Constitutive law BETON_DOUBLE_DP with double Drücker-Prager criterion for the cracking and the compression of the Summarized

concrete

The model presented in this document, behavior BETON_DOUBLE_DP, is a nonlinear constitutive law for the concrete. It leans on the theory of plasticity, it is valid for the three-dimensional stress states. The assumptions of modelization selected are the following ones:

- a field of reversibility of the stresses delimited by two criteria of the type Drücker Prager,
- a hardening of each criterion,
- in compression, a positive hardening to a peak, then a negative hardening,
- in tension, a negative hardening exclusively,
- a dependence of the shape of the curves post-peak in both cases (tension/compression) with the size of the finite element (the shape of this curve is related on negative hardening and the energy of cracking),
- of the normal plastic flow rules (associated plasticity) and an isotropic formulation of hardening,
- the taking into account of the dependence of the thresholds of elasticity compared to the temperature,
- the taking into account dependence of the Young modulus compared to the temperature.

Contents

1 Notations

$\boldsymbol{\sigma}$ indicates the stress tensor, arranged in the form of vector according to convention:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix}$$

One notes:

$I_1 = Trace(\boldsymbol{\sigma})$

$\sigma_H = \frac{1}{3} tr(\boldsymbol{\sigma})$ the hydrostatic stress

$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3} tr(\boldsymbol{\sigma}) \mathbf{I}$ the deviator of the stresses

$\varepsilon_H = \frac{1}{3} tr(\boldsymbol{\varepsilon})$ the voluminal strain

$\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \frac{1}{3} tr(\boldsymbol{\varepsilon}) \mathbf{I}$ the deviator of the strains

$\dot{\tilde{\varepsilon}}_{eq} = \sqrt{\frac{3}{2} \mathrm{trace}(\dot{\tilde{\boldsymbol{\varepsilon}}} . \dot{\tilde{\boldsymbol{\varepsilon}}})}$ strain rate are equivalent

$J_2 = \frac{1}{2} trace(\mathbf{s}^2)$ the second invariant of the stresses

$\sigma^{eq} = \sqrt{3 \mathrm{J}_2} = \sqrt{\frac{3}{2} trace(\mathbf{s}^2)}$ the initial

$\tau_{oct} = \sqrt{\frac{2}{3} J_2} = \sqrt{\frac{trace(\mathbf{s}^2)}{3}}$

$\sigma_{oct} = \sigma_H = \frac{I_1}{3} = \frac{trace(\boldsymbol{\sigma})}{3}$

f'_c limiting equivalent stress of fracture in simple compression
f'_{cc} limiting initial of fracture out of Bi compression
$\varphi f'_c$ elastic limit in initial
f'_t limiting compression of fracture in tension
$\alpha = \frac{f'_t}{f'_c}$ relationship between rupture limit in tension and compression
$\beta = \frac{f'_{cc}}{f'_c}$ relationship between rupture limit in bi--compression and compression simple
κ^p_t plastic strain in plastic
λ_t tension multiplier in tension

κ_c^p	plastic strain in plastic
λ_c	compression multiplier in curved
$f_c(\kappa_c^p)$	compression of hardening in curved
$f_t(\kappa_t^p)$	compression of hardening in ultimate
κ_t^u	tension plastic strain in ultimate
k_c^u	tension plastic strain in compression
G_c^f	energy of fracture in compression (characteristic of the material)
G_t^f	energy of fracture in tension (characteristic of the material)
θ	the maximum of temperature during the history of loading

2 Introduction

2.1 principal Characteristics of the model

The model presented into this document is a nonlinear constitutive law for the concrete. It leans on the theory of plasticity, it is valid for the three-dimensional stress states. The assumptions of modelization selected partly take again the models developed by G. Heinfling [bib2] and J.F. Georgin [bib1] and are the following ones:

- there exists a field of reversibility of the stresses delimited by two criteria of the type Drücker Prager,
- each criterion is hammer-hardened, the field of fracture corresponds to the maximum of the field of reversibility,
- in compression, hardening is positive to a peak, then it becomes negative,
- in tension, hardening is negative exclusively,
- the curves post-peak in both cases (tension/compression) vary with the size of the finite element (the shape of this curve is related on negative hardening and the energy of cracking),
- yielding is governed by a normality rule (associated plasticity) the formulation of hardenings is of isotropic type,
- the elasticity modulus and the thresholds of reversibility vary with the temperature.

Note:

The terminology of criterion of tension and criterion of compression is debatable. We will use it by practice, while being quite conscious that a stress state of tension can lead to the activation of the criterion known as of compression.

2.2 Why two criteria of Drücker Prager

the authors of the theses referred to [bib1] and [bib2] use a criterion of Drücker Prager in compression and a criterion of Rankine in tension. They justify these choices by physical considerations by showing that the field of reversibility thus obtained is close to experimental reality. On the other hand they limit their modelizations in stress states two-dimensional. We preferred to also replace the criterion of tension by a surface of the type Drücker Prager. By this choice, one frees oneself from certain difficulties particularly in the three-dimensional formulations. Surface 3D defining the acceptable stress states with respect to the tension is not any more one pyramid (Rankine 3D) but a conical surface whose top is located on the hydrostatic axis. The trace of the criterion "known as of tension" on the plane deviatoric is not any more one triangle, but a circle. The formulation obtained is simpler. The difference between the two criteria is tiny for stress states close to plane stress states. On the other hand, for the strongly confined stress states, the two approaches (of Rankine and Drücker Prager) are different, which is a limit of the model suggested.

3 Field of reversibility and functions thresholds

3.1 Pace of the field and the thresholds of reversibility

the field of reversibility is the field of the space of the stresses inside whose the ways of stress are reversible. Within the space of principal stresses $(\sigma_{1,}\sigma_{2,}\sigma_3)$, they are two cones whose axis is the trisecting one of equation $\sigma_1=\sigma_2=\sigma_3$. [Figure 3.1-a] a chart gives some.

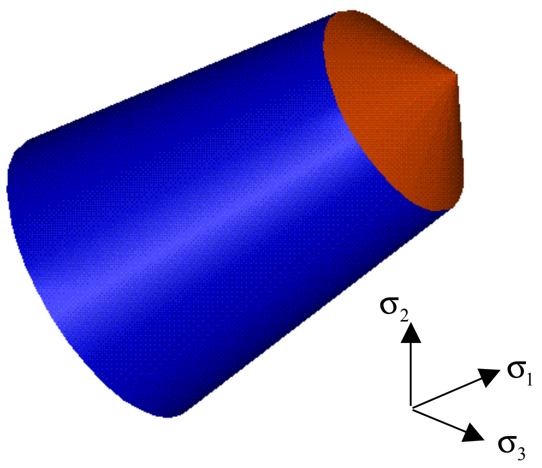


Figure 3.1 - has

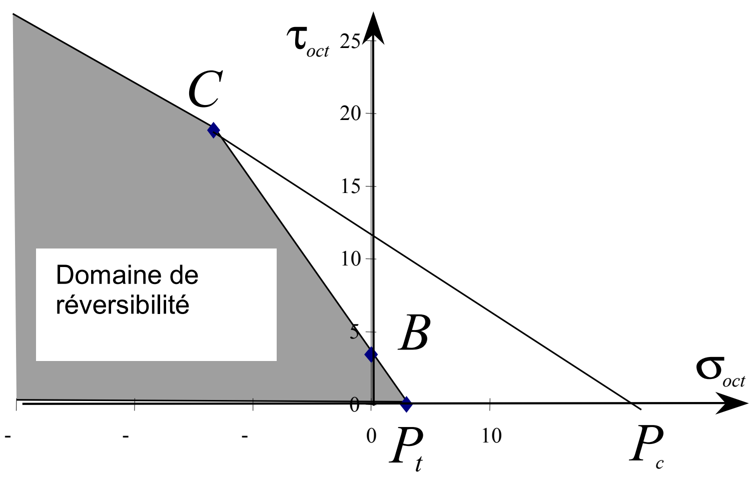


Figure 3.1 - B

In a plane $(\sigma_{oct}, \tau_{oct})$ the field of reversibility is determined by two straight lines as indicated on [Figure 3.1-b].
For a plane stress state, the field of reversibility is the cut of the three-dimensional field by a plane of equation $\sigma_3 = \mathrm{cste}$, as indicated on [Figure 3.1-c], result in a plane (σ_1, σ_2) being represented on the figure [Figure 3.1-d].

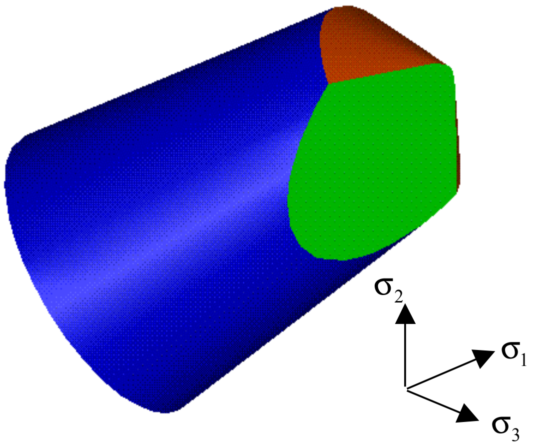


Figure 3.1 - C

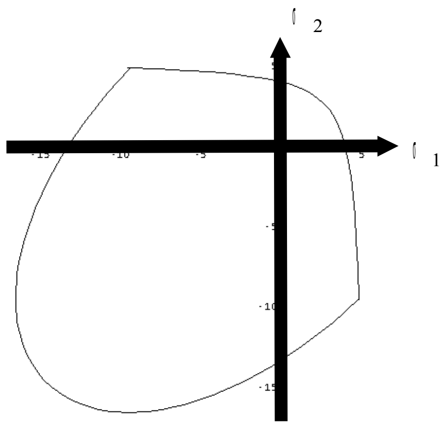


Figure 3.1 - D

3.2 Statement mathematical of the field of reversibility

It is defined by the inequation:

$$f(\boldsymbol{\sigma}, \boldsymbol{A}) \leq 0$$ **éq 3.2 -** 3.2-1

in which $\boldsymbol{A}$ represents the thermodynamic forces associated with the local variables (we note $\boldsymbol{\alpha}$ all the local variables).
For the model concrete that we present here, the equation [éq 3.2 - 3.2-1] take the particular shape

$$f_{comp}(\boldsymbol{\sigma}, A_c) = \frac{\tau_{oct} + a.\sigma_{oct}}{b} - \varphi\ f'_c + A_c \leq 0$$ **éq 3.2 -** 3.2-2

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\tau_{oct} + c.\sigma_{oct}}{d} - f'_t + A_t \leq 0$$ **éq 3.2 -** 3.2-3

$$f^H_{comp}(\boldsymbol{\sigma}, A_c) = \frac{a.\sigma_{oct}}{b} - \varphi\ f'_c + A_c \leq 0$$ **éq 3.2 -** 3.2-4

$$f^H_{trac}(\boldsymbol{\sigma}, A_t) = \frac{c.\sigma_{oct}}{d} - f'_t + A_t \leq 0$$ **éq 3.2 -** 3.2-5

equations [éq 3.2 - 3.2-2] and [éq 3.2 - 3.2-3] correspond respectively to the thresholds of “compression” and “tension”. The equations [éq 3.2 - 3.2-4] and [éq 3.2 - 3.2-5] limit the threshold of reversibility in the field of the isotropic tension, they amount excluding the x-axis on [Figure 3.1-b] beyond the points P_t or P_c. It is clear that only one of these two last conditions is enough. For the material not hammer-hardened, the choice of the coefficients is such that and $OP_t < OP_c$ the condition [éq 3.2 - 3.2-5] involve [éq 3.2 - 3.2-4].
We will see later that hardening can reverse the order of the points P_t and P_c, making the condition [éq 3.2 - 3.2-4] more constraining than [éq 3.2 - 3.2-5].

3.3 Rupture criterion. choice of the coefficients has, B, C and D

When the stress state reaches edge of the field of reversibility, of plastic strains develop thresholds and the move: they are hammer-hardened. The threshold of compression “increases” initially, then decreases, whereas the threshold of tension can only decrease. The threshold of fracture corresponds to the maximum field being able to be reached, it is represented on [Figure 3.3-a] in a diagram of plane stress:

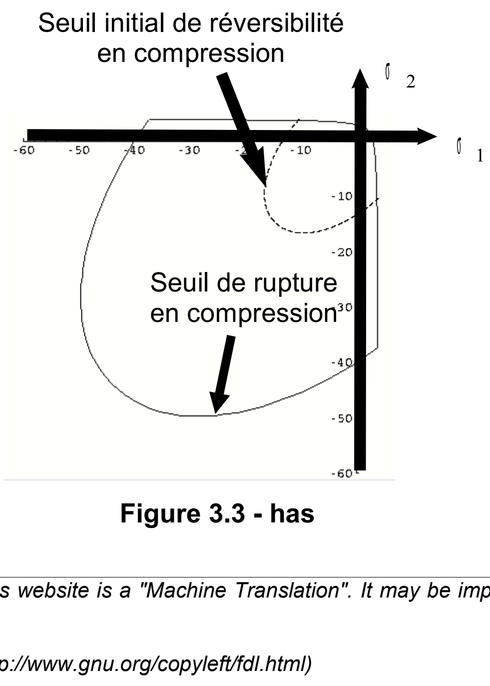


Figure 3.3 - has

the hardening of the thresholds results mathematically in the evolution of the quantitéset A_c A_t , the thresholds of fracture corresponding to the maximum of the fonctionset $f_c = \varphi f_c' - A_c$ $f_t = f_t' - A_t$. In the models selected, these functions are such as: $Max\, f_c = f_c'$ and $Max\, f_t = f_t'$;

The coefficients *has, B, C* , and *D* are thus defined from:

- F $_{you}$: strength in axial tension plain of the concrete,
- F $_{it}$: strength in axial compression plain of the concrete,
- F $_{DC}$ “: strength in axial compression Bi of the concrete,

One defines moreover coefficients: $\alpha = \frac{f_t'}{f_c'}$ and $\beta = \frac{f_{cc}'}{f_c'}$

to determine the coefficients has, B, C and D it is necessary to give oneself 4 equations which express in fact that the criteria are reached for particular and judiciously selected stress states.
A first possibility consists in writing that the two criteria are cut on the axes simple compression (points C of [Figure 3.3-b]).

