(p^-+\Delta p)=R(p^-)$
the scalar equation thus becomes: $F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)-\Delta p\left(3\mu+9KA^2\right)=0$
One finds:

$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)}{3\mu+9KA^2} \quad (2.2.2\text{-}12)$$

2) **Case where** $p^-\leq p_{ultm}$

2a) Hardening linear

One a: $R(p^-+\Delta p)=R(p^-)+h\Delta p$

the scalar equation thus becomes: $F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)-\Delta p\left(3\mu+9KA^2+h\right)=0$
One finds:

$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)}{3\mu+9KA^2+h} \quad (2.2.2\text{-}13)$$

2b) Hardening parabolic

While expressing in the same way $R(p^-+\Delta p)$ according to $R(p^-)$ and of Δp , one finds that the scalar equation is written:

$$F\left(\boldsymbol{\sigma}^{\boldsymbol{e}},p^-\right)+B\Delta p+G\Delta p^2=0$$

with:

$$\begin{cases} G=-\dfrac{\sigma_Y}{p_{ultm}^2}\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_Y}}\right)^2 \\ B=-3\mu-9KA^2+\dfrac{2\sigma_Y}{p_{ultm}}\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_Y}}\right)\left(1-\left(1-\sqrt{\dfrac{\sigma_{Yultm}}{\sigma_Y}}\right)\dfrac{p^-}{p_{ultm}}\right) \end{cases}$$

The only positive root of the polynomial is:

$$\Delta p=\frac{-B-\sqrt{B^2-4G\cdot F\left(\boldsymbol{\sigma}^e,\, p^-\right)}}{2G} \qquad (2.2.2\text{-}14)$$

2c) final Checking: Case where $(p^- + \Delta p) > p_{ultm}$

In the two preceding cases, once Δp calculated, it should be checked that $p^- + \Delta p \leq p_{ultm}$. If this inequality is not satisfied, one has then:

$$R(p^- + \Delta p) = R(p_{ultm})$$

The scalar equation thus becomes:

$$F\left(\boldsymbol{\sigma}^e,\, p_{ultm}\right)-\Delta p\left(3\mu+9KA^2\right)=0$$

$$\Delta p=\frac{F\left(\boldsymbol{\sigma}^e,\, p_{ultm}\right)}{3\mu+9KA^2} \qquad (2.2.2\text{-}15)$$

the principle of the analytical resolution presented above is equivalent to determine the point $\left(I_1,\mathbf{s}\right)$ like the projection of the point $\left(I_1^e,\mathbf{s}^{\mathbf{e}}\right)$ on the criterion (prediction plastic elastic-correction). This method thus comes from the flow model approximated on a finished increment, and can be represented by the following graph:

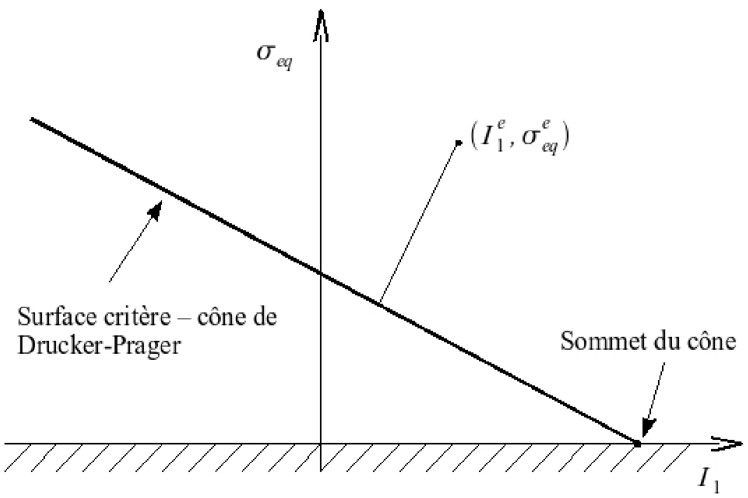


Figure 2.2.2-1: projection on the criterion.

3) Projection at the top of the cone

the integration of the model on a finished Δt increment can be complicated when the stress state is close to the top of the cone (see Figure 2.2.2-1), because of the nonsmooth character of surface criterion. There are then two cases:

- case of a pure hydrostatic state,
- case of projection in an NON-acceptable field.

In the typical case of a pure hydrostatic state, the derivative of the von Mises stress σ_{eq} compared to σ is not defined. The flow model (2.2.2-3) is undetermined (there is indeed a cone of possible norms to the criterion), and the equations (2.2.2-5), (2.2.2-7), (2.2.2-8), (2.2.2-9) cannot be written. There remains the definition of Δp on the trail of stresses (equation 2.2.2-4). As in the more general case, one must distinguish several cases:

1) **Case where** $p^{-}>p_{ultm}$: $R(p^{-}+\Delta p)=R(p^{-})$
The scalar equation with $\sigma_{eq}=0$ becomes : $A I_1^e-\Delta p\cdot 9KA^2=F(\boldsymbol{\sigma}^e, p^{-})-\Delta p\cdot 9KA^2=0$
One finds:

$$\Delta p=\frac{I_1^e}{9KA} \quad (2.2.2\text{-}16)$$

2) **Case where** $p^{-}\le p_{ultm}$

2a) Hardening linear

One a: $R(p^{-}+\Delta p)=R(p^{-})+h\Delta p$

the scalar equation with $\sigma_{eq}=0$ becomes:
$A I_1^e-\Delta p\cdot 9KA^2-R(p^{-})+h\Delta p=F(\boldsymbol{\sigma}^e, p^{-})-\Delta p\cdot 9KA^2=0$
One finds then:

$$\Delta p=\frac{A I_1^e}{9KA^2+h} \quad (2.2.2\text{-}17)$$

2b) Hardening parabolic

While expressing $R(p^{-}+\Delta p)$ according to $R(p^{-})$ and of Δp, one still finds the solution (2.2.2-14):

with the value of B modified compared to the preceding case:

$$B=-9KA^2+\frac{2\sigma_Y}{p_{ultm}}\left(1-\sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right)\left(1-\left(1-\sqrt{\frac{\sigma_{Yultm}}{\sigma_Y}}\right)\frac{p^{-}}{p_{ultm}}\right)$$

2c) final Checking: Case where $(p^{-}+\Delta p)>p_{ultm}$

In the cases 2a) and 2b), if the inequality $p^{-}+\Delta p\le p_{ultm}$ is not satisfied, one a:

$$R(p^- + \Delta p) = R(p_{ultm})$$

the increment Δp is given by the equation (2.2.2-16).

Because of the incremental resolution, it may be that the found solution is not acceptable, with $\sigma_{eq}<0$. That can happen when the stress state at time t^- is close to the top of the cone.

One then chooses to project the stress state found by elastic prediction on the top of the cone, that is to say to refer to a purely hydrostatic stress state. One makes a control a posteriori admissibility of the solution $\left(I_1, \mathbf{s}\right)$, and one makes possibly the correction.
In the details:

i) One brings up to date the stress state by the means as of equations (2.2.2-12), (2.2.2-13), (2.2.2-14), (2.2.2-15).
ii) One controls that the solution $\left(I_1, \mathbf{s}\right)$ found either acceptable, or that $\sigma_{eq}<0$ where, in an equivalent way, that I_1 or inside surface criterion:
$$I_1 \leqslant \frac{R(p)}{A}$$
iii) If this condition is not checked, one imposes the checking of the criterion with $\sigma_{eq}=0$ (top of the cone): $I_1 = \frac{R(p)}{A} \quad \Rightarrow A \cdot I_1 - R(p) = F(\sigma, R) = 0$
iv) One renews then the solution with the equations (2.2.2-16), (2.2.2-17), (2.2.2-14).

2.2.3 Computation of the tangent operator

2.2.3.1 total Computation of the tangent operator

One seeks to calculate the coherent matrix: $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}+\frac{1}{3}\mathbf{I}\otimes\frac{\partial I_1}{\partial \boldsymbol{\varepsilon}}$

By deriving the system of equations (2.2-7), one obtains:

$$\begin{cases}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial \mathbf{s}^{\mathbf{e}}}{\partial \boldsymbol{\varepsilon}}\left(1-3\frac{\mu}{\sigma_{eq}^{e}}.\Delta p\right)+\frac{3\mu}{\left(\sigma_{eq}^{e}\right)^2}.\Delta p.\left(\mathbf{s}^{\mathbf{e}}\otimes\frac{\partial \sigma_{eq}^{e}}{\partial \boldsymbol{\varepsilon}}\right)-\frac{3\mu}{\sigma_{eq}^{e}}.\left(\frac{\mathbf{s}^{\mathbf{e}}\otimes\partial\Delta p}{\partial \boldsymbol{\varepsilon}}\right)\\ \frac{\partial I_1}{\partial \boldsymbol{\varepsilon}}=\frac{\partial I_1^{e}}{\partial \boldsymbol{\varepsilon}}-9KA\frac{\partial \Delta p}{\partial \boldsymbol{\varepsilon}}\end{cases}$$

éq 2.2.3-1

Statement of $\frac{\partial \mathbf{s}^{\mathbf{e}}}{\partial \boldsymbol{\varepsilon}}$

$$\frac{\partial s_{ij}^{e}}{\partial \varepsilon_{pq}}=2\mu\left(\delta_{ip}\delta_{jq}-\frac{1}{3}\delta_{ij}\delta_{pq}\right)$$

Statement of $\frac{\partial I_1^{e}}{\partial \varepsilon}$

$$\frac{\partial I_1^{e}}{\partial \varepsilon_{\mathrm{pq}}}=3\mathrm{K}\,\delta_{pq}$$

Computation of $\frac{\partial \sigma_{eq}^{e}}{\partial \varepsilon}$

$$\frac{\partial \sigma_{\mathrm{eq}}^{e}}{\partial \varepsilon_{\mathrm{pq}}}=\frac{3\mu}{\sigma_{eq}^{e}}s_{pq}^{e}$$

Computation of $\frac{\partial \Delta p}{\partial \varepsilon}$

$$\boxed{\frac{\partial \Delta p}{\partial \varepsilon_{\mathrm{pq}}}=-\frac{1}{T(\Delta p)}.\left(\frac{3\mu}{\sigma_{eq}^{e}}s_{pq}^{e}+3AK\delta_{pq}\right)}$$

with:

$$T(\Delta p)=\begin{cases}-\left(3\mu+9KA^2\right) & \text{dans le cas } p^{-}+\Delta p\geq p_{ultm} \quad (\text{écrouissage linéaire ou parabolique})\\ -\left(3\mu+9KA^2+h\right) & \text{dans le cas } p^{-}+\Delta p< p_{ultm} \quad (\text{écrouissage linéaire})\\ B+2\mathrm{G}\Delta\mathrm{p} & \text{dans le cas } p^{-}+\Delta p< p_{ultm} \quad (\text{écrouissage parabolique})\end{cases}$$

where B and G have the same statement as in the paragraph [§2.2].