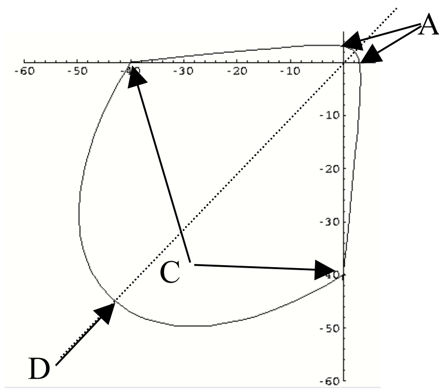


Figure 3.3 - B

By recalling that:

In simple compression: $\sigma < 0$; $\sigma_{oct} = \frac{\sigma}{3}$; $\tau_{oct} = -\frac{\sqrt{2}}{3}\sigma$

Out of Bi compression $\sigma < 0$; $\sigma_{oct} = 2\frac{\sigma}{3}$; $\tau_{oct} = -\frac{\sqrt{2}}{3}\sigma$

In simple tension $\sigma > 0$; $\sigma_{oct} = \frac{\sigma}{3}$; $\tau_{oct} = \frac{\sqrt{2}}{3}\sigma$

One obtains the following relations then:

Number of condition	Stress state	Criterion reaches	relation obtained
1	Compression simple	Compression	$a + 3\mathrm{b} = \sqrt{2}$
2	Bi compression	Compression	$2\mathrm{a} + \frac{3}{\beta} b = \sqrt{2}$

3	Tension simple	Tension	$-c+3\mathrm{d}=\sqrt{2}$
4	Compression simple	Tension	$c+3\alpha d=\sqrt{2}$

Table 3.3 - has

Which gives, while posing: $\alpha=\frac{f_t'}{f_c'}$ and $\beta=\frac{f_{cc}'}{f_c'}$

$$a=\sqrt{2}\frac{\beta-1}{2\beta-1} \quad b=\frac{\sqrt{2}}{3}\frac{\beta}{2\beta-1}$$ **éq 3.3 -** 3.3-1

$$c=\sqrt{2}\frac{1-\alpha}{1+\alpha} \quad d=\frac{2\sqrt{2}}{3}\frac{1}{1+\alpha}$$ **éq 3.3 -** 3.3-2

But this choice is problematic.

Indeed, after hardening of the criterion of tension, and for a limit of tension become null the field D" admissibility takes the shape indicated on [Figure 3.3-c], making nonacceptable of the Bi compressions states.

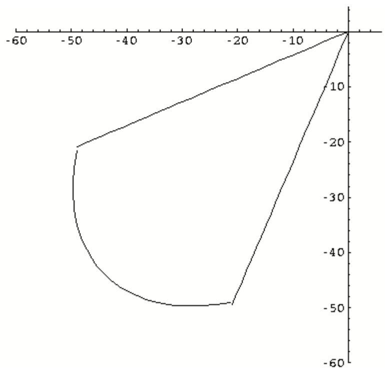


Figure 3.3 - C

Moreover, with this choice of the coefficients, certain ways of simple tension compression presented snap-back as indicated in appendix.

We then preferred to replace the condition number 4 of [Table 3.3-a] by a condition expressing that, after the limit of tension fell down to zero, the field of reversibility is that represented on [Figure 3.3-d].

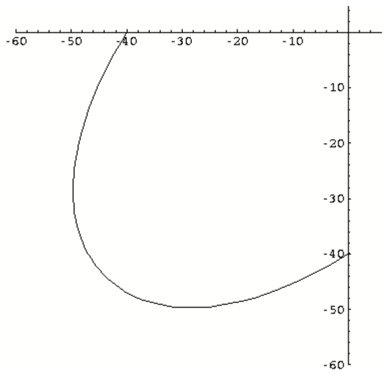


Figure 3.3 - D

This led to replace the relation $c+3\alpha d=\sqrt{2}$ by $c=\sqrt{2}$
the choice of the coefficients has, B, C and D is finally:

$$a=\sqrt{2}\frac{\beta-1}{2\beta-1} \quad b=\frac{\sqrt{2}}{3}\frac{\beta}{2\beta-1}$$ **éq 3.3 -** 3.3-3

$$c=\sqrt{2} \quad d=\frac{2\sqrt{2}}{3}$$ **éq 3.3 -** 3.3-4

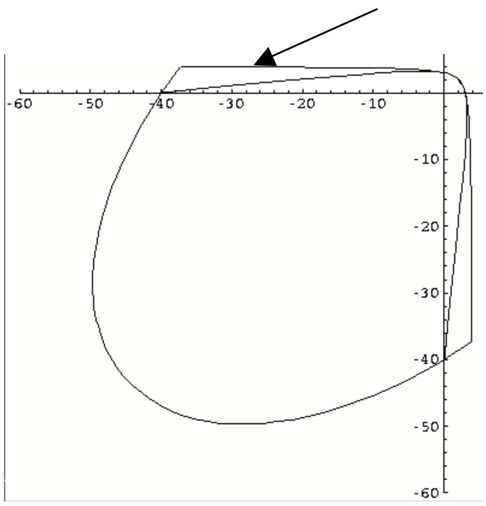


Figure 3.3 - E

It [Figure 3.3-e] shows the difference between the two models for a plane stress state.

3.4 Analyzes field of reversibility retained

In this chapter, we give indications on the order of magnitude of working stresses within the meaning of the criterion selected. We endeavour to give indications on tensile stresses, in particular for three-dimensional stress states.
[Figure 3.4-a] the watch initial fields (i.e. before hardening) for the following values of materials parameters:

f'_c	initial limit of fracture in simple compression:	f'_c = 40 limiting
f'_{cc}	Mpa initial of fracture out of Bi compression	f'_{cc} =44 Mpa
$\beta=\frac{f'_{cc}}{f'_c}$	relationship between rupture limit in bi--compression and simple compression	β = 1.1
$\varphi f'_c$	elastic limit in compression;	$\phi=0,33$
f'_t	initial limit of fracture in tension	f'_t = 4 Mpa

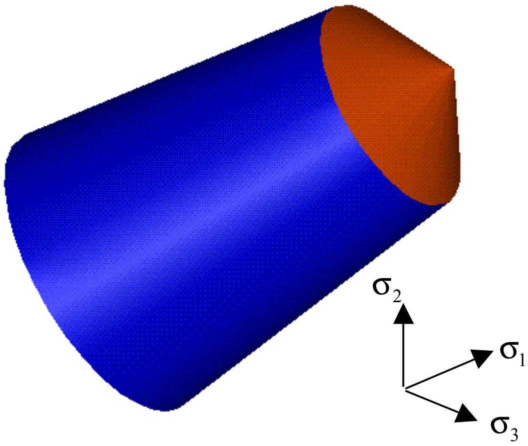


Figure 3.4 - has

the figures [Figure 3.4-b], [Figure 3.4-c] and [Figure 3.4-d] show the cuts of the three-dimensional field by the plane ones $\sigma_3=0$ and $\sigma_3=-25\,\text{Mpa}$

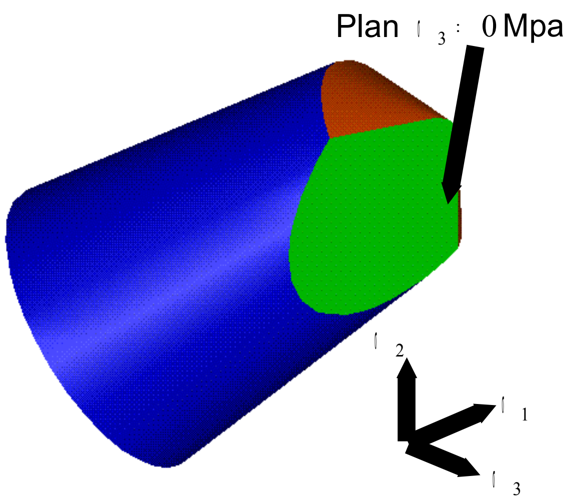


Figure 3.4 - B

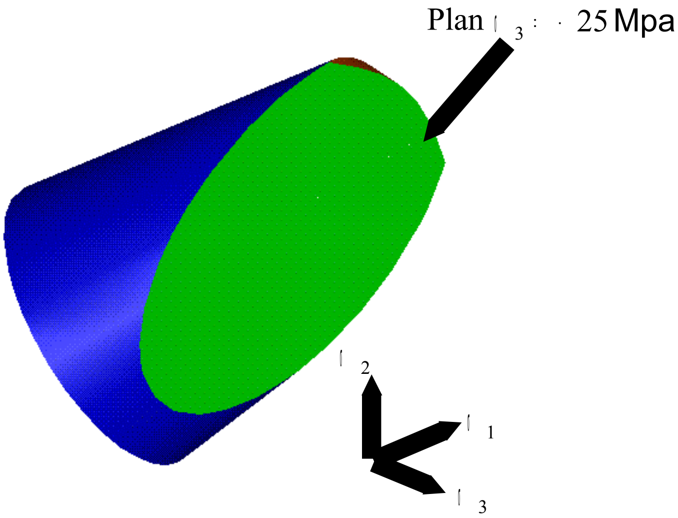


Figure 3.4 - C

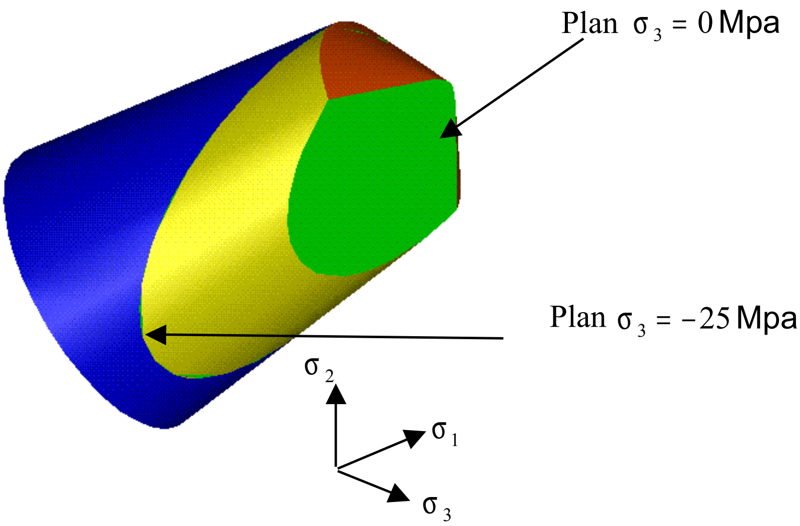


Figure 3.4 - D

It [Figure 3.4-e] shows the fields of reversibility in a plane (σ_1, σ_2) for constant σ_3 stress states, fields parameterized by the value of σ_3 . We represent the fields for $\sigma_3 = -25$ Mpa $\sigma_3 = 0$ Mpa $\sigma_3 = 4$ Mpa $\sigma_3 = 10$ Mpa $\sigma_3 = 15$ Mpa . It is seen there that for a containment of 25 Mpa of compression, tensile stresses can reach $15\,Mpa$, and that, in parallel, the field of reversibility for $\sigma_3 = 15$ Mpa N "is not empty and corresponds to order and compressive σ_1 stresses σ_2 of $-25\,Mpa$. It is also seen, that, for a value given of σ_3 , the maximum value of tension
Obtained for σ_1 and σ_2 is reached with L" intersection of the criteria of tension and compression.

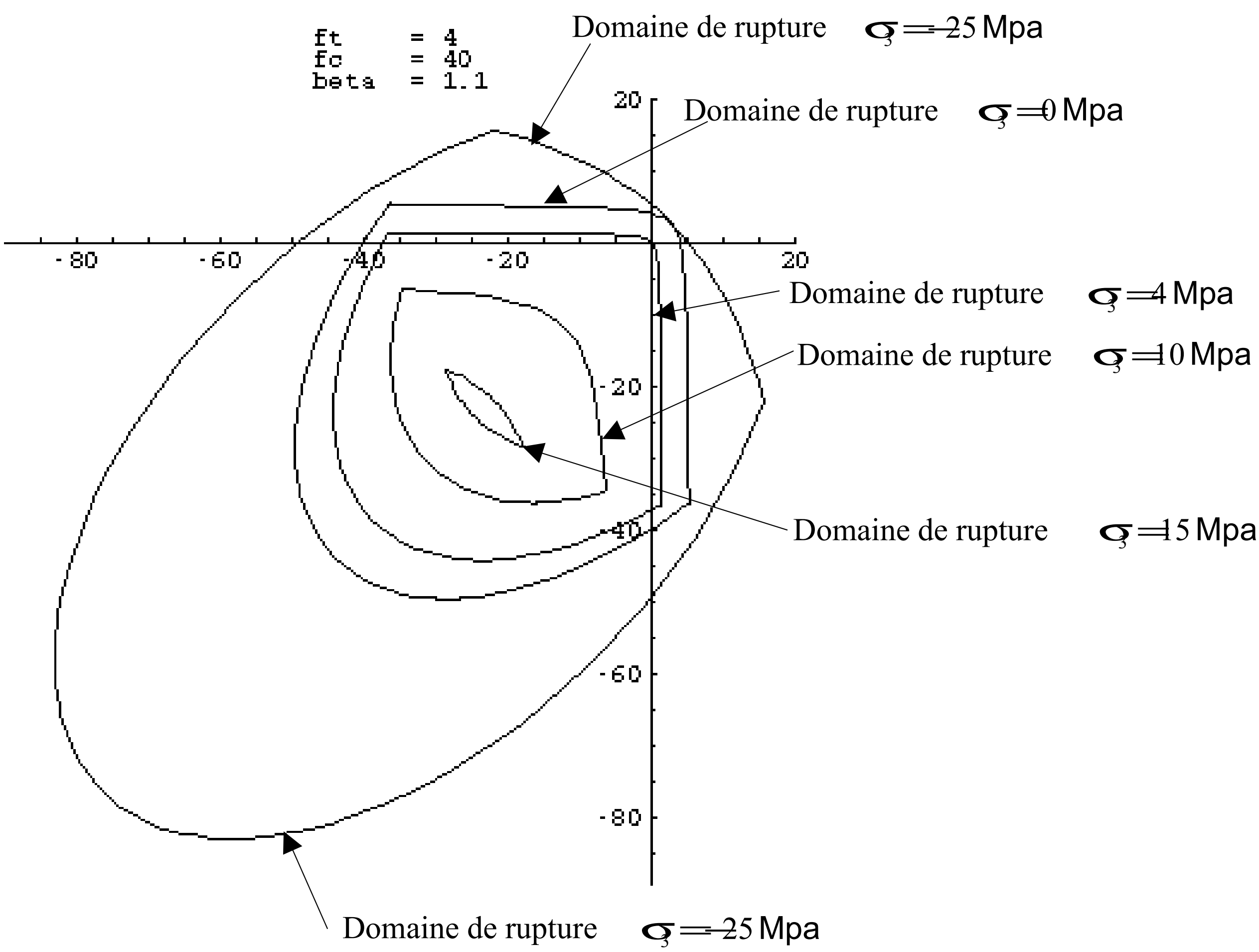


Figure 3.4 - E

We thus study the place of intersection of the criteria of tension and compression. We note $\left(\sigma_H^0, \sigma_0^{eq}\right)$ the point of intersection of the two criteria in the plane $\left(\sigma_H, \sigma^{eq}\right)$ (not C of [Figure 3.4-f]).

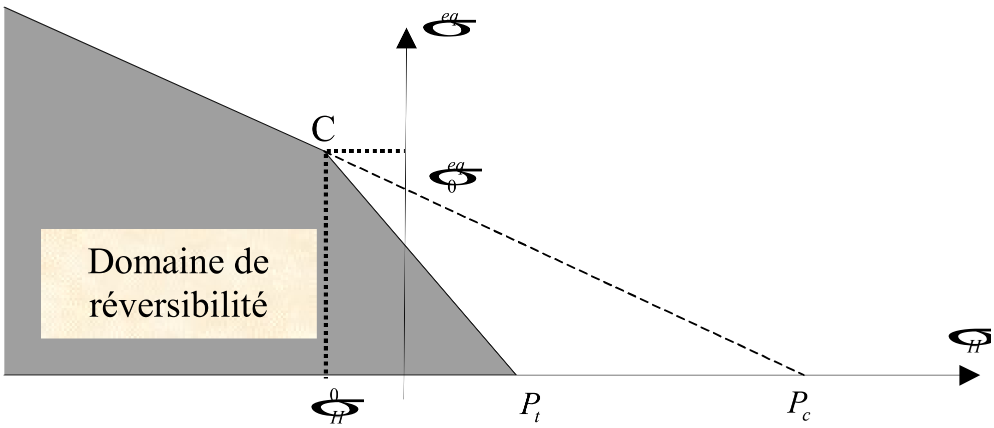


Figure 3.4 - F

the place of intersection of the two criteria within the space of stresses is given by:

$$\begin{cases} \sigma_1 = \frac{2}{3}\sigma_0^{\mathrm{eq}} \sin\left(\theta+\frac{\pi}{6}\right)+\sigma_H^0 \\ \sigma_2 = \frac{2}{3}\sigma_0^{\mathrm{eq}} \sin\left(-\theta+\frac{\pi}{6}\right)+\sigma_H^0 \\ \sigma_3 = 3\sigma_H^0-\sigma_1-\sigma_2 \end{cases}$$

Where θ is a parameter.

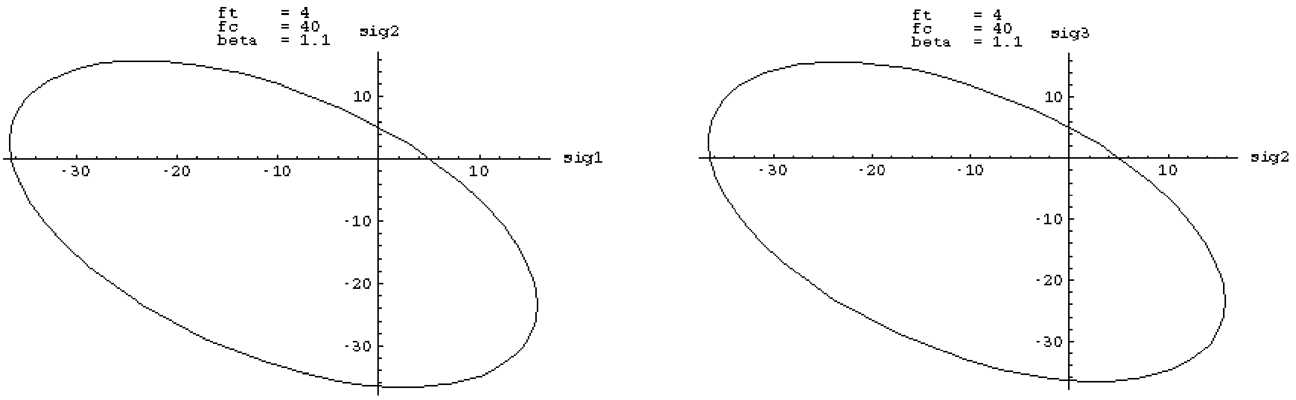


Figure 3.4 - G

It [Figure 3.4-g] shows projections of this place in the planes $(\sigma_{1,}\sigma_2)$ and $(\sigma_{2,}\sigma_3)$.
One can easily calculate the maximum value of the stress along this curve:

$$\sigma_{max}=\frac{f_c^{'}}{3}+\frac{2}{3\beta}f_t^{'}$$ **éq 3.4 -** 3.4-1

This equation shows that, whatever the value chosen for the rupture limit in tension, the maximum stress reached in tension is higher than the third of the rupture limit in compression.

[Figure 3.4-h] the watch three principal stresses according to the parameter θ .

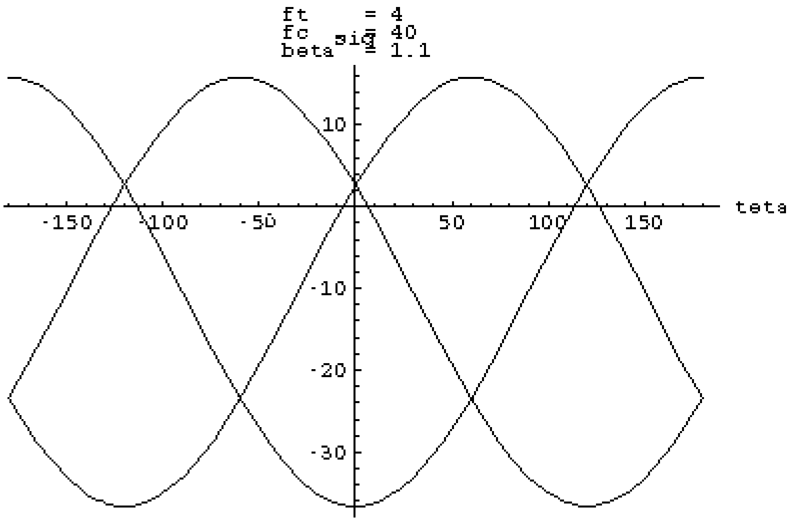


Figure 3.4 - H

One sees that one can reach a level of tension of $15\,Mpa$, but for a containment deet $\sigma_2=-25\,\mathrm{Mpa}$ $\sigma_3=-25\,\mathrm{Mpa}$.

To try to avoid this disadvantage, which is important, one can try to exploit the values of strength in compression and the parameter β .
As example, we chose the following clearance of parameters:

$f'_c=20\,Mpa$
$f'_{cc}=40\,Mpa$
$\beta=2$
$f'_t=4\,Mpa$

[Figure 3.4-i] the watch criteria with this choice of parameters. [Figure 3.4-j] the watch the value of the principal stresses at the intersection of the two criteria for this new choice of parameters. The maximum tension obtained is weaker ($8\,Mpa$), but it is reached for a level of containment also low ($-7\mathrm{Mpa}$).