Statement supplements

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}}=\left(1-\frac{3\mu}{\sigma_{eq}^{e}}\Delta p\right)\frac{\partial\mathbf{s}^{e}}{\partial\boldsymbol{\varepsilon}}+\left(\frac{3\mu}{\sigma_{eq}^{e}}\right)^{2}\left(\frac{\Delta p}{\sigma_{eq}^{e}}+\frac{1}{T}\right)\mathbf{s}^{e}\otimes\mathbf{s}^{e}+\frac{9\mu AK}{T\sigma_{eq}^{e}}(\mathbf{s}^{e}\otimes\mathbf{I}+\mathbf{I}\otimes\mathbf{s}^{e})+\left[K+\frac{9K^{2}A^{2}}{T}\right]\mathbf{I}\otimes\mathbf{I}$$

2.2.3.2 initial Computation of the tangent operator

One seeks has to express $\frac{\partial\sigma^{-}}{\partial\varepsilon^{-}}$. For that one will seek to calculate the tangent operator by a computation of velocity: $\frac{\partial\dot{\sigma}}{\partial\dot{\varepsilon}}$.

On the basis of the statement: $\dot{F}=\frac{\partial F}{\partial\dot{\boldsymbol{\sigma}}}+\frac{\partial F}{\partial p}\dot{p}=0$ it is shown that:

$$\dot{p}=\frac{3\mu}{\sigma_{eq}D}s.\dot{\varepsilon}+\frac{3\,AK}{D}\dot{\varepsilon}_{v}\quad\text{with}\quad D=3\mu+9KA^{2}+\frac{\partial R}{\partial p}$$

statements: $\dot{\boldsymbol{\sigma}}=\mathbf{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^{p}\right)$ and $\dot{\boldsymbol{\varepsilon}}^{p}=\dot{p}\frac{\partial\mathbf{F}}{\partial\boldsymbol{\sigma}}$ it is shown then that:

$$\frac{\partial\dot{\boldsymbol{\sigma}}}{\partial\dot{\boldsymbol{\varepsilon}}}=H-\left(\frac{3\,\mu}{\sigma_{eq}}\boldsymbol{s}+3\,AK\,\boldsymbol{I}\right)\frac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}}$$

who is not other than the form of the coherent matrix of the total system of the preceding paragraph where $\Delta p=0$.

2.3 Formulation in non-aligned version

the non-aligned version of the model Drucker-Prager introduced into Code_Aster does not have as a claim to model a realistic physical behavior finely. The goal is to represent most simply possible physics (coarsely) realistic, in particular in the case of the soil mechanics for which the angle of dilatancy varies with the plastic strain.

The plastic potential is thus different from the surface of load in this new formulation. Numerical integration was introduced only for the statement of behavior with parabolic hardening.

The plastic potential is the following: $G(\boldsymbol{\sigma},p)=\sigma_{eq}+\beta(p)I_{1}$

where $\beta(p)$ is a function which decrease linearly with the evolution of the plastic strain according to the relation

$$\beta(p)=\begin{cases}\beta(\psi^{0})\left(1-\dfrac{p}{p_{ult}}\right) & \text{si}\quad p\in\left[0,p_{ult}\right]\\ 0 & \text{si}\quad p>p_{ult}\end{cases}$$

where ψ^0 indicates the initial angle of dilatancy and $\beta(\psi^0)=\frac{2\sin(\psi^0)}{3-\sin(\psi^0)}$.

Yielding is written now

$$d\,\varepsilon_{ij}^{p}=dp\frac{\partial G(\boldsymbol{\sigma}\,,p)}{\partial\sigma_{ij}}$$

knowing that one always has the criterion defining the surface of load: $F(\sigma\,,p)=\sigma_{eq}+AI_1-R(p)\le 0$

2.3.1 Analytical resolution

the method of resolution being similar to that of chapter 2.2.2 one points out below only the statements of the new equations

$$\begin{cases}\Delta e_{ij}^{p} & = & \frac{3}{2}\frac{s_{ij}}{\sigma_{eq}}\Delta p\\ \Delta\varepsilon_{V}^{p} & = & 3\beta(p)\Delta p\end{cases}$$

$$\begin{cases}s_{ij}=s_{ij}^{e}\left(1-3\mu\frac{\Delta p}{\sigma_{eq}^{e}}\right)\\ I_1=I_1^{e}-9K\beta(p)\Delta p\end{cases}$$

2.3.1.1 Case where $p^{-}>p_{ult}$

$$\Delta p=\frac{F(\sigma^{e},p^{-})}{3\mu}$$

2.3.1.2 Case where $p^{-}\le p_{ult}$

In this case Δp is solution of a polynomial equation of the second order of which the roots will depend on the increment of strain and the data characterizing materials parameters. The polynomial in question is the following

$$F(\sigma^{e},p^{-})+C^{1}\Delta p+C^{2}\Delta p^{2}=0$$

where $F(\sigma^{e},p^{-})>0$, and the two constants C^1 and C^2 are defined by

$$C^1=-3\mu-9KA\beta(p^{-})+2\frac{\sigma_Y}{p_{ult}}\left(1-\left(1-\sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)\frac{p^{-}}{p_{ult}}\right)\left(1-\sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)$$

$$C^2=-\frac{\sigma_Y}{p_{ult}^{2}}\left(1-\sqrt{\frac{\sigma_{Yult}}{\sigma_Y}}\right)^{2}+9AK\frac{\beta(\psi^0)}{p_{ult}}$$

the root Δp is then characterized according to the following code:

1 so $C^2<0$ then $\Delta p=\frac{-C^1-\sqrt{(C^1)^2-4F(\boldsymbol{\sigma}^e,p^-)C^2}}{2\,C^2}$

2 if $C^2>0$ and $F(\boldsymbol{\sigma}^e,p^-)>\frac{(C^1)^2}{4C^2}$ then there is no solution. A recutting of time step is possible if the request were made in command `STAT_NON_LINE`.

3 if $C^2>0$ and $F(\boldsymbol{\sigma}^e,p^-)<\frac{(C^1)^2}{4C^2}$ $C^1<0$ then the polynomial admits two solutions. One chooses smallest positive of them. $\Delta p=\frac{-C^1-\sqrt{(C^1)^2-4F(\boldsymbol{\sigma}^e,p^-)C^2}}{2\,C^2}$

4 if $C^2>0$ and $F(\boldsymbol{\sigma}^e,p^-)<\frac{(C^1)^2}{4C^2}$ $C^1>0$ then there is no solution. A recutting of time step is possible if the request were made in command `STAT_NON_LINE`.

2.3.2 Computation of the tangent operator

the formulation is modified very little compared to the associated case: equations 2.2.3-1 become:

$$\begin{cases}\frac{\partial s}{\partial\varepsilon}=\frac{\partial s^e}{\partial\varepsilon}\left(1-\frac{3\mu}{\sigma^e_{eq}}.\Delta p\right)+\frac{3\mu}{\left(\sigma^e_{eq}\right)^2}.\Delta p.\left(s^e\otimes\frac{\partial\sigma^e_{eq}}{\partial\varepsilon}\right)-\frac{3\mu}{\sigma^e_{eq}}.\left(s^e\otimes\frac{\partial\Delta p}{\partial\varepsilon}\right)\\ \frac{\partial I_1}{\partial\varepsilon}=\frac{\partial I_1^e}{\partial\varepsilon}-9\mathrm{K}\left(\beta-\frac{\beta(\Psi^0)\Delta p}{p_{ult}}\right)\frac{\partial\Delta p}{\partial\varepsilon}\end{cases}$$

2.3.2.1 Statement of $\frac{\partial s^e_{ij}}{\partial\varepsilon_{pq}}$

$$\frac{\partial s^e_{ij}}{\partial\varepsilon_{pq}}=2\mu\left(\delta_{ip}\delta_{jq}-\frac{1}{3}\delta_{ij}\delta_{pq}\right)$$

2.3.2.2 Statement of $\frac{\partial I_1^e}{\partial\varepsilon_{pq}}$

$$\frac{\partial I_1^e}{\partial\varepsilon_{pq}}=3K\delta_{pq}$$

2.3.2.3 Computation of $\frac{\partial\sigma^e_{eq}}{\partial\varepsilon_{pq}}$

$$\frac{\partial\sigma^e_{eq}}{\partial\varepsilon_{pq}}=\frac{3\mu}{\sigma^e_{eq}}s^e_{pq}$$

2.3.2.4 Computation of $\frac{\partial \Delta p}{\partial \varepsilon_{pq}}$

$$\frac{\partial \Delta p}{\partial \varepsilon_{pq}}=-\frac{1}{T(\Delta p)}\cdot\left(\frac{3\mu}{\sigma_{eq}^{e}}s_{pq}^{e}+3\,AK\delta_{pq}\right)$$

with:

$$T(\Delta p)=\begin{cases}-3\mu & \text{si } p^{-}+\Delta p\geq p_{ult}\\ C^{1}+2C^{2}\Delta p & \text{si } p^{-}+\Delta p<p_{ult}\end{cases}$$

where C^1 and C^2 are constants defined in paragraph 2.3.

2.3.2.5 Statement supplements

$$\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}+\frac{1}{3}.\mathbf{I}\otimes\frac{\partial I_1}{\partial \boldsymbol{\varepsilon}}$$

$$\begin{cases}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\varepsilon}}=\frac{\partial \mathbf{s^e}}{\partial \boldsymbol{\varepsilon}}\left(1-\frac{3\mu}{\sigma_{eq}^{e}}.\Delta p\right)+\frac{3\mu}{\left(\sigma_{eq}^{e}\right)^{2}}.\Delta p.\left(\mathbf{s^e}\otimes\frac{\partial \sigma_{eq}^{e}}{\partial \boldsymbol{\varepsilon}}\right)-\frac{3\mu}{\sigma_{eq}^{e}}.\left(\mathbf{s^e}\otimes\frac{\partial \Delta p}{\partial \boldsymbol{\varepsilon}}\right)\\ \frac{\partial I_1}{\partial \boldsymbol{\varepsilon}}=\frac{\partial I_1^{e}}{\partial \boldsymbol{\varepsilon}}-9K\left(\beta-\frac{\beta(\Psi^{0})\Delta p}{p_{ult}}\right)\frac{\partial \Delta p}{\partial \boldsymbol{\varepsilon}}\end{cases}$$

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{pq}}=\left(1-\frac{3\mu}{\sigma_{eq}^{e}}\Delta p\right)\cdot\frac{\partial s_{ij}^{e}}{\partial \varepsilon_{pq}}+\frac{1}{3}\frac{\partial I_1^{e}}{\partial \varepsilon_{pq}}\delta_{ij}+\frac{\partial \sigma_{eq}^{e}}{\partial \varepsilon_{pq}}\left(\frac{3\mu}{\left(\sigma_{eq}^{e}\right)^{2}}s_{ij}^{e}\Delta p\right)+\frac{\partial \Delta p}{\partial \varepsilon_{pq}}\left(-3\mu\frac{s_{ij}^{e}}{\sigma_{eq}^{e}}-3K\beta(p)\delta_{ij}+3K\frac{\beta(\psi^{0})}{p_{ult}}\Delta p\delta_{ij}\right)$$

2.4 Local variables of the Drucker-Prager models associated and nonassociated

These models comprise 3 local variables:

- $V1$ is the cumulated plastic deviatoric strain p
- $V2$ is the cumulated plastic voluminal strain $\sum \Delta\varepsilon_V^p$
- $V3$ is the indicator of state (1 if $\Delta p>0$, 0 in the contrary case).

3 Indicator of localization of Rice for the model Drucker-Prager

One defines the indicator of localization of the criterion of Rice in the frame of the Drucker-Prager constitutive law. But the definition of an indicator of localization perhaps used, a more general way, in studies in fracture mechanics, damage mechanics, theory of the bifurcation, soil mechanics and rock mechanics (and overall in the frame as of materials with lenitive constitutive law).

This definite indicator a state from which the evolution of the studied mechanical system (equations, of equilibrium, constitutive law) can lose its character of unicity. This theory allows, in other words:

1. the computation of the possible state of initiation of the localization which is perceived like the limit of validity of computations by conventional finite elements;
2. "qualitative" determination of the orientation angles of the zones of localization.

The criterion of localization constitutes a limit of reliability of computations by "classical" finite elements.

3.1 The various ways of studying the localization

In the frame as of studies conducted in soil mechanics, one noted a strong dependence of the numerical solution according to the discretization by finite elements. He appears a concentration of high values of plastic strains cumulated on the level of the finite elements and it is noted that this "zone of localization" changes brutally with the refinement of the mesh. This phenomenon of localization is source of numerical problems and generates problems of convergences within the meaning of the finite elements.

The localization can be interpreted like an unstable, precursory phenomenon of mechanism of fracture, characterizing certain types of materials requested in the inelastic field. To study instabilities related to the localization one distinguishes, on the one hand, the classes of materials with behavior depend on time and on the other hand, those not depending on time. For the materials with behavior independent of time, the approach commonly used is the method called by bifurcation (it is with this method that one is interested in this note). It consists in analyzing the losses of unicity of the problem out of velocities. For the materials with behavior depend on time, the unicity of the problem out of velocities is often guaranteed and this does not prevent the observation of instabilities during their strain. For these materials, one must then resort to other approaches. Most usually used is the approach by disturbance. This approach will not be treated in this note, but for more information to consult the notes [bib1], [bib2].

Rudnicki and Rice [bib3] showed that the study of the localization of the strains in rock mechanics fits in the frame of the theory of the bifurcation. This one is based on the notion of unstable equilibrium. Rice [bib4] considers that the bifurcation point marks the end of the stable mode. The beginning of the localization is associated with a rheological instability of the system and this instability corresponds locally to the loss of ellipticity of the equations which control the continuous incremental equilibrium out of velocities. Rice thus proposes a criterion known as of "bifurcation by localization" which makes it possible to detect the state from which, the solution of the mathematical equations which control the problem in extreme cases considered and the evolution of the studied mechanical system (equations, of equilibrium, constitutive law) lose their character of unicity. This theory allows the computation of the state of initiation of the localization which is perceived like the limit of validity of computations by conventional finite elements.

3.2 Theoretical approach

3.2.1 Writing of the problem of velocity

One considers a structure occupying, at one time t, the open one Ω of $\Re^3$. The problem of velocity consists in finding the field rates of travel v when the structure is subjected at the speeds of volume forces $\dot{f}_d$, the rates of travel imposed v_d on part $\partial_1\Omega$ of the border and at the speeds of surface forces $\dot{F}_d$ on the complementary part $\partial_2\Omega$.

In the local writing of the problem, the field rates of travel v must thus check problem:

1 v sufficient regular and $v=v_d$ on $\partial_1 \Omega$

2 the balance equations:

$$\mathrm{div}[\mathbf{L}:\boldsymbol{\varepsilon}(v)]+\dot{\mathbf{f}}_d=0 \text{ on } \Omega$$

$$\mathbf{L}:\boldsymbol{\varepsilon}(v).\mathbf{n}=\dot{\mathbf{F}}_d \text{ on } \partial_2 \Omega$$

$\mathbf{n}$ being the outgoing unit norm with $\partial_2 \Omega$.

•Compatibility conditions (one limits oneself here to the small disturbances):

$$\boldsymbol{\varepsilon}(v)=\frac{1}{2}\left[\nabla \mathbf{v}+(\nabla \mathbf{v})^T\right]$$

where the operator $\mathbf{L}$ is defined in a general way for the constitutive laws written in incremental form by the relation:

$$\dot{\boldsymbol{\sigma}}=\mathbf{L}(\boldsymbol{\varepsilon},\mathbf{V}):\dot{\varepsilon}$$

with:

$$\mathbf{L}=\begin{cases}\mathbf{E} & \text{si } F<0 \text{ ou } F=0 \text{ et } \dfrac{\mathbf{b}:\mathbf{E}:\dot{\boldsymbol{\varepsilon}}}{h}\leq 0\\ \mathbf{H}= \quad \mathbf{E}-\dfrac{(\mathbf{E}:\mathbf{a})\otimes(\mathbf{b}:\mathbf{E})}{h} & \text{si } F=0 \text{ et } \dfrac{\mathbf{b}:\mathbf{E}:\dot{\boldsymbol{\varepsilon}}}{h}>0\end{cases}$$

where $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{\varepsilon}$ the total deflection, $\mathbf{V}$ a set of local variables and F surface threshold of plasticity. The statements of a, b, E and H depend on the formulation of the constitutive law.

3.2.2 Results of existence and unicity, Loss of ellipticity

We give in this chapter some results without demonstrations. The reference for these demonstrations however is specified.

A sufficient condition of existence and unicity of the preceding problem is: $\dot{\sigma}:\dot{\varepsilon}>0$. This inequality can be interpreted like a definition, in the three-dimensional case, of NON-softening. The demonstration is made by Hill [bib5] for the standard materials and by Benallal [bib1] for the materials NON-standards.

The loss of ellipticity corresponds to the time for which the operator $\mathbf{N.H.N}$ becomes singular for a direction $\mathbf{N}$ in a point of structure. This condition is equivalent to the condition: $\det(\mathbf{N.H.N})=0$. It is the condition of "bifurcation continues"[1] within the meaning of Rice also called acoustic tensor. Rice and Rudnicki [bib3] show that this condition of loss of ellipticity of the local problem velocity is a requirement with the "continuous or discontinuous" bifurcation[2] for solid. The boundary conditions do not play any part, only the constitutive law defines the conditions of localization (threshold of localization and directional sense of the surface of localization.

The continuous bifurcations thus provide the lower limit of the range of strain for which the discontinuous bifurcations can occur.

3.2.3 Analytical resolution for the case 2d.

One poses $\mathrm{N}=(\mathrm{N}_1,\mathrm{N}_2,0)$ with $\mathrm{N}_1^2+\mathrm{N}_2^2=1$

1 a continuous bifurcation, a plastic strain occurs inside and outside the zone of localization and one has the same constitutive law inside and outside the tape.

2 a discontinuous bifurcation, one has on both sides of the tape a continuity of displacement but there is not the same behavior. An elastic discharge occurs with external of the zone of localization, while a loading and an elastoplastic strain continue occur inside.