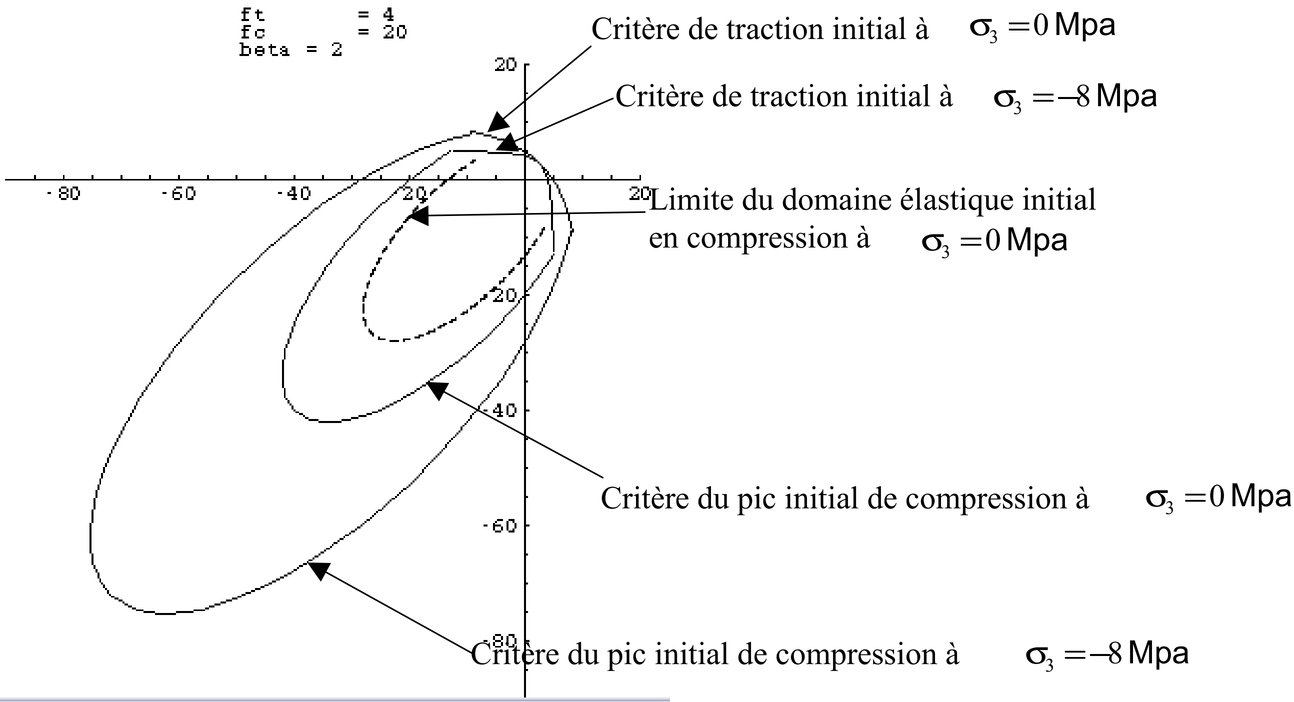


Figure 3.4 - I

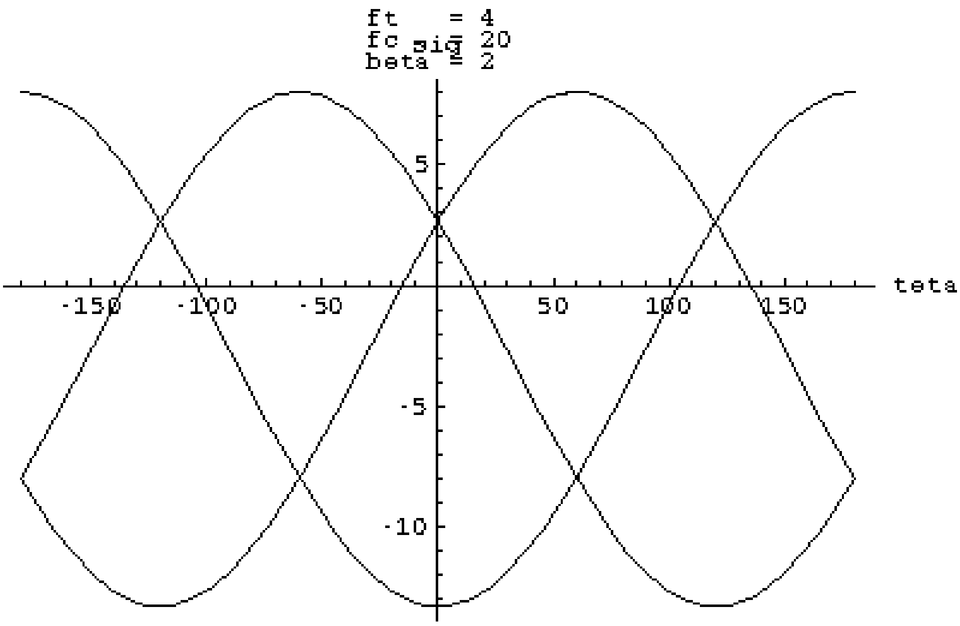


Figure 3.4 - J

3.5 Hardening

As we already mentioned in the paragraph [§ 3.3], when the stress state reaches edge of the field of reversibility, plastic strains and the local variables develop, the thresholds move: they are hammer-hardened. For our model, the local variables are two, they are noted κ_c^p for the local variable "known as of compression" and κ_t^p for that "known as of tension". These variables determine the evolution of the thresholds of compression and of tension respectively, the thermodynamic forces theirs are connected by the relations:

$$A_c=\varphi\ f'_c-f_c\left(\kappa_c^p\right)$$ **éq 3.5 -** 3.5-1

and

$$A_t=f'_t-f_t\left(\kappa_t^p\right)$$ **éq 3.5 -** 3.5-2

where $f_c\left(\kappa_c^p\right)$ and $f_t\left(\kappa_t^p\right)$ represent the values of strength in compression and tension respectively.

3.5.1 Functions of hardening

the function $f_c\left(\kappa_c^p\right)$ is initially increasing then decreasing, the decreasing part being either linear [Figure 3.5.1-a], or quadratic [Figure 3.5.1-b],

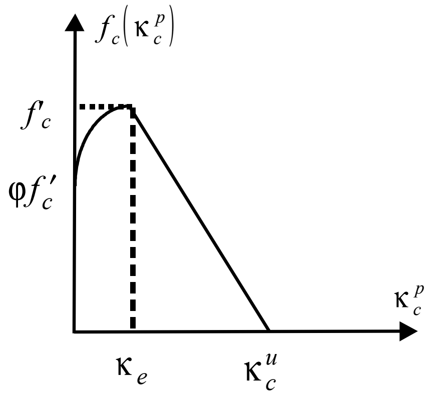


Figure 3.5.1-a

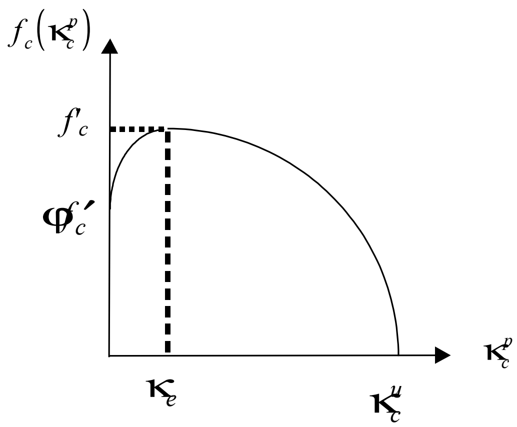
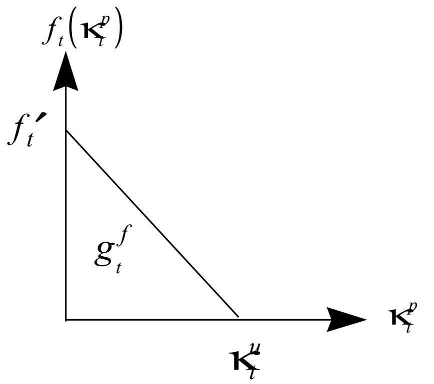


Figure 3.5.1-b

φ is a data of the model. The shape of the curve enters κ_c^e and κ_c^e (negative hardening) depends on the element, and more precisely on its dimensions, according to a criterion similar to that chosen by G. Heinfling, [ ] for the taking into account of the localization of the strains.
In tension, the shape of the curve giving the value of the elastic limit $f_t(\kappa_t^p)$ according to the cumulated plastic strain κ_t^p does not comprise a part “pre-peak”, the part “post-peak” being either linear [Figure 3.5.1-c], or exponential [Figure 3.5.1-d].



Appear 3.5.1-c

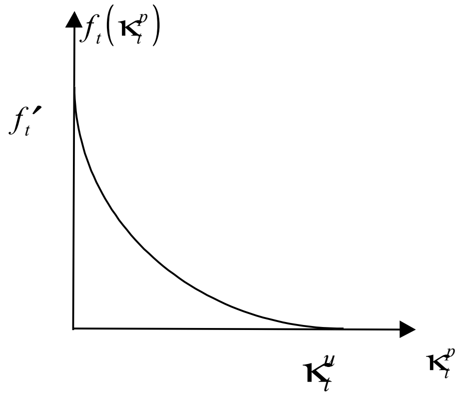


Figure 3.5.1-d

3.5.2 Curved of hardening and moduli post Model

3.5.2.1 peak of cracking distributed

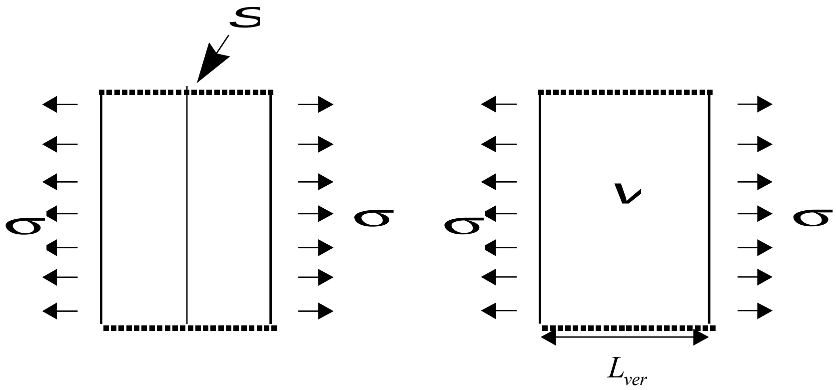
the introduction of a behavior softening post-peak into the relations stress-strains poses major problems. Under stress statical, beyond a certain level of stress, corresponding to the starter of the lenitive behavior, the equations governing the equilibrium of structure lose their elliptic nature. These equations of the mechanical problem then form a system of equations with partial derivatives evil posed of which the number of solutions is multiple. This problem results in an NON-objectivity compared to the mesh. It results from this a pathological sensitivity of the numerical solution to the smoothness and the directional sense of the mesh.
In order to solve this problem, or at least, to limit the consequences of them on the reliability of the predicted solution, it is necessary to use techniques known as of regularization. The object of these techniques is to enrich the mechanical description of the medium, to be able to describe nonhomogeneous states of strain, and to preserve the mathematical nature of the problem. One

operates this regularization by introducing, in the constitutive law, a characteristic length or internal length, connected to the width of the zone of localization. Several techniques are possible to improve the mechanical description of the lenitive medium. They constitute limitings device of localization. The implementation of these techniques requires in general, of the delicate numerical developments. An intermediate approach between the use of the classical models and the placement of these limitings device of localization consists in making depend the slope post-peak on the relation stress-strain, of the size of the element, so as to dissipate with the fracture a constant energy. This approach constitutes a step towards a nonlocal description of the continuum.

Let us consider initially a real crack of surface S whose measurement is A [3.5.2.1 Figure - has]. S is a surface of discontinuity of the field of displacement $\mathbf{u}$. It is supposed that to create this discontinuity, it is necessary to spend an energy W whose statement is: $\mathrm{W}=\int_S G_f(\mathbf{x})dS$, G_f being a property of the material.

Let us consider now that one wants to represent the same phenomenon, by not representing a discontinuity of displacement but a plastic strain uniformly distributed in a volume V .

Dissipated energy will be: $\mathrm{W}=\int_v dV\int_0^{t_r}\sigma^{ij}\frac{d\,\varepsilon_{ij}^p}{dt}dt$, where one noted t_r the “time-to-failure”.



Appear 3.5.2.1 - has

By making the series of following assumptions:

- the crack is plane,
  - G_f is constant along crack and thus $\mathrm{W}=A.G_f$,
  - V is a basic cylinder S and a height L_{ver} ,
  - $g_{f}=\int_0^{t_t}\sigma^{ij}\frac{d\,\varepsilon_{ij}^p}{dt}dt$ is constant in V .

One leads finally to the relation:

$$W=Vg_f=V\int_0^{t_t}\sigma^{ij}\frac{d\,\varepsilon_{ij}^p}{dt}dt=A.G_f \quad \text{éq 3.5.2.1 - 1}$$

Or:

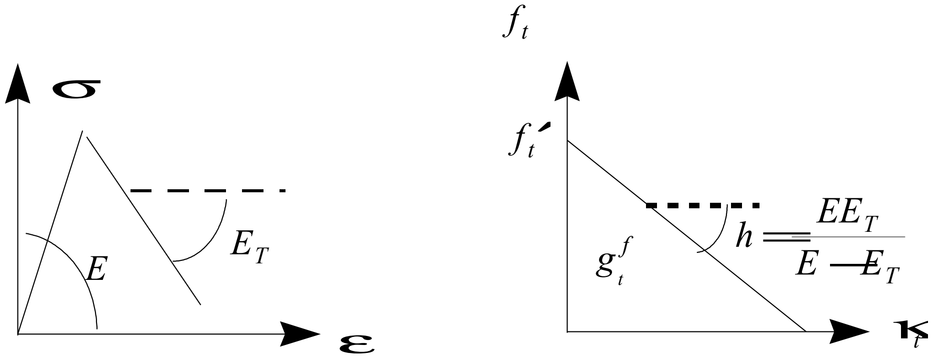
$$g^f=\int_0^{t_r}\sigma^{ij}\frac{d\,\varepsilon_{ij}^p}{dt}dt=\frac{G_f}{L_{ver}} \quad \text{éq 3.5.2.1 - 2}$$

It is seen easily that: $g^f=\int_0^{\kappa_u}f(\kappa)\,d\,\kappa$, writing in which the quantities (g^f,f,κ_u,κ) represent respectively $(g_t^f,f_t,\kappa_t^u,\kappa_t^p)$ in tension and $(g_c^f,f_c,\kappa_c^u,\kappa_c^p)$ compression. The data of g^f thus determines k_u , this in tension as in compression:

$$g_c^f=\int_0^{k_c^u}f(\kappa_c^p)d\,\kappa_c^p=\frac{G_c^f}{L_{ver}}$$

$$g_t^f=\int_0^{\kappa_t^u} f(\kappa_t^p)d\,\kappa_t^p=\frac{G_t^f}{L_{ver}}$$

The quantity g^f is thus related to the slope of the curve post peak in a diagram force-variable of hardening, which is related to the forced slope post peak in a diagram strain.
Let us suppose for example that the forced relation strain is linear in mode post peak. Let us call $E_T<0$ the slope post peak on the diagram $(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ and $h<0$ the corresponding slope in diagram (f,κ) [3.5.2.1 Figure - B]. There is the relation $h=\frac{EE_T}{E-E_T}\Leftrightarrow E_T=\frac{hE}{E+h}$ which shows that one must have:
$-h<E$, or else the diagram $(\boldsymbol{\sigma},\boldsymbol{\varepsilon})$ presents a snap back.