One has then: $\mathbf{N.H.N}=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & C \end{bmatrix}$ where Ortiz [bib6] shows that:

$$C=N_1^2 H_{1313}+N_2^2 H_{2323}>0$$
$$A_{11}=N_1^2 H_{1111}+N_1 N_2 (H_{1112}+H_{1211})+N_2^2 H_{1212}$$
$$A_{22}=N_1^2 H_{1212}+N_1 N_2 (H_{1222}+H_{2212})+N_2^2 H_{2222}$$
$$A_{12}=N_1^2 H_{1112}+N_1 N_2 (H_{1122}+H_{1212})+N_2^2 H_{1222}$$
$$A_{21}=N_1^2 H_{1211}+N_1 N_2 (H_{1212}+H_{2211})+N_2^2 H_{2212}$$

It is thus enough to study the sign of $\det(A)$ as specified by Doghri [bib7]:

$$\det(A)=a_0 N_1^4+a_1 N_1^3 N_2+a_2 N_1^2 N_2^2+a_3 N_1 N_2^3+a_4 N_2^4$$

with:

$$a_0=H_{1111} H_{1212}-H_{1112} H_{1211}$$
$$a_1=H_{1111}(H_{1222}+H_{2212})-H_{1112} H_{2211}-H_{1122} H_{1211}$$
$$a_2=H_{1111} H_{2222}+H_{1112} H_{1222}+H_{1211} H_{2212}-H_{1122} H_{1212}-H_{1122} H_{2211}-H_{1212} H_{2211}$$
$$a_3=H_{2222}(H_{1112}+H_{1211})-H_{1122} H_{2212}-H_{1222} H_{2211}$$
$$a_4=H_{1212} H_{2222}-H_{1222} H_{2212}$$

One poses then $N_1=\cos\theta$ and $N_2=\sin\theta$ with $\theta\in]-\frac{\pi}{2};\ +\frac{\pi}{2}]$. Two cases then are distinguished:

- so $\theta=+\frac{\pi}{2}$ then $\det(A)=0$ if $a_4=0$;
- so $\theta\neq+\frac{\pi}{2}$ then $\det(A)=0$ so $f(x)=a_4 x^4+a_3 x^3+a_2 x^2+a_1 x+a_0=0$ with $x=\tan\theta$.

3.2.4 Computation of the roots

to solve a polynomial of degree N (like that definite above, where n=4) one proposes to use the method known as “Companion Matrix Polynomial”. The principle of this method consists in seeking the eigenvalues of the matrix (of Hessenberg type) of order N associated with the polynomial. If the polynomial is considered $P(x)=x^n+a_{n-1} x^{n-1}+\ldots+a_k x^k+\ldots+a_1 x+a_0$. To seek the roots of this polynomial amounts seeking the eigenvalues of the matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\mathrm{a}_0 \\ 1 & 0 & 0 & 0 & 0 & -\mathrm{a}_1 \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & -\mathrm{a}_k \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & -\mathrm{a}_{n-1} \end{bmatrix}$$

This indicator is calculated by option INDL_ELGA of CALC_CHAMP [U4.81.04]. It produces in each point of integration 5 components: the first is the indicator of localization being worth 0 if $det(N.H.N)>0$

(not localization), and being worth 1 if not, which corresponds has a possibility of localization. The other components provide the directions of localization.

4 Sensitivity analyzes

the analysis of sensitivity relates only to the version associated with the formulation described with chapter 2.2.

4.1 Sensitivity to the data materials

4.1.1 the direct problem

We place ourselves in this part in the frame of the resolution of nonlinear computations.
In *Code_Aster*, any nonlinear static computation is solved incrémentalement. It thus requires with each step of load $i \in \{1, I\}$ the resolution of the nonlinear system of equations:

$$\begin{cases} R(u_i, t_i) + B^t \lambda_i = L_i \\ \mathbf{Bu}_i = u_i^d \end{cases}$$ **éq 4.1.1-1**

with

$$(\mathbf{R}(\mathbf{u_i}, t_i))_k = \int_\Omega \boldsymbol{\sigma}(\mathbf{u_i}) : \boldsymbol{\varepsilon}(\mathbf{w_k}) \, d\Omega$$ **éq 4.1.1-2**

- $\mathbf{w_k}$ is the shape function of k the ième degree of freedom of modelled structure,
- $(\mathbf{R}(\mathbf{u_i}, t_i))$ is the vector of the nodal forces.

The resolution of this system is done by the method of Newton-Raphson:

$$\begin{cases} \mathbf{K}_i^n \delta\mathbf{u}_i^{n+1} + \mathbf{B}^t \delta\lambda_i^{n+1} & = & \mathbf{L}_i - \mathbf{R}(\mathbf{u}_i^n, t_i) + \mathbf{B}^t \lambda_i^n \\ \mathbf{B}\delta u_i^{n+1} & = & -\mathbf{B}u_{i-1}^n \end{cases}$$ **éq 4.1.1-3**

where $\mathbf{K_i^n} = \frac{\partial \mathbf{R}}{\partial \mathbf{u}}\Big|_{(u_i^n, t_i)}$ is the tangent matrix with the step of load i and the iteration of Newton n.

The solution is thus given by:

$$\begin{cases} \mathbf{u}_i & = & \mathbf{u}_{i-1} + \sum_{n=0}^{N} \delta\mathbf{u}_i^n \\ \lambda_i & = & \lambda_{i-1} + \sum_{n=0}^{N} \delta\lambda_i^n \end{cases}$$ **éq 4.1.1-4**

with N, the nombre of iterations of Newton which was necessary to reach convergence.

4.1.2 The computation derived

4.1.2.1 Preliminaries

In the frame from the sensitivity analysis, it is necessary to insist on the dependences of a quantity compared to the others. We will thus clarify that the results of preceding computation depend on a given Φ parameter (elastic limit, Young modulus, density,…) and that in the following way:

$$u_i = u_i(\Phi) \quad \lambda_i = \lambda_i(\Phi) .$$

But that is not sufficient. Also we place ourselves in the frame of an incremental computation with constitutive law of the Drucker-Prager type. If one considers the interdependences of the parameters on an algorithmic level, one can write:

$$\mathbf{R} = \mathbf{R}(\boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi),\ p_{i-1}(\Phi), \Delta\mathbf{u}(\Phi))$$

$$\boldsymbol{\sigma}_{\mathbf{i}} = \boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi) + \Delta\boldsymbol{\sigma}(\sigma_{i-1}(\Phi),\ p_{i-1}(\Phi),\ \Delta\mathbf{u}(\Phi),\ \Phi)$$

$$p_i = p_{i-1}(\Phi) + \Delta p(\boldsymbol{\sigma}_{\mathbf{i-1}}(\Phi),\ p_{i-1}(\Phi),\ \Delta\mathbf{u}(\Phi),\ \Phi)$$

Where $\Delta\mathbf{u}$ is the displacement increment with convergence with the step of load i .

Let us specify the meaning of the notations which we will use for derivatives:

- $\dfrac{\partial X}{\partial Y}$ indicate explicit partial **derivative** from X ratio with Y ,
- $X_{,Y}$ indicates the total **variation** from X ratio with Y .

4.1.2.2 Derivative of the equilibrium

Taking into account the preceding remarks, let us express the total variation of [éq 2.1-1] compared to Φ :

$$\begin{cases} \dfrac{\partial \mathbf{R}}{\partial \Phi} + \dfrac{\partial \mathbf{R}}{\partial \Delta\mathbf{u}} \cdot \Delta\mathbf{u}_{,\Phi} + \dfrac{\partial \mathbf{R}}{\partial \boldsymbol{\sigma}_{i-1}} \cdot \boldsymbol{\sigma}_{i-1,\Phi} + \dfrac{\partial \mathbf{R}}{\partial p_{i-1}} \cdot p_{i-1,\Phi} + \mathbf{B}^t \boldsymbol{\lambda}_{i,\Phi} & = & 0 \\ \mathbf{B}\Delta\mathbf{u}_{,\Phi} & = & -\mathbf{B}\mathbf{u}_{i-1,\Phi} \end{cases} \qquad \textbf{éq 4.1.2.2 - 1}$$

Let us notice that here $\dfrac{\partial \mathbf{R}}{\partial \Phi} = 0$: $\mathbf{R}$ does not depend explicitly on Φ but implicitly as we will see it in detail in the continuation.

That is to say:

$$\begin{cases} \mathbf{K}_i^N \Delta\mathbf{u}_{,\Phi} + \mathbf{B}^t \boldsymbol{\lambda}_{i,\Phi} & = & -\mathbf{R}_{,\Phi}\big|_{\Delta\mathbf{u} \neq \Delta\mathbf{u}(\Phi)} \\ \mathbf{B}\Delta\mathbf{u}_{,\Phi} & = & -\mathbf{B}\mathbf{u}_{i-1,\Phi} \end{cases} \qquad \textbf{éq 4.1.2.2 - 2}$$

Where

- $\mathbf{K}_{\mathbf{i}}^{\mathbf{N}}$ is the last tangent matrix used to reach convergence in the iterations of Newton,
- $\mathbf{R}_{,\Phi}\big|_{\Delta u \neq \Delta u(\Phi)}$ is the total variation of $\mathbf{R}$, without taking account of the dependence from $\Delta\mathbf{u}$ ratio with $\boldsymbol{\Phi}$.

The problem lies now in the computation of $\mathbf{R}_{,\Phi}\big|_{\Delta u \neq \Delta u(\Phi)}$.

Note:

In [éq 4.1.2.2 - 2], one used the fact that $\mathbf{K_i^N}=\dfrac{\partial \mathbf{R}(\mathbf{u_i},t_i)}{\partial \Delta \mathbf{u}}$ *whereas in [éq 4.1.1-3] one defined it par.* $\mathbf{K_i^N}=\dfrac{\partial \mathbf{R}(\mathbf{u_i},t_i)}{\partial \mathbf{u_i^N}}$ *There is well equivalence of these two definitions insofar as* $\mathbf{u}_i=\mathbf{u}_{i-1}+\Delta\mathbf{u}$ *and that* $\mathbf{R}$ *depends indeed on* $\Delta\mathbf{u}$ *(and as well sure of* $\boldsymbol{\sigma}_{\mathbf{i-1}}$ *and* p_{i-1} *).*

Note:

If one derives compared to $\boldsymbol{\Phi}$ *directly [éq 4.1.1-3], one finds* $\mathbf{K}^n=\dfrac{\partial \mathbf{u}^{n+1}}{\partial \boldsymbol{\Phi}}+\mathbf{B}^t\boldsymbol{\lambda},\boldsymbol{\Phi}=-\mathbf{R},_{\boldsymbol{\Phi}/\Delta\mathbf{u}\neq\Delta\mathbf{u}/\boldsymbol{\Phi}}-\mathbf{K}^n,_{\boldsymbol{\Phi}}\delta\mathbf{u}^{n+1}$ *. What is the same thing* ***with convergence*** *and reveals that the error on* $\dfrac{\partial \mathbf{u}}{\partial \Phi}$ *depends on* $\mathbf{K^{-1}}\mathbf{K},_{\Phi}$ *.*

4.1.2.3 Computation of derivative of the constitutive law

In the continuation, by preoccupation with a clearness, we will give up the indices $i-1$.
According to [éq 4.1.1-2], one can rewrite $\mathbf{R},_{\Phi}|_{\Delta u\neq \Delta u(\Phi)}$ in the form:

$$\mathbf{R},_{\Phi}|_{\Delta u\neq \Delta u(\Phi)}=\int_{\Omega}\left(\boldsymbol{\sigma},_{\Phi}+\Delta\boldsymbol{\sigma},_{\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}\right):\boldsymbol{\varepsilon}(\mathbf{w_k})\,d\Omega \qquad \textbf{éq 4.1.2.3 - 1}$$

One must thus calculate $\Delta\boldsymbol{\sigma},_{\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$. With this intention, we will use the statements which intervene in the numerical integration of the constitutive law.

4.1.2.4 Case of linear elasticity

In the frame of linear elasticity, the constitutive law is expressed by:

$$\begin{cases}\Delta\tilde{\boldsymbol{\sigma}}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})\\ Tr(\Delta\boldsymbol{\sigma})=3\mathrm{K}.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}))\end{cases}$$

or:

$$\Delta\boldsymbol{\sigma}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \qquad \textbf{éq 4.1.2.4 - 1}$$

where $\mathbf{Id}$ is the tensor identity of order 2.

Then, by calculating the total variation of [éq 4.1.2.4 - 1] compared to $\boldsymbol{\Phi}$, one obtains:

$$\Delta\boldsymbol{\sigma},_{\Phi}=2\mu,_{\Phi}.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K,_{\Phi}.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id}+2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u},_{\Phi})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u},_{\Phi})).\mathbf{Id} \qquad \textbf{éq 4.1.2.4 - 2}$$

Is:

$$\Delta\boldsymbol{\sigma},_{\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}=2\mu,_{\Phi}.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K,_{\Phi}.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \qquad \textbf{éq 4.1.2.4 - 3}$$

4.1.2.5 Cases of the elastoplasticity of the Drucker-Prager type

the constitutive law of the Drucker-Prager type are written:

$$\begin{cases} \boldsymbol{\varepsilon}(\Delta \mathbf{u}) - \mathbf{S} : \boldsymbol{\sigma} & = \dfrac{3}{2} \cdot \Delta p \cdot \dfrac{\tilde{\boldsymbol{\sigma}} + \Delta\tilde{\boldsymbol{\sigma}}}{(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq}} \\ (\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})_{eq} + A \cdot Tr(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) & \leq R(p + \Delta p) \end{cases}$$

éq 4.1.2.5 - 1

where $\mathbf{S}$ is the tensor of the elastic flexibilities and R is the plasticity criterion defined by:

in the case of a linear hardening:

$$R(p) = h \cdot p + \sigma^y \text{ pour } 0 \leq p \leq p_{ultm}$$
$$R(p) = h \cdot p_{ultm} \text{ pour } p \geq p_{ultm}$$

in the case of a parabolic hardening:

$$R(p) = \sigma^y \cdot \left(1 - \left(1 - \sqrt{\frac{\sigma^y_{\text{ultm}}}{\sigma^y}}\right) \cdot \frac{p}{p_{\text{ultm}}}\right)^2 \text{ pour } 0 \leq p \leq p_{ultm}$$
$$R(p) = \sigma^y_{ultm} \text{ pour } p \geq p_{ultm}$$

In numerical terms, this constitutive law is integrated using an algorithm of radial return: one makes an elastic prediction (noted $\boldsymbol{\sigma}^{\mathbf{e}}$) which one corrects if the threshold is violated. One thus writes:

$$\begin{cases} \Delta\tilde{\boldsymbol{\sigma}} & = 2\mu\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) - 3\mu \cdot \dfrac{\Delta p}{\boldsymbol{\sigma}^e_{eq}} \cdot \tilde{\boldsymbol{\sigma}}^e \\ Tr(\Delta\boldsymbol{\sigma}) & = 3K \cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})) - 9K \cdot A \cdot \Delta p \\ \Delta p & = \text{solution de } \sigma^e_{eq} - (3\mu + 9K \cdot A^2) \cdot \Delta p + A \cdot Tr(\sigma^e) - R(p^- + \Delta p) = 0 \end{cases}$$

éq 4.1.2.5 - 2

We will distinguish two cases.

1st case : $\Delta p = 0$
What amounts saying that during these step of load, the Gauss point considered did not see an increase in its plasticization. One finds oneself then in the case of linear elasticity:

$$\Delta\boldsymbol{\sigma}_{,\Phi}\big|_{\Delta\mathbf{u} \neq \Delta\mathbf{u}(\Phi)} = 2\mu_{,\Phi} . \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) + K_{,\Phi} . Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})) . \mathbf{Id}$$ **éq 4.1.2.5 - 3**

2nd cases : $\Delta p > 0$
Taking into account the dependences between variables in [éq 4.1.2.5 - 1], one can write:

$$\begin{cases} \Delta\boldsymbol{\sigma}_{,\Phi} & = \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}_{,\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial p} \cdot p_{,\Phi} + \dfrac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}(\Delta\mathbf{u})} \cdot \boldsymbol{\varepsilon}(\Delta\mathbf{u}_{,\Phi}) \\ \Delta p_{,\Phi} & = \dfrac{\partial\Delta p}{\partial\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\sigma}} \cdot \boldsymbol{\sigma}_{,\Phi} + \dfrac{\partial\Delta p}{\partial p} \cdot p_{,\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\varepsilon}(\Delta\mathbf{u})} \cdot \boldsymbol{\varepsilon}(\Delta\mathbf{u})_{,\Phi} \end{cases}$$ **éq 4.1.2.5 - 4**

Moreover, in agreement with the algorithmic integration of the model, we will separate parts deviatoric and hydrostatic.

$$\begin{cases} \Delta\boldsymbol{\sigma},_{\Phi}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} & = \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\Phi}\cdot\mathbf{Id} \\ & + \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}}\cdot\mathbf{Id}\cdot\boldsymbol{\sigma},_{\Phi} \\ & + \dfrac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial p}\cdot p,_{\Phi} + \dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial p}\cdot\mathbf{Id}\cdot p,_{\Phi} \\ \Delta p,_{\Phi}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)} & = \dfrac{\partial\Delta p}{\partial\Phi} + \dfrac{\partial\Delta p}{\partial\boldsymbol{\sigma}}\cdot\boldsymbol{\sigma},_{\Phi} + \dfrac{\partial\Delta p}{\partial p}\cdot p,_{\Phi} \end{cases}$$ **éq 4.1.2.5 - 5**

And thus, one calculates:

$$\boxed{\frac{\partial\Delta\boldsymbol{\sigma}}{\partial\Phi}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\Phi} = \frac{\partial 2\mu}{\partial\Phi}\cdot\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) - \frac{\partial 3\mu}{\partial\Phi}\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\frac{\partial\Delta p}{\partial\Phi}}{\sigma^e_{eq}}\cdot\tilde{\boldsymbol{\sigma}}^e + 3\mu\cdot\frac{\Delta p\cdot\frac{\partial\sigma^e_{eq}}{\partial\Phi}}{{\sigma^e_{eq}}^2}\cdot\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\frac{\partial\tilde{\boldsymbol{\sigma}}^e}{\partial\Phi}$$

$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\Phi} = \frac{\partial 3\mathrm{K}}{\partial\Phi}\cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})) - \frac{\partial 9\mathrm{K}}{\partial\Phi}\cdot A\cdot\Delta p - 9\mathrm{K}\cdot\frac{\partial A}{\partial\Phi}\cdot\Delta p - 9\mathrm{K}\cdot A\cdot\frac{\partial\Delta p}{\partial\Phi}$$

$$\boxed{\frac{\partial\Delta\boldsymbol{\sigma}}{\partial\sigma}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}} = -3\mu\cdot\frac{\frac{\partial\Delta p}{\partial\sigma}}{\sigma^e_{eq}}\otimes\tilde{\boldsymbol{\sigma}}^e + 3\mu\cdot\frac{\Delta p}{{\sigma^e_{eq}}^2}\cdot\frac{\partial\sigma^e_{eq}}{\partial\boldsymbol{\sigma}}\otimes\tilde{\boldsymbol{\sigma}}^e - 3\mu\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot\mathbf{J}$$

where $\mathbf{J}$ is the operator deviatoric defined by: $\mathbf{J}:\boldsymbol{\sigma}=\tilde{\boldsymbol{\sigma}}$

$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\boldsymbol{\sigma}} = -9\mathrm{K}\cdot A\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}$$

$$\boxed{\frac{\partial\Delta\sigma}{\partial p}}$$

$$\frac{\partial\Delta\tilde{\boldsymbol{\sigma}}}{\partial p} = -\frac{3\mu}{\boldsymbol{\sigma}^e_{eq}}\cdot\frac{\partial\Delta p}{\partial p}\cdot\tilde{\boldsymbol{\sigma}}^e$$

$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial p} = -9\cdot K\cdot\mathbf{A}\cdot\frac{\partial\Delta p}{\partial p}$$

$$\boxed{\Delta p,_{\Phi}}$$

The fact is used that: $(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq} = (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^e_{eq} - 3\mu\cdot\Delta p$

$$\Delta p_{,\Phi}=\frac{1}{3\mu}\cdot((\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq,\Phi}-(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq,\Phi}-\frac{\partial 3\mu}{\partial\Phi}\cdot\Delta p)$$

Note:

In these computations were or must be used the following results:

$$\frac{\partial\tilde{\boldsymbol{\sigma}}^{e}}{\partial\Phi}=\frac{\partial 2\mu}{\partial\Phi}\cdot\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})$$

Tensor of order 2

$$\frac{\partial\sigma^{e}_{eq}}{\partial\Phi}=\frac{3}{2}\cdot\frac{(\frac{\partial 2\mu}{\partial\Phi}\cdot\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})):(\tilde{\boldsymbol{\sigma}}+2\mu\,\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}))}{\sigma^{e}_{eq}}$$

Tensor

$$\frac{\partial\sigma^{e}_{eq}}{\partial\boldsymbol{\sigma}}=\frac{3}{2}\cdot\frac{\tilde{\boldsymbol{\sigma}}^{e}}{\sigma^{e}_{eq}}$$

Scalar of order 2

$$\frac{\partial\tilde{\boldsymbol{\sigma}}^{e}}{\partial\boldsymbol{\sigma}}=\mathbf{J}$$

Tensor of order 4

$$\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\boldsymbol{\sigma}}=Id$$

Tensor of order 2

$$\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\Phi}=\frac{\partial 3K}{\partial\Phi}\cdot Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}))$$

Scalar

One must also calculate derivatives partial of the increment of plastic strain cumulated compared to materials parameters, with the stresses and with the cumulated plastic strain (cf Annexes)

Those are obtained by deriving the equation solved to compute: the increment from plastic strain cumulated during direct computation.