Appear 3.5.2.1 - B

the condition $-h<E$ is known as condition of applicability, it will result in an inequality on g^f and thus on L_{ver} .
In the frame of a resolution by the finite element method, ground volume representative of the fissured medium can be comparable to an element of the mesh. The characteristic length (noted thereafter l_c) introduced into the method of the energy of equivalent fracture corresponds to the length $Lver$.
During a computation corresponding to an unspecified structure, the determination this characteristic length is delicate. It depends on the position of the plane of crack, of dimensions and of the type of the elements…
a simple estimate for the two-dimensional cases can be expressed in the form:
$l_c=r\sqrt{A_e}$ where A_e is the area of the element considered, and r , a correct factor, being worth 1 for the quadratic elements, and $\sqrt{2}$ the linear elements.
One can extend this formulation to 3D case: $l_c=r\sqrt[3]{V_e}$ where V_e the volume of the element indicates.
Concerning the evolution of hardening with the temperature, we regard as in [bib2] that energies of fracture and the strength to fracture depend not on the current temperature T of the material point considered at time t , but of the maximum temperature reached in this point since the beginning of the loading until time t. When we need to show the dependence of the quantities compared to the temperature, we will note:

- θ the maximum of temperature since the beginning of loading,
- $f_c^{'}(\theta)$ strength in compression,
- $f_t^{'}(\theta)$ indicates strength in tension,
- $f_c(\theta,\kappa_c^p)$ the curve of hardening in compression,
- $f_t(\theta,\kappa_t^p)$ the curve of hardening in tension.

3.5.2.2 Behavior concrete in tension and linear curve post-peak

In this modelization, the concrete is supposed to be elastic until its strength in tension f'_t. The traction $f_t(\kappa_t^p)$ diagram is represented on [Figure 3.5.1-c] and is entirely defined by strength in tension of the material, the energy of cracking G_t^f, and the characteristic length l_c.
The mathematical statement of this curve is:

$$f_t(\theta,\kappa_t^p)=f'_t(\theta)\left(1-\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right) \qquad \textbf{éq 3.5.2.2 - 1}$$

the equivalence of dissipated energy makes it possible to write:

$$G_t^f(\theta)=l_c\int_0^{\kappa_t^u} f_t(\theta,\kappa_t^p)=l_c f'_t(\theta)\int_0^{\kappa_t^u}\left(1-\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right)d\kappa_t^p$$

from where

$$G_t^f(\theta)=\frac{l_c.f'_t(\theta).\kappa_t^u(\theta)}{2} \qquad \textbf{éq 3.5.2.2 - 2}$$

and

$$\kappa_t^u(\theta)=\frac{2.G_t^f(\theta)}{l_c.f'_t(\theta)} \qquad \textbf{éq 3.5.2.2 - 3}$$

the condition of applicability is written:

$$l_c \le \frac{2.E(\theta).G_t^f(\theta)}{f_t'^{\,2}(\theta)} \qquad \textbf{éq 3.5.2.2 - 4}$$

3.5.2.3 Behavior of the concrete in tension and exponential curve post-peak

In this modelization, the concrete is supposed to be elastic until its strength in tension f'_t. The traction $f_t(\kappa_t^p)$ diagram is represented on [Figure 3.5.1-d] and is entirely defined by strength in tension of the material, the energy of cracking G_t^f, and the characteristic length l_c.
The mathematical statement of this curve is:

$$f_t(\theta,\kappa_t^p)=f'_t(\theta).\exp\left(-a\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right) \qquad \textbf{éq 3.5.2.3 - 1}$$

the equivalence of dissipated energy makes it possible to write:

$$G_t^f(\theta)=l_c\int_0^{\infty} f_t(\theta,\kappa_t^p)=l_c f_{t'}(\theta)\int_0^{\infty}\exp\left(-a\frac{\kappa_t^p}{\kappa_t^u(\theta)}\right)d\kappa_t^p$$

From where:

$$G_t^f(\theta)=\frac{l_c.f'_t(\theta).\kappa_t^u(\theta)}{a} \qquad \textbf{éq 3.5.2.3 - 2}$$

and

$$\frac{\kappa_t^u(\theta)}{a}=\frac{G_t^f(\theta)}{l_c.f'_t(\theta)} \qquad \textbf{éq 3.5.2.3 - 3}$$

Is still: $f_t(\theta,\kappa_t^p)=f_t^{'}(\theta).\exp\left(-l_c.f_t^{'}(\theta)\frac{\kappa_t^p}{G_t^f(\theta)}\right)$

The maximum slope of the curve is then $h_{\max}(\theta)=\frac{-l_c.f_t^{'^2}(\theta)}{G_t^f(\theta)}$

and the condition of applicability is written:

$$l_c\le\frac{E(\theta).G_t^f(\theta)}{f_t^{'^2}(\theta)}\qquad\forall\theta$$ **éq 3.5.2.3 - 4**

3.5.2.4 Behavior of the concrete in compression and linear curve post-peak

In this modelization, the behavior of the concrete is supposed to be elastic until the elastic limit, given by a proportionality factor (noted φ. expressed as a percentage of strength to the peak $f_c^{'}(\theta)$. For the standard concretes φ is about 30%. The curve $f_c(\kappa_c^p)$ in compression is represented on [Figure 3.5.1-a] and is entirely defined by strength in tension of the material, the energy of cracking G_c^f , and the characteristic length l_c .
The mathematical statement of this curve is:

$$\begin{cases} f_c(\theta,\kappa_c^p)=f'_c(\theta)\left(\varphi+(2-2\phi)\frac{\kappa_c^u}{\kappa_e(\theta)}+(\varphi-1)\frac{\kappa_c^{p^2}}{\kappa_{e^2}(\theta)}\right) & \text{si}\quad \kappa_c^p\le\kappa_e(\theta) \qquad \textit{éq 3.5.2.4-1}\\ f_c(\theta,\kappa_c)=f'_c(\theta)\left(\frac{(\kappa_c^p-\kappa_c^u(\theta))}{(\kappa_e(\theta)-\kappa_c^u(\theta))}\right) & \text{si}\quad \kappa_e(\theta)\le\kappa_c^p\le\kappa_c^u(\theta) \qquad \textit{éq 3.5.2.4-2}\end{cases}$$

Strength in maximum compression is reached when: $\kappa_e(\theta)=(2-2\varphi)\frac{f_c^{'}(\theta)}{E(\theta)}$

The equivalence of dissipated energy makes it possible to write:

$$G_c^f(\theta)=l_c\int_0^{\kappa_c^u}f_c(\theta,\kappa_c^p)\,d\kappa_c^p$$

from where

$$G_c^f(\theta)=l_c.f_c^{'}(\theta)\left(\frac{2\varphi+1}{6}\kappa_e(\theta)+\frac{1}{2}\kappa_c^u(\theta)\right)$$ **éq 3.5.2.4 - 3**

and

$$\kappa_c^u(\theta)=\frac{2.G_c^f(\theta)}{l_c.f_c^{'}(\theta)}-\frac{2\varphi+1}{3}\kappa_e(\theta)$$ **éq 3.5.2.4 - 4**

the slope of the curve is then $h(\theta)=-\frac{f_c^{'}(\theta)}{\kappa_c^u(\theta)-\kappa_e(\theta)}$

and the condition of applicability is written:

$$l_c\le\frac{E(\theta).G_c^f(\theta)}{f_c^{'^2}(\theta)}\frac{6}{11-4\varphi-4\varphi^2}\qquad\forall\theta$$ **éq 3.5.2.4 - 5**

3.5.2.5 Behavior of the concrete in compression and nonlinear curve post-peak

In this modelization, the behavior of the concrete is supposed to be elastic until the elastic limit, given by a proportionality factor (noted φ. expressed as a percentage of strength to the peak $f'_c(\theta)$. For the standard concretes φ is about 30%. The curve $f_c(\kappa_c^p)$ in compression is represented on [Figure 3.5.1-b] and is entirely defined by strength in tension of the material, the energy of cracking G_c^f , and the characteristic length l_c .
The mathematical statement of this curve is:

$$\begin{cases} f_c(\theta,\kappa_c^p)=f'_c(\theta)\left(\varphi+(2-2\varphi)\dfrac{\kappa_c^p}{\kappa_e(\theta)}+(\varphi-1)\dfrac{\kappa_c^{p^2}}{\kappa_{e^2}(\theta)}\right) & \text{si} \quad \kappa_c^p\le\kappa_e(\theta) \quad \textit{éq 3.5.2.5-1} \\ f_c(\theta,\kappa_c)=f'_c(\theta)\left(1-\dfrac{\left(\kappa_c^p-\kappa_e(\theta)\right)^2}{\left(\kappa_c^u(\theta)-\kappa_e(\theta)\right)^2}\right) & \text{si} \quad \kappa_e(\theta)\le\kappa_c^p\le\kappa_c^u(\theta) \quad \textit{éq 3.5.2.5-2} \end{cases}$$

Strength in maximum compression is reached when: $\kappa_e(\theta)=(2-2\varphi)\dfrac{f'_c(\theta)}{E(\theta)}$

The equivalence of dissipated energy makes it possible to write:

$$G_c^f(\theta)=l_c\int_0^{\kappa_c^u} f_c(\theta,\kappa_c^p)\,d\kappa_c^p$$

from where:

$$G_c^f(\theta)=l_c.f'_c(\theta)\left(\frac{2}{3}\kappa_c^u(\theta)+\frac{\varphi}{3}\kappa_e(\theta)\right) \quad \textbf{éq 3.5.2.5 - 3}$$

and:

$$\kappa_c^u(\theta)=\frac{3}{2}\frac{G_c^f(\theta)}{l_c.f'_c(\theta)}-\frac{\varphi}{2}\kappa_e(\theta) \quad \textbf{éq 3.5.2.5 - 4}$$

the maximum slope of the curved post-peak is then $h_{\max}(\theta)=-\dfrac{2.f'_c(\theta)}{\kappa_c^u(\theta)-\kappa_e(\theta)}$

and the condition of applicability is written:

$$l_c\le\frac{3}{2}\frac{E(\theta).G_c^f(\theta)}{f_c'^2(\theta)}\frac{1}{4-\varphi-\varphi^2} \qquad \forall\theta \quad \textbf{éq 3.5.2.5 - 5}$$

4 Yielding

In this paragraph, we give the plastic form of the strainrates, by distinguishing the case says general where the stress state is located on a “regular” zone of edge of the field of reversibility and the case where it is at the top of one of the cones.

4.1 General form of the normality rule

In space $(\boldsymbol{\sigma}, A)$, the inequalities [éq 3.2 - 3.2-2], [éq 3.2 - 3.2-3], [éq 3.2 - 3.2-4], [éq 3.2 - 3.2-5], define a convex field which we will note $C_{(\sigma, A)}$. We will note Ψ_c the indicating function of this convex:

$$\psi_c(\boldsymbol{\sigma}, A) = \begin{cases} 0 \text{ si } (\boldsymbol{\sigma}, A) \in C_{(\sigma, A)} \\ \infty \text{ sinon} \end{cases} \qquad \text{éq 4.1 - 4.1-1}$$

When the border of the field of reversibility is reached, of plastic strains irreversible develop, according to the classical theory of plasticity.

For a standard material, [bib4] the flow model checks the maximum plastic work principle, which results in the equation:

$$\left(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}\right) \in \partial \Psi_c \qquad \textbf{éq 4.1 -} \text{ 4.1-2}$$

where $\partial \Psi_c$ note under differential of the function Ψ_c . We point out [bib3] that under differential of a convex function in a point x is all the vectors z such as: $f(x^*) \geq f(x) + \langle z, x^* - x \rangle \quad \forall x^*$

It is then seen easily that [éq 4.1 - 4.1-2] involves:

$$\Psi_c(\boldsymbol{\sigma}^*, \mathbf{A}^*) \geq \Psi_c(\boldsymbol{\sigma}, \mathbf{A}) + \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{\sigma}^* - \boldsymbol{\sigma}) + \dot{\boldsymbol{\alpha}}(\mathbf{A}^* - \mathbf{A}) \qquad \forall \boldsymbol{\sigma}^* \, et \, \mathbf{A}^* \qquad \textbf{éq 4.1 -} \text{ 4.1-3}$$

Taking into account the definition of the characteristic function, one to see easily that [éq 4.1 - 4.1-3] is equivalent to:

$$\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma} + \dot{\boldsymbol{\alpha}} \mathbf{A} \geq \dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^* \qquad \forall \boldsymbol{\sigma}^* \, et \, \mathbf{A}^* \in C_{(\sigma, A)} \qquad \textbf{éq 4.1 -} \text{ 4.1-4}$$

In other words yielding is such that the couple $(\boldsymbol{\sigma}, \mathbf{A})$ carries out the maximum of plastic dissipation among the acceptable thermodynamic forces.

4.2 Partly current statement of yielding

When the function f is differentiable at the point considered $(\boldsymbol{\sigma}, \mathbf{A})$ the normality rule writes simply

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad \textbf{éq 4.2 -} \text{ 4.2-1}$$

$$\dot{\boldsymbol{\alpha}} = \dot{\lambda} \frac{\partial f}{\partial \mathbf{A}} \qquad \textbf{éq 4.2 -} \text{ 4.2-2}$$

$\dot{\lambda}$ and f checking the conditions of Kuhn-Tucker:

$$\left.\begin{array}{c} \dot{\lambda} \geq 0 \\ f \leq 0 \\ \dot{\lambda} . f = 0 \end{array}\right\} \qquad \textbf{éq 4.2 -} \text{ 4.2-3}$$

the variable of hardening is related to the plastic multiplier by the model of hardening. By means of plastic work, one can write: $\dot{\kappa} f = \boldsymbol{\sigma} \dot{\boldsymbol{\varepsilon}}^p$.
If f is a tensorial function homogeneous of order 1 compared to the variable σ , one has $\boldsymbol{\sigma} \frac{\partial f}{\partial \boldsymbol{\sigma}} = f$, which leads to the equality: $\dot{\lambda} = \dot{\kappa}$ and thus finally with the equations:

$$\dot{\boldsymbol{\varepsilon}}_c^p = \dot{\kappa}_c^p \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}} \qquad \textbf{éq 4.2 -} \text{ 4.2-4}$$

$$\dot{\boldsymbol{\varepsilon}}_t^p = \dot{\kappa}_t^p \frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}$$ **éq 4.2 -** 4.2-5

4.3 Statement of yielding at the top of a cone

We give two presentations of same result. The first presentation uses the theory of the standard materials generalized and the under differentials, the second share of an equality posed a priori on plastic work.