4.1.2.6 Computation of derivative of Once

calculated $\Delta\boldsymbol{\sigma}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$ displacement, one can constitute the second member $\mathbf{R}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}$ by means of [éq 4.1.2.3 - 1]. One then solves the system [éq 4.1.2.2 - 2] and one obtains the derived displacement increment compared to Φ .

4.1.2.7 Computation of derivative of the other quantities

Now that one has $\Delta\mathbf{u}_{,\Phi}$, one must calculate derivative of the other quantities. One separates two more cases:

Linear elasticity
According to [éq 4.1.2.5 - 1], one as follows calculates derivative of the increment of stress:

$$\Delta\boldsymbol{\sigma}_{,\Phi}=\Delta\boldsymbol{\sigma}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}+2\mu\,.\,\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}_{,\Phi})+K\,.\,Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}_{,\Phi}))\,.\,\mathbf{Id}$$

The increment of cumulated plastic strain, as for him, does not see evolution:

$$\Delta p_{,\Phi}=0$$

Elastoplasticity of the Drucker-Prager type
If $\Delta p=0$, the preceding case is found.
If not, one obtains according to [éq 4.1.2.5 - 2]:

$$\Delta\boldsymbol{\sigma}_{,\Phi}=\Delta\boldsymbol{\sigma}_{,\Phi}|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\Phi)}+\frac{\partial\Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}(\Delta\mathbf{u})}:\boldsymbol{\varepsilon}(\Delta\mathbf{u}_{,\Phi})$$

And for the cumulated plastic strain, one uses the following relation:

$$(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq}=(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq}-3\mu\cdot\Delta p$$

This one enables us to write that:

$$\Delta p_{,\Phi} = \frac{1}{3\mu} \cdot ((\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq,\Phi} - (\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq,\Phi} - \frac{\partial 3\mu}{\partial\Phi} \cdot \Delta p)$$

The significant equivalent stresses are calculated as follows:

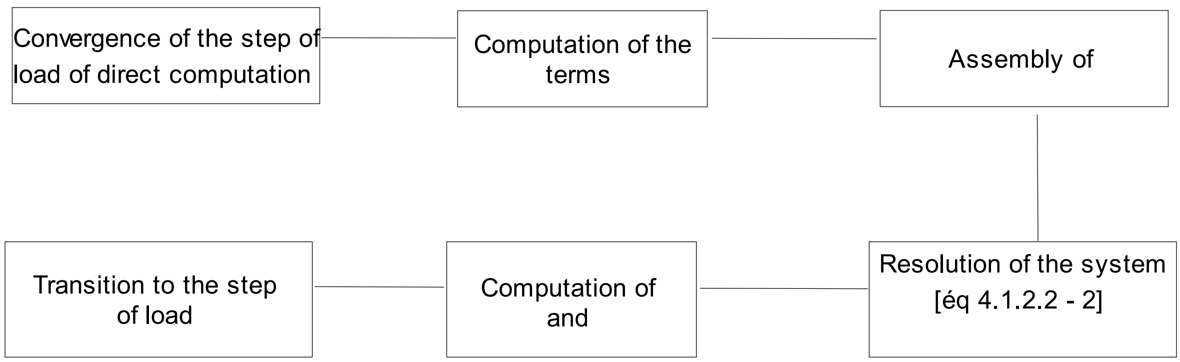
$$(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq,\Phi} = \frac{3}{2(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})^{e}_{eq}} \cdot (\tilde{\boldsymbol{\sigma}}_{,\Phi} + \frac{\partial 2\mu}{\partial\Phi} \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}) + 2\mu \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}_{,\Phi})) : (\tilde{\boldsymbol{\sigma}} + 2\mu \cdot \tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}))$$

$$(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq,\Phi} = \frac{3}{2(\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma})_{eq}} \cdot (\tilde{\boldsymbol{\sigma}}_{,\Phi} + \Delta\tilde{\boldsymbol{\sigma}}_{,\Phi}) : (\tilde{\boldsymbol{\sigma}} + \Delta\tilde{\boldsymbol{\sigma}})$$

Once all these computations are finished, all the derived quantities are reactualized and one passes to the step of load according to.

4.1.2.8 Synthesis

to summarize the preceding paragraphs, one represents the various stages of computation by the following diagram:



4.2 Sensitivity to the loading

the approach is here rather close to that of the preceding paragraph. We develop it nevertheless entirely in a preoccupation with a clearness, so that this paragraph can be read independently.

4.2.1 The direct problem: statement of the loading

Until now we expressed the direct problem in the form:

$$\begin{cases} \mathbf{R}(\mathbf{u}_i, t_i) + \mathbf{B}^t \lambda_i & = & \mathbf{L}_i \\ \mathbf{B}\mathbf{u}_i & = & \mathbf{u}_i^d \end{cases}$$

éq the 4.2.1-1

loadings are gathered with the second member and understand the forces imposed $\mathbf{L_i}$ and the imposed displacements $\mathbf{u_i^d}$.

Let us suppose that the loading in imposed force $\mathbf{L_i}$ depends on a scalar parameter α in the following way:

$$\mathbf{L_i}(\alpha)=\mathbf{L_i^1}+\mathbf{L_i^2}(\alpha)$$ **éq 4.2.1-2**

Where

- $\mathbf{L_i^1}$ is a vector independent of α ,
- $\mathbf{L_i^2}$ depends linearly on α .

One wishes compute the sensitivity of the results of direct computation to a variation of the parameter α .

4.2.2 The problem derived

4.2.2.1 Derivative from the equilibrium

As in the preceding chapter, by taking account as of dependences between the various fields, one derives the equilibrium [éq 4.2.1-1] compared α :

$$\begin{cases} \dfrac{\partial \mathbf{R}}{\partial \alpha}+\dfrac{\partial \mathbf{R}}{\partial \Delta \mathbf{u}}\cdot \Delta \mathbf{u},_{\alpha}+\dfrac{\partial \mathbf{R}}{\partial \boldsymbol{\sigma}_{i-1}}\cdot \boldsymbol{\sigma}_{i-1},_{\alpha}+\dfrac{\partial \mathbf{R}}{\partial p_{i-1}}\cdot p_{i-1},_{\alpha}+\mathbf{B}^t \boldsymbol{\lambda}_i,_{\alpha} & = \mathbf{L}_i^2(1) \\ \mathbf{B}\Delta \mathbf{u},_{\alpha} & = -\mathbf{B}\mathbf{u}_{i-1,\alpha} \end{cases}$$ **éq 4.2.2.1 - 1**

One used the fact that $\mathbf{L_i^2}$ depends linearly on α .

That is to say:

$$\begin{cases} \mathbf{K}_i^N \Delta \mathbf{u},_{\alpha}+\mathbf{B}^t \boldsymbol{\lambda}_i,_{\alpha} & = \mathbf{L}_i^2(1)-\mathbf{R},_{\alpha}\big|_{\Delta \mathbf{u}\neq \Delta \mathbf{u}(\alpha)} \\ \mathbf{B}\Delta \mathbf{u},_{\alpha} & = -\mathbf{B}\mathbf{u}_{i-1,\alpha} \end{cases}$$

éq 4.2.2.1 - 2

Where

- $\mathbf{K_i^N}$ is the last tangent matrix used to reach convergence in the iterations of Newton,
- $\mathbf{R},_{\alpha}\big|_{\Delta \mathbf{u}\neq \Delta \mathbf{u}(\alpha)}$ is the total variation of $\mathbf{R}$, without taking account of the dependence from $\Delta \mathbf{u}$ ratio with α .

The problem lies like previously in the computation of $\mathbf{R},_{\alpha}\big|_{\Delta \mathbf{u}\neq \Delta \mathbf{u}(\alpha)}$.

4.2.2.2 Computation of derivative of the constitutive law

According to [éq 4.1.1-2], one can rewrite $\mathbf{R},_{\alpha}\big|_{\Delta \mathbf{u}\neq \Delta \mathbf{u}(\alpha)}$ in the form:

$$\mathbf{R}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=\int_\Omega\left(\boldsymbol{\sigma}_{,\alpha}+\Delta\boldsymbol{\sigma}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}\right):\boldsymbol{\varepsilon}(\mathbf{w_k})\,d\Omega \qquad \text{éq 4.2.2.2 - 1}$$

With this intention, we will use the statements which intervene in the numerical integration of the constitutive law to compute: $\Delta\boldsymbol{\sigma}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$.

4.2.2.3 Case of linear elasticity

In the frame of linear elasticity, the constitutive law is expressed by:

$$\Delta\boldsymbol{\sigma}=2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u})).\mathbf{Id} \qquad \text{éq 4.2.2.3 - 1}$$

where $\mathbf{Id}$ is the tensor identity of order 2.

Then, by calculating the total variation of [éq 4.2.2.3 - 1] compared to α . one obtains:

$$\begin{aligned}\Delta\sigma_{,\alpha}&=2\mu_{,\alpha}.\tilde{\varepsilon}(\Delta u)+K_{,\alpha}.Tr(\varepsilon(\Delta u)).\mathbf{Id}+2\mu.\tilde{\varepsilon}(\Delta u_{,\alpha})+K.Tr(\varepsilon(\Delta u_{,\alpha})).\mathbf{Id}\\&=0.\qquad +0.\qquad +2\mu.\tilde{\varepsilon}(\Delta u_{,\alpha})+K.Tr(\varepsilon(\Delta u_{,\alpha})).\mathbf{Id}\end{aligned} \qquad \text{éq 4.2.2.3 - 2}$$

Is:

$$\Delta\boldsymbol{\sigma}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=0.$$

4.2.2.4 Case of the elastoplasticity of the Drucker-Prager type

Like previously, we will distinguish two cases.

1st case : $\Delta p=0$
What amounts saying that during these step of load, the Gauss point considered did not see an increase in its plasticization. One finds oneself then in the case of linear elasticity:

$$\Delta\boldsymbol{\sigma}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}=0.$$

2nd case : $\Delta p>0$
Taking into account the dependences between variables, one can write:

$$\begin{cases}\Delta\sigma_{,\alpha} = \dfrac{\partial\Delta\sigma}{\partial\alpha}+\dfrac{\partial\Delta\sigma}{\partial\sigma}\cdot\sigma_{,\alpha}+\dfrac{\partial\Delta\sigma}{\partial p}\cdot p_{,\alpha}+\dfrac{\partial\Delta\sigma}{\partial\varepsilon(\Delta u)}\cdot\varepsilon(\Delta u_{,\alpha})\\ \Delta p_{,\alpha} = \dfrac{\partial\Delta p}{\partial\alpha}+\dfrac{\partial\Delta p}{\partial\sigma}\cdot\sigma_{,\alpha}+\dfrac{\partial\Delta p}{\partial p}\cdot p_{,\alpha}+\dfrac{\partial\Delta p}{\partial\varepsilon(\Delta u)}\cdot\varepsilon(\Delta u)_{,\alpha}\end{cases}$$

Moreover, in agreement with the algorithmic integration of the model, we will separate parts deviatoric and hydrostatic.

$$\begin{cases} \Delta\sigma,_{\alpha}\big|_{\Delta u\neq \Delta u(\alpha)} &= \dfrac{\partial \Delta\tilde{\sigma}}{\partial\alpha} \quad +\dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial\alpha}\cdot\mathbf{Id} \\ & + \quad \dfrac{\partial \Delta\tilde{\sigma}}{\partial\sigma}\cdot\sigma,_{\alpha}+\dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial\sigma}\cdot\mathbf{Id}\cdot\sigma,_{\alpha} \\ & + \quad \dfrac{\partial \Delta\tilde{\sigma}}{\partial p}\cdot p,_{\alpha}+\dfrac{1}{3}\cdot\dfrac{\partial Tr(\Delta\sigma)}{\partial p}\cdot\mathbf{Id}\cdot p,_{\alpha} \\ \Delta p,_{\alpha}\big|_{\Delta u\neq \Delta u(\alpha)} &= \dfrac{\partial \Delta p}{\partial\alpha}+\dfrac{\partial \Delta p}{\partial\sigma}\cdot\sigma,_{\alpha}+\dfrac{\partial \Delta p}{\partial p}\cdot p,_{\alpha} \end{cases}$$

And thus, one calculates:

$$\boxed{\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\alpha}}$$

Insofar as there is not explicit dependence from $\Delta\boldsymbol{\sigma}$ ratio with α , one obtains:

$$\frac{\partial \Delta\tilde{\boldsymbol{\sigma}}}{\partial\alpha}=0.$$
$$\frac{\partial Tr(\Delta\boldsymbol{\sigma})}{\partial\alpha}=0.$$

$$\boxed{\frac{\partial \Delta\boldsymbol{\sigma}}{\partial\boldsymbol{\sigma}}}$$

$$\frac{\partial \Delta\tilde{\sigma}}{\partial\sigma}=-3\mu\cdot\frac{\dfrac{\partial \Delta p}{\partial\sigma}}{\sigma^e_{eq}}\otimes\tilde{\sigma}^e+3\mu\cdot\frac{\Delta p}{\sigma^{e^2}_{eq}}\cdot\frac{\partial\sigma^e_{eq}}{\partial\sigma}\otimes\tilde{\sigma}^e-3\mu\cdot\frac{\Delta p}{\sigma^e_{eq}}\cdot J$$

where $\mathbf{J}$ is the operator deviatoric.

$$\frac{\partial Tr(\Delta\sigma)}{\partial\sigma}=-9\mathrm{K}\cdot A\cdot\frac{\partial \Delta p}{\partial\sigma}$$

$$\boxed{\frac{\partial \Delta\boldsymbol{\sigma}}{\partial p}}$$

$$\frac{\partial \Delta\tilde{\sigma}}{\partial p}=-\frac{3\mu}{\sigma^e_{eq}}\cdot\frac{\partial \Delta p}{\partial p}\cdot\tilde{\sigma}^e$$

$$\frac{\partial Tr(\Delta\sigma)}{\partial p}=-9\cdot K\cdot A\cdot\frac{\partial \Delta p}{\partial p}$$

$$\boxed{\Delta p,_{\alpha}}$$

The fact is used that: $(\sigma+\Delta\sigma)_{eq}=(\sigma+\Delta\sigma)^e_{eq}-3\mu\cdot\Delta p$

$$\Delta p_{,\alpha}=\frac{1}{3\mu}\cdot\left(\left(\sigma+\Delta\sigma\right)_{eq}^{e,\alpha}-\left(\sigma+\Delta\sigma\right)_{eq_{,\alpha}}-\frac{\partial 3\mu}{\partial\alpha}\cdot\Delta p\right)$$

One will refer again to the remark at the end of [§ 4.1.2.5] for the quantities whose computation was not here detailed.

4.2.2.5 Computation of derivative of Once

calculated $\Delta\boldsymbol{\sigma}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$ displacement, one can constitute the second member $\mathbf{R}_{,\alpha}\big|_{\Delta\mathbf{u}\neq\Delta\mathbf{u}(\alpha)}$. One then solves the system [éq 4.2.2.1 - 1] and one obtains the derived displacement increment compared to α .

4.2.2.6 Computation of derivative of the other quantities

Now that one has $\Delta\mathbf{u}_{,\alpha}$, one must calculate derivative of the other quantities. One separates two more cases:

Linear elasticity
According to [éq 4.2.2.3 - 1], one as follows calculates derivative of the increment of stress:

$$\Delta\boldsymbol{\sigma}_{,\alpha}=0.+2\mu.\tilde{\boldsymbol{\varepsilon}}(\Delta\mathbf{u}_{,\alpha})+K.Tr(\boldsymbol{\varepsilon}(\Delta\mathbf{u}_{,\alpha})).\mathbf{Id}$$

The increment of cumulated plastic strain, as for him, does not see evolution:

$$\Delta p_{,\alpha}=0$$

Elastoplasticity of the type Drucker Prager
If $\Delta p=0$, the preceding case is found.
If not, one obtains:

$$\Delta\sigma_{,\alpha}=\Delta\sigma_{,\alpha}\big|_{\Delta u\neq\Delta u(\alpha)}+\frac{\partial\Delta\sigma}{\partial\varepsilon(\Delta u)}:\varepsilon(\Delta u_{,\alpha})$$

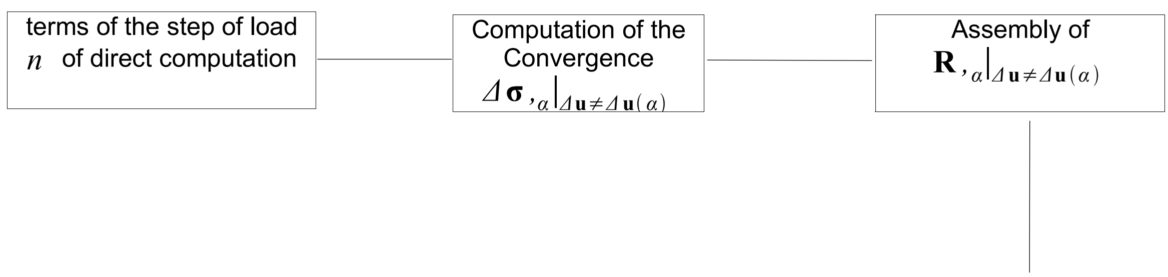
And for the cumulated plastic strain:

$$\Delta p_{,\alpha}=\frac{1}{3\mu}\cdot\left(\left(\sigma+\Delta\sigma\right)_{eq}^{e,\alpha}-\left(\sigma+\Delta\sigma\right)_{eq_{,\alpha}}-\frac{\partial 3\mu}{\partial\alpha}\cdot\Delta p\right)$$

Once all these computations are finished, all the derived quantities are reactualized and one passes to the step of load according to.

4.2.2.7 Synthesis

to summarize the preceding paragraphs, one represents the various stages of computation by the following diagram:



Transition to the step of load $n+1$ — Computation of $\Delta \boldsymbol{\sigma},_{\alpha}$ and $\Delta p,_{\alpha}$ — Resolution of the system [éq 4.2.2.1 - 1] $\Rightarrow \Delta \mathbf{u},_{\alpha}$

5 Functionalities and checking

the constitutive law can be defined by key words DRUCK_PRAG and DRUCK_PRAG_N_A for the non-aligned version (command STAT_NON_LINE, key word factor COMP_INCR). They are associated with materials DRUCK_PRAG and DRUCK_PRAG_FO (command DEFI_MATERIAU).