4.3.1 Demonstration by the general theory of the standard materials

the field $C_{(\sigma, A)}$ consists of two cones. The function Ψ_c is not differentiable either at the intersection of these two cones, or at the top of each one of these cones. When the point $(\boldsymbol{\sigma}, \mathbf{A})$ belongs to the intersection of the two cones, the preceding equations remain valid, with the accuracy that the strains figure of compression and tension develop at the same time. This case known as "multi criterion" moreover is treated in [bib4]. We will be satisfied here to treat the case where $(\boldsymbol{\sigma}, \mathbf{A})$ is at the top of a cone, and we will choose the most frequent case of the top of the cone of tension, knowing that the case of the top of the cone of compression is treated exactly in the same way.
The criteria are rewritten by means of the variables σ^{eq} and σ_H, more practical in the development analytical.

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq} + \frac{c}{d} \sigma_H - f_t' + A_t \leq 0$$ 4.3.1-1

$$f_{trac}^H(\boldsymbol{\sigma}, A_t) = \frac{c}{d} \sigma_H - f_t' + A_t \leq 0$$ **4.3.1-2**

We thus consider a case where:

$$\left.\begin{array}{l} \sigma^{eq} = 0 \\ \dfrac{c}{d} \sigma_H - f_t' + A_t = 0 \end{array}\right\}$$ **4.3.1-3**

On the basis of [éq 4.1 - 4.1-4], we will calculate plastic dissipation as being the maximum of $\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^*$ for all couples $\boldsymbol{\sigma}^*, \mathbf{A}^* \in C_{(\sigma, A)}$

$$D^p = \underset{\boldsymbol{\sigma}^*, \boldsymbol{A}^* \in C_{(\sigma, A)}}{Max} (\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \mathbf{A}^*)$$ **4.3.1-4**

By writing whereas this maximum is finished and reached when $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}$ and $\boldsymbol{A}^* = \boldsymbol{A}$, we find conditions on $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\boldsymbol{\alpha}}$. In fact, the finished character will be enough.
By means of the decomposition of the tensors partly isotropic and déviatoire, and the particular form of the variables of hardening, one finds easily:

$$\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{\sigma}^* + \dot{\boldsymbol{\alpha}} \boldsymbol{A}^* = \dot{\tilde{\boldsymbol{\varepsilon}}}^p \boldsymbol{s}^* + 3 \sigma_H^* \dot{\varepsilon}_H^p + \dot{\kappa}_t^p A_t^*$$ 4.3.1-5

then Let us consider all Σ_1 the vectors forced of trace null and whose equivalent stress of Von Mises is worth 1: $\Sigma_1 = \{ \boldsymbol{\sigma}, \sigma^{eq} = 1, \mathrm{trace}(\boldsymbol{\sigma}) = 0 \}$

$$(\boldsymbol{\sigma}, \mathbf{A}) \in C_{(\sigma, A)} \Leftrightarrow \left\{ \begin{array}{l} \boldsymbol{\sigma} = \sigma^{eq} \mathbf{s_1} + \sigma_H \boldsymbol{I} \\ \mathbf{s_1} \in \Sigma_1 \\ \dfrac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq} + \dfrac{c}{d} \sigma_H - f_t' + A_t \leq 0 \\ \dfrac{c}{d} \sigma_H - f_t' + A_t \leq 0 \end{array} \right.$$ **4.3.1-6**

In other words, the "direction" of the deviator of the stresses is unspecified for a couple $(\boldsymbol{\sigma}, \boldsymbol{A}) \in C_{(\sigma, A)}$.
One can thus write:

$$D^p = \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^*, s_1^* \in \Sigma_1 \\ \left\{\begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \end{array}\right.}}{Max} \left(\sigma^{eq^*} \underset{s_1 \in \Sigma_1}{Max} \dot{\boldsymbol{\varepsilon}}^p \mathbf{s}_1^* + 3\sigma_H^* \dot{\varepsilon}_H^p + \dot{\kappa}_t^p A_t^* \right) \quad \mathbf{4.3.1\text{-}7}$$

It is clear that the maximum of $\dot{\boldsymbol{\varepsilon}}^p \boldsymbol{s}_1^*$ is reached when $\boldsymbol{s}_1^*$ is "parallel" with $\dot{\boldsymbol{\varepsilon}}^p$ and that one has then: $\underset{s_1 \in \Sigma_1}{Max} \dot{\boldsymbol{\varepsilon}}^p \boldsymbol{s}_1^* = \frac{2}{3} \dot{\varepsilon}_{eq}^p$.

[éq 4.3.1-7] can thus be written:

4.3.1-8

$$D^p = \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{\begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \end{array}\right.}}{Max} \left(\frac{2}{3} \dot{\varepsilon}_{eq}^p \sigma^{eq^*} \right) + \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{\begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \end{array}\right.}}{Max} \left(3 \boldsymbol{\sigma}_H^* \dot{\boldsymbol{\varepsilon}}_H^p \right) + \underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{\begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \end{array}\right.}}{Max} \left(\dot{\kappa}_t^p A_t^* \right)$$

Like $\dot{\varepsilon}_{eq}^p \ge 0$, one has for the first term:

$$\underset{\substack{\sigma^{eq^*}, \sigma_H^*, A_t^* \\ \left\{\begin{array}{l} \frac{\sqrt{2}}{3\mathrm{d}} \sigma^{eq^*} + \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \\ \frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0 \end{array}\right.}}{Max} \left(\frac{2}{3} \dot{\varepsilon}_{eq}^p \sigma^{eq^*} \right) = \frac{2}{3} \dot{\varepsilon}_{eq}^p \left(\frac{-3\mathrm{d}}{\sqrt{2}} A_t^* + \frac{3\mathrm{d}}{\sqrt{2}} f_t' \frac{3\mathrm{c}}{\sqrt{2}} \sigma_H^* \right) \quad \mathbf{4.3.1\text{-}9}$$

[éq 4.3.1-9] deferred in [éq 4.3.1-8] gives:

$$D^p = \sqrt{2}\, d\, \dot{\varepsilon}_{eq}^{p} f_t' + \underset{\frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0}{\underset{\sigma_H^*, A_t^*}{Max}} \left(\sigma_H^* \left(3\dot{\varepsilon}_H^{p} - \sqrt{2}\, c\, \dot{\varepsilon}_{eq}^{p} \right) \right) + \underset{\frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0}{\underset{\sigma_H^*, A_t^*}{Max}} \left(A_t^* \left(\dot{\kappa}_t^p - \sqrt{2}\, d\, \dot{\varepsilon}_{eq}^{p} \right) \right)$$

4.3.1-10

Let us pose then:

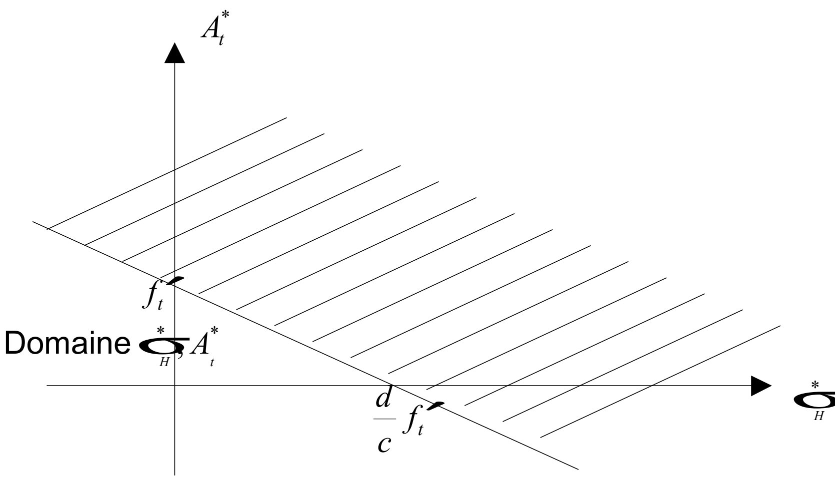
$m = 3\dot{\varepsilon}_H^{p} - \sqrt{2}\, c\, \dot{\varepsilon}_{eq}^{p}$, $n = \dot{\kappa}_t^p - \sqrt{2}\, d\, \dot{\varepsilon}_{eq}^{p}$ and $q = \sqrt{2}\, d\, \dot{\varepsilon}_{eq}^{p} f'$

With these notations, [éq 4.3.1-10] becomes:

$$D^p = q + \underset{\frac{c}{d}\sigma_H^* - f_t' + A_t^* \le 0}{\underset{\sigma_H^*, A_t^*}{Max}} \left(m\, \sigma_H^* + n\, A_t^* \right)$$

4.3.1-11

It is about a problem of type “simplex”. The field of σ_H^*, A_t^* is represented on [Figure 4.3.1 - has].



Appear 4.3.1-a

As the field σ_H^*, A_t^* extends towards $-\infty$ at the same time for σ_H^* and A_t^* , so that D^p is finished, it is necessary that m and n are positive. The maximum of $m\, \sigma_H^* + n\, A_t^*$ is reached for a couple σ_H^*, A_t^* located on edge of the field of σ_H^*, A_t^* .

One has then: $D^p = q + n f_t' + \underset{\sigma_H^*}{Max} \left(\sigma_H^* \left(m - n\frac{c}{d} \right) \right)$

So that D^p is finished, it is necessary that:

$$m = n\frac{c}{d}$$

Taking again the definitions of m and n , this relation gives:

$$3\, \dot{\varepsilon}_H^p = \frac{c}{d} \dot{\kappa}_t^p$$ **éq 4.3.1-12**

In addition, the stresses $m \ge 0$ and $n \ge 0$ give:

$$3\, \dot{\varepsilon}_H^p \ge \sqrt{2}\, c\, \dot{\varepsilon}_{eq}^p$$ **éq 4.3.1-13**

and

$$\dot{\kappa}_t^p \geq \sqrt{2}\, d\, \dot{\tilde{\varepsilon}}_{eq}^p \quad \textbf{éq 4.3.1-14}$$

these two last inequalities being equivalent because of [éq 4.3.1-11].
The equations [éq 4.3.1-11] and [éq 4.3.1-12] define yielding in the top of one of the cones of the field of reversibility.

4.3.2 Demonstration by plastic work

the starting point is to consider that compared to the developments made into cubes regular points, they are primarily the relations [éq 4.2-1] and [éq 4.2-2], known as normality rules, which cannot be written any more. However the relation [éq 4.2-1] implies the equality $\dot{\kappa}_t^p\, f_t(\kappa_t^p) = \boldsymbol{\sigma}\, \dot{\boldsymbol{\varepsilon}}^{\boldsymbol{p}}$, which can, it, being maintained.
We will thus leave the equation:

$$\dot{\kappa}_t^p\, f_t(\kappa_t^p) = \boldsymbol{\sigma}\, \dot{\boldsymbol{\varepsilon}}^{\boldsymbol{p}} \quad \textbf{éq 4.3.2-1}$$

We use partly isotropic decomposition and déviatoire tensors and find:

$$\dot{\kappa}_t^p\, f_t(\kappa_t^p) = \dot{\tilde{\boldsymbol{\varepsilon}}}^{\boldsymbol{p}}\, \boldsymbol{s} + 3\,\sigma_H\, \dot{\varepsilon}_H \quad \textbf{éq 4.3.2-2}$$

At the top of the cone of tension, one has the relations [éq 4.3.1-3], which, carried in [éq 4.3.2-2] give, by means of also [éq 3.5 - 3.5-2]:

$$\dot{\kappa}_t^p\, f_t(\kappa_t^p) = 3\,\frac{d}{c}\, f_t(\kappa_t^p)\, \dot{\varepsilon}_H \quad \textbf{éq 4.3.2-3}$$

And one thus finds the relation [éq 4.3.1-12]: . $3\,\dot{\varepsilon}_H = \frac{c}{d}\,\dot{\kappa}_t^p$

4.4 Together equations of behavior (summarized)

H the matrix of elasticity is noted:

$$H = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix}$$

With:

$\lambda = \nu \frac{E}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$, and the $K = \frac{3\lambda+2\mu}{3}$

forced relations strains are written finally:

$$\boldsymbol{\sigma} = \boldsymbol{H}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_c^p - \boldsymbol{\varepsilon}_t^p\right) \quad \textbf{éq 4.4 -}\ 4.4\text{-}1$$