Model HOEK_BROWN is checked by the cases following tests:

SSND104	[V6.08.104]	Validation of behavior DRUCK_PRAG_N_A
SSNP124	[V6.03.124]	biaxial Test drained with a behavior DRUCK_PRAGER softening
SSNP125	non-existent Documentation	Validation of option INDL_ELGA for behavior DRUCK_PRAGER
SSNV168	[V6.04.168]	triaxial Compression test drained with a behavior DRUCK_PRAGER softening
WTNA101	[V7.33.101]	triaxial Compression test NON-drained with a behavior DRUCK_PRAGER softening
WTNP114	[V7.32.114]	Case test of reference for the computation of the mechanical strains

the tests according to specifically check the sensitivity analysis with the parameters of the model:

SENSM12	[V1.01.190]	Plates under pressure in plane strains (plasticity of DRUCK_PRAGER)
SENSM13	[V1.01.192]	triaxial Compression test with the model of type 3D
SENSM14	[V1.01.193]	Cavity 2D sensitivity analysis (Model DRUCK_PRAGER)

6 Bibliography

- BENALLAL A. and COMI C.: The role of deviatoric and volumetric non-associativities in strain localization (1993).
- CANO V: Instabilities and fracture in the solids élasto-visco-plastics (1996).
- RICE JR and RUDNICKI JW: A note one adds features of the theory of localization of strain (1980).
- RICE JR: The localization of plastic strains, in Theoretical and Applied Mechanics (1976).
- HILL R: A general theory of uniqueness and stability in elastic-plastic solids (1958).
- ORTIZ M: Year analytical study of the localized concrete failure of modes (1987).
- DOGHRI I: Study of the localization of the damage (1989).
- TARDIEU N: Sensitivity analysis in mechanics, Documentation of reference of *the Code_Aster* [R4.03.03].

7 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
7.4	R.FERNANDES, P. OF	initial Text

	BONNIERES, C.CHAVANT EDF R & D/AMA	
9.4	R.FERNANDES	Addition of the model nonassociated

Annexe 1 Computation of derivatives partial of Δp

A1.1 Computation of derivative partial of the increment of plastic strain in the case of a linear hardening

$R(p) = h \cdot p + \sigma^y$ for $0 \leq p < p_{ultm}$

$$\Delta p = \frac{\sigma^e_{eq} + \mathrm{A} \cdot Tr(\boldsymbol{\sigma}^e) - h \cdot p^- - \sigma^Y}{9K \cdot \mathrm{A}^2 + 3\mu + h}$$

thus:

$$\frac{\partial \Delta p}{\partial \Phi} = \frac{1}{9K.\mathrm{A}^2 + 3\mu + h} \cdot \Big(\frac{\partial \sigma^e_{eq}}{\partial \Phi} + \frac{\partial \mathrm{A}}{\partial \Phi} \cdot Tr(\boldsymbol{\sigma}^e) + \mathrm{A} \cdot \frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial \Phi} - \frac{\partial h}{\partial \Phi} \cdot p^- - \frac{\partial \sigma^y}{\partial \Phi}$$
$$- \Delta p \cdot \Big(9 \cdot \frac{\partial K}{\partial \Phi} \cdot \mathrm{A}^2 + 18 \cdot K \cdot A \cdot \frac{\partial \mathrm{A}}{\partial \Phi} + \frac{\partial 3\mu}{\partial \Phi} + \frac{\partial h}{\partial \Phi}\Big)\Big)$$

$$\frac{\partial \Delta p}{\partial \boldsymbol{\sigma}} = \frac{1}{3\mu + 9K \cdot \mathrm{A}^2 + h} \cdot \Big(A \cdot \frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial \boldsymbol{\sigma}} + \frac{\partial \sigma^e_{eq}}{\partial \boldsymbol{\sigma}}\Big)$$

$$\frac{\partial \Delta p}{\partial p} = -h \cdot \frac{1}{3\mu + 9K \cdot \mathrm{A}^2 + h}$$

$R(p) = h \cdot p_{ultm} + \sigma^y$ for $p > p_{ultm}$

$$\Delta p = \frac{\sigma^e_{eq} + \mathrm{A} \cdot Tr(\boldsymbol{\sigma}^e) - h \cdot p_{ultm} - \sigma^Y}{9K \cdot \mathrm{A}^2 + 3\mu}$$

thus:

$$\frac{\partial \Delta p}{\partial \Phi} = \frac{1}{9K \cdot \mathrm{A}^2 + 3\mu} \cdot \Big(\frac{\partial \sigma^e_{eq}}{\partial \Phi} + \frac{\partial \mathrm{A}}{\partial \Phi} \cdot Tr(\boldsymbol{\sigma}^e) + \mathrm{A} \cdot \frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial \Phi} - \frac{\partial h}{\partial \Phi} \cdot p_{ultm} - h \cdot \frac{\partial p_{ultm}}{\partial \Phi} - \frac{\partial \sigma^y}{\partial \Phi}$$
$$- \Delta p \cdot \Big(9 \cdot \frac{\partial K}{\partial \Phi} \cdot \mathrm{A}^2 + 18 \cdot K \cdot A \cdot \frac{\partial \mathrm{A}}{\partial \Phi} + \frac{\partial 3\mu}{\partial \Phi}\Big)\Big)$$

$$\frac{\partial \Delta p}{\partial \boldsymbol{\sigma}} = \frac{1}{3\mu + 9K \cdot \mathrm{A}^2} \Big(A \cdot \frac{\partial Tr(\boldsymbol{\sigma}^e)}{\partial \boldsymbol{\sigma}} + \frac{\partial \sigma^e_{eq}}{\partial \boldsymbol{\sigma}}\Big)$$

$$\frac{\partial \Delta p}{\partial p} = 0$$

A1.2 Computation of derivative partial of the increment of plastic strain in the case of a parabolic hardening

$$R(p)=\sigma^{y}\cdot(1-(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}})\cdot\frac{p}{p_{ultm}})^{2} \text{ for } 0\le p<p_{ultm}$$

$$\frac{\partial\sigma_{eq}^{e}}{\partial\Phi}-(\frac{\partial 3\mu}{\partial\Phi}+9\mathrm{A}^{2}.\frac{\partial K}{\partial\Phi}+18K.\mathrm{A}.\frac{\partial\mathrm{A}}{\partial\Phi}).\Delta p-(3\mu+9K.\mathrm{A}^{2}).\frac{\partial\Delta p}{\partial\Phi}+\frac{\partial\mathrm{A}}{\partial\Phi}.Tr(\sigma^{e})+\mathrm{A}.\frac{\partial Tr(\sigma^{e})}{\partial\Phi}$$
$$-\frac{\partial\sigma^{y}}{\partial\Phi}.(1-(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}).\frac{p^{-}+\Delta p}{p_{ultm}})^{2}$$
$$-2\sigma^{y}.(1-(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}).\frac{p^{-}+\Delta p}{p_{ultm}}).$$

$$(\frac{\partial\sigma_{ultm}^{y}}{\partial\Phi}.\frac{p^{-}+\Delta p}{2p_{ultm}.\sqrt{\sigma_{ultm}^{y}.\sigma^{y}}}-\frac{\partial\sigma^{y}}{\partial\Phi}.\frac{p^{-}+\Delta p}{2p_{ultm}.\sigma^{y}}.\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}+\frac{\partial p_{ultm}}{\partial\Phi}.(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}).\frac{p^{-}+\Delta p}{p_{ultm}{}^{2}}-\frac{\partial\Delta p}{\partial\Phi}\frac{1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}}{p_{ultm}})$$
$$=0$$

$$\frac{\partial\sigma_{eq}^{e}}{\partial\boldsymbol{\sigma}}-(3\mu+9K\cdot\mathrm{A}^{2})\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}+A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\boldsymbol{\sigma}}+2\sigma^{y}\cdot(1-(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}})\cdot\frac{p^{-}+\Delta p}{p_{ultm}})\cdot\frac{1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}}{p_{ultm}}\cdot\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=0$$

$$-(3\mu+9K\cdot\mathrm{A}^{2})\cdot\frac{\partial\Delta p}{\partial p}+2\sigma^{y}\cdot(1-(1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}})\frac{p^{-}+\Delta p}{p_{ultm}})\cdot\frac{1-\sqrt{\frac{\sigma_{ultm}^{y}}{\sigma^{y}}}}{p_{ultm}}\cdot(1+\frac{\partial\Delta p}{\partial p})=0$$

$$R(p)=\sigma_{ultm}^{y}\quad p>p_{ultm}$$

$$\frac{\partial\Delta p}{\partial\Phi}=\frac{1}{3\mu+9K\cdot\mathrm{A}^{2}}(\frac{\partial\sigma_{eq}^{e}}{\partial\Phi}-(\frac{\partial 3\mu}{\partial\Phi}+\frac{\partial 9K}{\partial\Phi}\cdot\mathrm{A}^{2}+18K\cdot\frac{\partial\mathrm{A}}{\partial\Phi}\cdot A)\cdot\Delta p+\frac{\partial\mathrm{A}}{\partial\Phi}\cdot Tr(\boldsymbol{\sigma}^{e})+\mathrm{A}\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\Phi}-\frac{\partial\sigma_{ultm}^{y}}{\partial\Phi})$$

$$\frac{\partial\Delta p}{\partial\boldsymbol{\sigma}}=\frac{1}{3\mu+9K\cdot\mathrm{A}^{2}}(\frac{\partial\sigma_{eq}^{e}}{\partial\boldsymbol{\sigma}}+A\cdot\frac{\partial Tr(\boldsymbol{\sigma}^{e})}{\partial\boldsymbol{\sigma}})$$

$$\frac{\partial\Delta p}{\partial p}=0$$

A1.3 Case of projection at the top of the cone

the principle of the analytical resolution consists in determining the effective stresses like the projection of the elastic stresses on the criterion.

It may be that there is no solution.

If the condition $\Delta p \leq \frac{\sigma_{eq}^{e}}{3\mu}$ is not observed, it is necessary to find the effective stresses by projection at the top of the cone $\Delta p = \frac{\sigma_{eq}^{e}}{3\mu}$.

In this case, one obtains:

$$\frac{\partial \Delta p}{\partial \Phi} = \frac{1}{3\mu} \cdot \left(\frac{\partial \sigma_{eq}^{e}}{\partial \Phi} - \Delta p \cdot \frac{\partial 3\mu}{\partial \Phi} \right)$$

$$\frac{\partial \Delta p}{\partial \boldsymbol{\sigma}} = \frac{1}{3\mu} \cdot \frac{\partial \sigma_{eq}^{e}}{\partial \boldsymbol{\sigma}}$$

$$\frac{\partial \Delta p}{\partial p} = 0$$