For a regular point of the cone of compression:

$$f_{comp}(\boldsymbol{\sigma}, A_c) = \frac{\sqrt{2}}{3b}\sigma^{eq} + \frac{a}{b}\sigma_H - \varphi\, f_c^{'} + A_c \leq 0 \quad \textbf{éq 4.4 -}\ 4.4\text{-}2$$

$$\dot{\kappa}_c^p\, f_{comp} = 0 \quad ; \quad \dot{\varepsilon}_c^p = \dot{\kappa}_c^p \frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}} \quad \textbf{éq 4.4 -}\ 4.4\text{-}3$$

For a regular point of the cone of tension:

$$f_{trac}(\boldsymbol{\sigma}, A_t) = \frac{\sqrt{2}}{3\mathrm{d}}\sigma^{eq} + \frac{c}{d}\sigma_H - \varphi f'_t + A_t \leq 0$$ **éq 4.4 -** 4.4-4

$$\dot{\kappa}_t^p f_{trac} = 0 \quad ; \quad \dot{\varepsilon}_t^p = \dot{\kappa}_t^p \frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}$$ **éq 4.4 - 5**

For a point at the top of the cone of compression:

$$\mathbf{s}=0 \quad \textbf{éq 4.4 - 6}$$

$$f^{H}_{comp}(\boldsymbol{\sigma}, A_c)=\frac{a}{b}\sigma_H-\varphi\ f'_c+A_c=0 \quad \textbf{éq 4.4 - 7}$$

$$3\,\dot{\varepsilon}^{p}_{c\ H}=\frac{a}{b}\dot{\kappa}^{p}_{c} \quad \textbf{éq 4.4 - 8}$$

$$3\,\dot{\varepsilon}^{p}_{c\ H}\geq\sqrt{2}\,a\,\dot{\tilde{\varepsilon}}^{p}_{c\ eq} \quad \textbf{éq 4.4 - 9}$$

For a point at the top of the cone of tension:

$$\mathbf{s}=0 \quad \textbf{éq 4.4 - 10}$$

$$f^{H}_{trac}(\sigma, A_t)=\frac{c}{d}\sigma_H-f'_t+A_t=0 \quad \textbf{éq 4.4 - 11}$$

$$3\,\dot{\varepsilon}^{p}_{t\ H}=\frac{c}{d}\dot{\kappa}^{p}_{t} \quad \textbf{éq 4.4 - 12}$$

$$3\,\dot{\varepsilon}^{p}_{t\ H}\geq\sqrt{2}\,c\,\dot{\tilde{\varepsilon}}^{p}_{t\ eq} \quad \textbf{éq 4.4 - 13}$$

5 Numerical integration of the constitutive law

5.1 the total problem and the local problem: recalls

For a given structure (geometry and material), and for a given loading, the fields of displacement, stress and local variables are by solving a set of partial derivative equations nonlinear formed starting from the balance equations and of the constitutive laws. The document [bib5] presents the algorithm of which we give abstract here:

$\mathbf{u_0}$ and $\boldsymbol{\sigma_0}$ known

Buckles urgent t_i : known $\mathbf{L}_i=\mathbf{L}(t_i)$

- $\mathbf{u}_{i-1}$ loading; computation of the prediction $\Delta\mathbf{u}_i^0$
- Iterations of equilibrium of known Newton
  - $\mathbf{u}_i^n$ N; $\Delta\mathbf{u}_i^n=\mathbf{u}_i^n-\mathbf{u}_{i-1}$
  - Buckle elements el
    - Boucle points of gauss G
      - computation $\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}=\boldsymbol{\varepsilon}^{el}_{g}(\Delta\mathbf{u}_i^n)$
      - constitutive law :
        - computation of: $\boldsymbol{\sigma}^{el^n}_{g_i}$ and $\boldsymbol{\alpha}^{el^n}_{g_i}$ from $\boldsymbol{\sigma}^{el}_{g_{i-1}}$, $\boldsymbol{\alpha}^{el}_{g_{i-1}}$ and $\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}$
        - computation of $\dfrac{\partial\boldsymbol{\sigma}^{el^n}_{g_i}}{\partial\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}}$ (according to option)
      - Accumulation in vectors and matrixes assembled :
        - Accumulation of $\boldsymbol{Q}_g^{T^{el}}.\boldsymbol{\sigma}^{el^n}_{g_i}$ in $\boldsymbol{Q}^T.\boldsymbol{\sigma}_i^n$
        - Accumulation of computation inside $\boldsymbol{Q}_g^{T^{el}}\dfrac{\partial\boldsymbol{\sigma}^{el^n}_{g_i}}{\partial\Delta\boldsymbol{\varepsilon}^{el^n}_{g_i}}\boldsymbol{Q}_g^{el}\ \ \boldsymbol{K}_i^n$ (according to option)
  - Computation from $\delta\mathbf{u}_i^n$:
  - $\boldsymbol{K}_i^n.\delta\mathbf{u}_i^n=-\boldsymbol{Q}^T.\boldsymbol{\sigma}_i^n+\mathbf{L_i}$
  - linear iteration of search to determine ρ

Actualization :
$\Delta \mathbf{u}_i^{n+1} = \Delta \mathbf{u}_i^n + \rho\, \delta \mathbf{u}_i^n$
IF test convergence OK
| fine Newton: time step according to I = i+1
If not
| N = n+1

The computation of the stresses and local variables $\boldsymbol{\alpha}_{g_i}^{el^n}$, $\boldsymbol{\sigma}_{g_i}^{el^n}$ with the iteration of Newton n and the time t_i starting from the stresses and local variables $\boldsymbol{\sigma}_{g_{i-1}}^{el}$, $\boldsymbol{\alpha}_{g_{i-1}}^{el}$ time t_{i-1} and of the value and $\Delta \boldsymbol{\varepsilon}_{g_i}^{el^n}$ the increase in strain in the time interval estimated with the iteration of Newton n consists in integrating the equations [éq 4.4-1], [éq 4.4-2] with [éq 4.4-5] or [éq 4.4-6] with [éq 4.4-9] or [éq 4.4-10] with [éq 4.4-13] according to the cases with the initial conditions:

$$\boldsymbol{\alpha}\left(t_{i-1}\right) = \boldsymbol{\alpha}_{g_{i-1}}^{el}$$ **éq 5.1 -** 5.1-1

$$\boldsymbol{\sigma}\left(t_{i-1}\right) = \boldsymbol{\sigma}_{g_{i-1}}^{el}$$ **éq 5.1 -** 5.1-2

$$\boldsymbol{\varepsilon}^p\left(t_{i-1}\right) = \boldsymbol{\varepsilon}_{g_{i-1}}^{p^{el}}$$ **éq 5.1 -** 5.1-3

With the condition of loading in imposed strain:

$$\boldsymbol{\varepsilon}\left(t_i\right) = \boldsymbol{\varepsilon}_{g_i}^{el^n}$$ **éq 5.1 -** 5.1-4

result of this integration will provide:

$$\boldsymbol{\sigma}_{g_i}^{el^n} = \boldsymbol{\sigma}\left(t_i\right)$$
$$\boldsymbol{\varepsilon}_{g_i}^{p^{el^n}} = \boldsymbol{\varepsilon}^p\left(t_i\right)$$
$$\boldsymbol{\alpha}_{g_i}^{el^n} = \boldsymbol{\alpha}\left(t_i\right)$$

The object of this chapter is to present the numerical integration of these equations. It is about a system of equations nonlinear differentials which we solve by an implicit method of Eulerian. From now, the quantity at the beginning of time step (known) will be noted with an index˜ , whereas the unknowns at the end of time step (all unknowns except $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{g_i}^{el^n}$) are noted without index. For a quantity quelconqueon a note $\Delta a = a - a^-$.
One always starts by calculating an elastic solution $\boldsymbol{\sigma}^e$, by supposing that there is no evolution of plastic strains and local variables. So at least one of the criteria is violated by this elastic solution, it has reasons there to calculate yieldings. It is then necessary to distinguish the regular cases for which the solutionest $\boldsymbol{\sigma}$ on the regular part of one of the cones or at their intersection of the cases known as singular where the solutionest $\boldsymbol{\sigma}$ at the top of one of the two cones. Logic allowing to examine and choose these various cases, and the algorithm which results from this are relatively complex. We thus present in first the processing of each case and explain their sequence subsequently, in the chapter [§ 5.7].

5.2 Digital processing of the regular case.

One presents in detail only the case where develop at the same time plastic strains in tension and compression and where thus solutionappartient it $\boldsymbol{\sigma}$ at the intersection of the two cones. Let us note however that, even if $\boldsymbol{\sigma}^e$ violates at the same time the two criteria, for as much the final solution $\boldsymbol{\sigma}$ can belong very well finally only to one of the cones hammer-hardened. One is thus brought to search balanced which one applies that they belong to one of the two cones or both. The case or it belongs to only one of the two cones results easily from the more general case presented here. The equations which we have to solve are finally:

$$\mathbf{s}^{e}=\frac{\mu}{\mu^{-}}\mathbf{s}^{-}+2\mu\Delta\tilde{\boldsymbol{\varepsilon}}$$ **éq 5.2 -** 5.2-1

$$\sigma_{H^e}=\frac{K}{K^{-}}\sigma_{H^-}+3K\Delta\varepsilon_H$$ **éq 5.2 -** 5.2-2

$$\mathbf{s}=\mathbf{s}^{e}-2\mu\left(\Delta\tilde{\boldsymbol{\varepsilon}}_c^p+\Delta\tilde{\boldsymbol{\varepsilon}}_t^p\right)$$ **éq 5.2 -** 5.2-3

$$\sigma_H=\sigma_H^e-3K\left(\Delta\varepsilon_{H_c}^p+\Delta\varepsilon_{H_t}^p\right)$$ **éq 5.2 -** 5.2-4

$$f_{comp}(\boldsymbol{\sigma},\kappa_c^p)=\frac{\sqrt{2}}{3b}\sigma^{eq}+\frac{a}{b}\sigma_H-f_c(\kappa_c^p)=0$$ **éq 5.2 -** 5.2-5

$$\Delta\varepsilon_c^p=\Delta\kappa_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}$$ **éq 5.2 -** 5.2-6

$$f_{trac}(\boldsymbol{\sigma},\kappa_t^p)=\frac{\sqrt{2}}{3d}\sigma^{eq}+\frac{c}{d}\sigma_H-f_t(\kappa_t^p)=0$$ **éq 5.2 -** 5.2-7

$$\Delta\varepsilon_t^p=\Delta\kappa_t^p\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}$$ **éq 5.2 -** 5.2-8

By taking the isotropic and deviatoric parts plastic strains, the equations [éq 5.2-6] and [éq 5.2 - 5.2-8] give:

$$\Delta\tilde{\boldsymbol{\varepsilon}}_c^p=\frac{\Delta\kappa_c^p}{\sqrt{2}b}\frac{\mathbf{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-9

$$\Delta\boldsymbol{\varepsilon}_{H_c}^p=\Delta\kappa_c^p\frac{a}{3b}$$ **éq 5.2 -** 5.2-10

$$\Delta\tilde{\boldsymbol{\varepsilon}}_t^p=\frac{\Delta\kappa_t^p}{\sqrt{2}d}\frac{\mathbf{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-11

$$\Delta\varepsilon_{H_t}^p=\Delta\kappa_t^p\frac{c}{3d}$$ **éq 5.2 -** 5.2-12

While deferring [éq 5.2-9] and [éq 5.2-11] in [éq 5.2 - 5.2-3], one finds:

$$\mathbf{s}=\mathbf{s}^{e}-2\mu\left(\frac{\Delta\kappa_c^p}{\sqrt{2}b}+\frac{\Delta\kappa_t^p}{\sqrt{2}d}\right)\frac{\mathbf{s}}{\sigma^{eq}}$$ **éq 5.2 -** 5.2-13

which shows that $\mathbf{s}$ is parallel to $\mathbf{s}^e$ from where one deduces:

$$\frac{\mathbf{s}}{\sigma^{eq}}=\frac{\mathbf{s}^e}{\sigma^{e^{eq}}}$$ **éq 5.2 -** 5.2-14

While deferring [éq 5.2-14] in [éq 5.2-9] and [éq 5.2-11] one finds:

$$\Delta\tilde{\boldsymbol{\varepsilon}}_c^p=\frac{\Delta\kappa_c^p}{\sqrt{2}b}\frac{\mathbf{s}^e}{\sigma^{e^{eq}}}$$ éq 5.2 - 5.2-15

$$\Delta\tilde{\boldsymbol{\varepsilon}}_t^p=\frac{\Delta\kappa_t^p}{\sqrt{2}d}\frac{\mathbf{s}^e}{\sigma^{e^{eq}}}$$ éq 5.2 - 5.2-16

One defers then [éq 5.2 - 5.2-15] and [éq 5.2 - 5.2-16] in [éq 5.2 - 5.2-3] and [éq 5.2 - 5.2-4], and one expresses the criteria [éq 5.2-5] and [éq 5.2 - 5.2-7] with these new results. That led to two equations having like inconnueset $\Delta\kappa_c^p$ $\Delta\kappa_t^p$:

$$\frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)-f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)=0$$ éq 5.2 - 5.2-17

$$\frac{\sqrt{2}}{3\mathrm{d}}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e-\Delta\kappa_c^p\left(\frac{2\mu}{3\,bd}+\frac{Kac}{bd}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3\mathrm{d}^2}+\frac{Kc^2}{d^2}\right)-f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)=0$$ éq 5.2 - 5.2-18

It is this system of two equations to two unknowns who should finally be solved. If the linear f_c f_t fonctionsetsont, i.e. if one is in linear mode post-peak in compression as in tension, it is about a linear system which will thus be solved in an iteration. In the case, is mode pre-peak in compression (which is always nonlinear), that is to say modelizations with nonlinear modes post peak, the system [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] is solved by a method of Newton:

One notes $f^*_{comp}\left(\Delta\kappa_c^p, \Delta\kappa_t^p\right)$ the criterion of compression regarded as function of the only variables $\Delta\kappa_c^p$ and $\Delta\kappa_t^p$, in the same way for the tension:

$$f^*_{comp}\left(\Delta\kappa_c^p, \Delta\kappa_t^p\right)=\frac{\sqrt{2}}{3\mathrm{b}}\sigma^{e^{eq}}+\frac{a}{b}\sigma^e_H-\Delta\kappa_c^p\left(\frac{2\mu}{3\mathrm{b}^2}+\frac{Ka^2}{b^2}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)-f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)$$

$$f^*_{trac}\left(\Delta\kappa_c^p, \Delta\kappa_t^p\right)=\frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma^e_H-\Delta\kappa_c^p\left(\frac{2\mu}{3bd}+\frac{Kac}{bd}\right)-\Delta\kappa_t^p\left(\frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}\right)-f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)$$

The ième iteration of Newton for system [éq 5.2 - 5.2-17] - [éq 5.2 - 5.2-18] is:

$$\begin{Bmatrix}\Delta\kappa_c^p\\ \Delta\kappa_t^p\end{Bmatrix}^{i+1}=\begin{Bmatrix}\Delta\kappa_c^p\\ \Delta\kappa_t^p\end{Bmatrix}^{i}-\boldsymbol{J}_i^{-1}\begin{Bmatrix}f^*_{comp}\left(\Delta\kappa_c^p, \Delta\kappa_t^p\right)\\ f^*_{tracp}\left(\Delta\kappa_c^p, \Delta\kappa_t^p\right)\end{Bmatrix}^{i}$$

The jacobian $\boldsymbol{J}_i$ is worth:

$$\boldsymbol{J}_i=\begin{bmatrix}\dfrac{\partial f^*_{comp}}{\Delta\kappa_c^p} & \dfrac{\partial f^*_{comp}}{\Delta\kappa_t^p}\\ \dfrac{\partial f^*_{trac}}{\Delta\kappa_c^p} & \dfrac{\partial f^*_{trac}}{\Delta\kappa_t^p}\end{bmatrix}$$

With:

$$\frac{\partial f^*_{comp}}{\Delta\kappa_c^p}=-\left(\frac{2\mu}{3\mathrm{b}^2}+\frac{Ka^2}{b^2}\right)-\frac{\partial f_c\left(\kappa_c^{p^-}+\Delta\kappa_c^p\right)}{\partial\Delta\kappa_c^p}$$

$$\frac{\partial f^*_{comp}}{\partial\Delta\kappa_t^p}=-\left(\frac{2\mu}{3\mathrm{bd}}+\frac{Kac}{bd}\right)$$

$$\frac{\partial f^*_{trac}}{\partial\Delta\kappa_c^p}=-\left(\frac{2\mu}{3\mathrm{bd}}+\frac{Kac}{bd}\right)$$

$$\frac{\partial f^*_{trac}}{\partial\Delta\kappa_t^p}=-\left(\frac{2\mu}{3\mathrm{d}^2}+\frac{Kc^2}{d^2}\right)-\frac{\partial f_t\left(\kappa_t^{p^-}+\Delta\kappa_t^p\right)}{\partial\Delta\kappa_t^p}$$

The initial Jacobian of the system results from the values of derivatives in $\Delta\kappa_c^p=0$ and $\Delta\kappa_t^p=0$, which amounts solving the nonlinear system on the basis of the solution null. Nonthe linearities are introduced by the curves of softening. In the post-peak part, when they are linear, convergence is done in an iteration. When they are nonlinear, convergence requires only some iterations. To leave the solution null thus does not pose problem of convergence. That returns from the linearization of the criteria in the vicinity of the elastic prediction.

5.3 Existence of a solution and condition of applicability

We point out that the solution from the problem [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] must check the conditions [éq 4.2-3] and thus inter alia the positivity of the increases in the plastic multipliers.

$$\Delta \kappa_c^p \geq 0 \quad \textbf{éq 5.3 -} \text{ 5.3-1}$$

$$\Delta \kappa_t^p \geq 0 \quad \textbf{éq 5.3 -} \text{ 5.3-2}$$

Let us suppose that we are in a case of linear behavior post peak in tension as in compression and call respectively h_c and the h_t slopes of the parts post peak. The increases in the plastic multipliers are obtained by solving the linear system:

$$\begin{bmatrix} \frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}+h_c & \frac{2\mu}{3bd}+\frac{Kac}{bd} \\ \frac{2\mu}{3bd}+\frac{Kac}{bd} & \frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}+h_t \end{bmatrix} \begin{pmatrix} \Delta \kappa_c^p \\ \Delta \kappa_t^p \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e - f_c\left(\kappa_c^{p^-}\right) \\ \frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e - f_t\left(\kappa_t^{p^-}\right) \end{pmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-3}$$

Since the criteria of tension and compression were activated in tension as in compression, the second member of this system is positive. But nothing ensures in so far as the solution of [éq 5.3 - 3] will be positive.
If one poses:

$$\boldsymbol{HTC}=\begin{bmatrix} \frac{2\mu}{3b^2}+\frac{Ka^2}{b^2}+h_c & \frac{2\mu}{3bd}+\frac{Kac}{bd} \\ \frac{2\mu}{3bd}+\frac{Kac}{bd} & \frac{2\mu}{3d^2}+\frac{Kc^2}{d^2}+h_t \end{bmatrix} \quad \text{éq } \textbf{5.3 -} \text{ 5.3-4}$$

One a:

$$\begin{Bmatrix} \Delta \kappa_c^p \\ \Delta \kappa_t^p \end{Bmatrix} = \boldsymbol{HTC}^{-1} \begin{pmatrix} \frac{\sqrt{2}}{3b}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e - f_c\left(\kappa_c^{p^-}\right) \\ \frac{\sqrt{2}}{3d}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e - f_t\left(\kappa_t^{p^-}\right) \end{pmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-5}$$

With:

$$|\mathbf{HTC}|=3\frac{h_t}{\beta^2}(3K+\mu)-6\frac{h_t}{\beta}+h_c h_t+\frac{9}{4}h_c K+9 h_t K+\frac{3\mu}{4}(h_c+16 h_t+9K) \quad \textbf{éq 5.3 -} \text{ 5.3-6}$$

$$\mathbf{HTC}^{-1}=\frac{1}{|\mathbf{HTC}|}\begin{bmatrix} h_t+\frac{3}{4}(3K+\mu) & \frac{9K}{2\beta}-\frac{9K}{2}+\frac{3\mu}{2\beta}-3\mu \\ \frac{9K}{2\beta}-\frac{9K}{2}+\frac{3\mu}{2\beta}-3\mu & h_c+9K-18\frac{K}{\beta}+12\mu\frac{(\beta-1)}{\beta} \end{bmatrix} \quad \textbf{éq 5.3 -} \text{ 5.3-7}$$

One sees that conditions from positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2] lead to relatively complicated relations. If the solution from the problem [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18] does not check the conditions of positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2], that can correspond is with the fact that the coefficientsetsont h_c h_t such as there is no positive solution (that would correspond to a snap-back in a diagram (σ, K)), that is to say with the fact that the solution activates finally only one of the two criteria.
Let us examine the simpler case of only one activated criterion. Let us suppose to fix the notations that the only activated criterion is the criterion of tension.
One must have:

$$\Delta\kappa_t^p\left(\frac{2\mu}{3\mathrm{d}^2}+\frac{Kc^2}{d^2}+h_t\right)=\frac{\sqrt{2}}{3\mathrm{d}}\sigma^{e^{eq}}+\frac{c}{d}\sigma_H^e-f_t\left(\kappa_t^{p-}\right)$$ **éq 5.3 -** 5.3-8

One sees reappearing the condition known as of applicability:

$$\left(\frac{2\mu}{3\mathrm{d}^2}+\frac{Kc^2}{d^2}+h_t\right)>0$$ **éq 5.3 -** 5.3-9

This condition is the generalization of the condition $-h<E$ presented to the paragraph [§ 3.5.2.1] in a typical case of axial request plain.
The following strategy will thus be retained:

If $\frac{\sqrt{2}}{3\mathrm{b}}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e-f_c\left(\kappa_c^{p-}\right)>0$ and $\frac{\sqrt{2}}{3\mathrm{b}}\sigma^{e^{eq}}+\frac{a}{b}\sigma_H^e-f_c\left(\kappa_c^{p-}\right)>0$

Activation a priori of the two criteria: resolution problem [éq 5.2 - 5.2-17] and [éq 5.2 - 5.2-18]
So not convergence or so not checking conditions of positivity [éq 5.3 - 5.3-1] and [éq 5.3 - 5.3-2]

Seeks solution with only one criterion activated
So not convergence or not checking condition positivity

Stop on diagnostic of nonchecking of condition of applicability of the type [éq 5.3 - 5.3-9]

5.4 Processing of the nonregular cases

In this paragraph, we describe the discrete processing of the equations corresponding to projection at the top of the cone of tension, [éq 4.4-10] with [éq 4.4-13], knowing that projection at the top of the cone of compression is done in the same way. The equations [éq 4.4-10] and [éq 4.4-12] define yielding in this case, whereas the equation [éq 4.4-13] is a condition of acceptability of projection at the top of the cone.

5.4.1 Computation of the stresses and plastic strains

the discrete forms of [éq 4.4-10] with [éq 4.4-13], are:

$$\mathbf{s}=0$$ **éq 5.4.1-1**

$$\sigma_H=\frac{d}{c}f_t\left(\kappa_t^{p-}+\Delta\kappa_t^p\right)$$ **éq 5.4.1-2**

$$3\Delta\varepsilon_{t_H}^p=\frac{c}{d}\Delta\kappa_t^p$$ **éq 5.4.1-3**

the relation [éq 5.2 - 5.2-4] established in the regular case is always valid, one jointly uses it with [éq 5.4.1-3] in [éq 5.4.1-1] and one obtains:

$$\sigma_H^e-\frac{c}{d}K\Delta\kappa_t^p=\frac{d}{c}f_t\left(\kappa_t^{p-}+\Delta\kappa_t^p\right)$$ **éq 5.4.1-4**

the relation [éq 5.4.1-4] is a nonlinear equation compared to the variable $\Delta\kappa_t^p$ which one solves by an algorithm of Newton, which makes it possible to calculate $\Delta\varepsilon_{t_H}^p$ by [éq 5.4.1-3] and σ_H by [éq 5.4.1-2]. Taking into account [5.4.1-1], the stresses are thus completely known. [éq 5.4.1-1] still gives:

$$\mathbf{s}=\mathbf{s^e}-2\mu\Delta\tilde{\boldsymbol{\varepsilon}}_t^p=0$$ **éq 5.4.1-5**

This last equation makes it possible to calculate $\Delta\tilde{\boldsymbol{\varepsilon}}_t^p$ and the plastic strains are completely known.

5.4.2 Acceptability

the discrete form of the relation [éq 4.4-13] is:

$$3\Delta\varepsilon_{t\ H}^p\geq\sqrt{2}\,c\,\Delta\tilde{\boldsymbol{\varepsilon}}_{t\ eq}^p$$ éq 5.4.2-1

[éq 5.4.1-5] gives:

$$\Delta \tilde{\varepsilon}^{p}_{t\,eq}=\frac{\sigma^{e^{eq}}}{2\mu} \quad \text{éq 5.4.2-2}$$

By means of [éq 5.2 - 5.2-4] and [éq 5.4.2-2], [éq 5.4.2-1] is written:

$$\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\le\sigma^{e}_{H}-\sigma_{H} \quad \text{éq 5.4.2-3}$$

5.4.2.1 Acceptability a priori and a posteriori

For the criterion of tension and the part post peak of the criterion of compression, $\sigma_H=\frac{d}{c}f_t\left(\kappa^{p^-}_t+\Delta\kappa^p_t\right)$ is a decreasing function of the variable of hardening $\Delta\kappa^p_t$. One from of deduced that $\sigma^e_H-\sigma^-_H\le\sigma^e_H-\sigma_H$ and thus that:

$$\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\le\sigma^{e}_{H}-\sigma^{-}_{H} \quad\Rightarrow\quad \sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\le\sigma^{e}_{H}-\sigma_{H}$$

The condition $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\le\sigma^{e}_{H}-\sigma^{-}_{H}$ is known as condition of acceptability a priori because it can be calculated as of the elastic prediction. The condition $\sigma^{e^{eq}}\frac{cK}{\sqrt{2}\mu}\le\sigma^{e}_{H}-\sigma_{H}$ is known as condition of acceptability a posteriori.

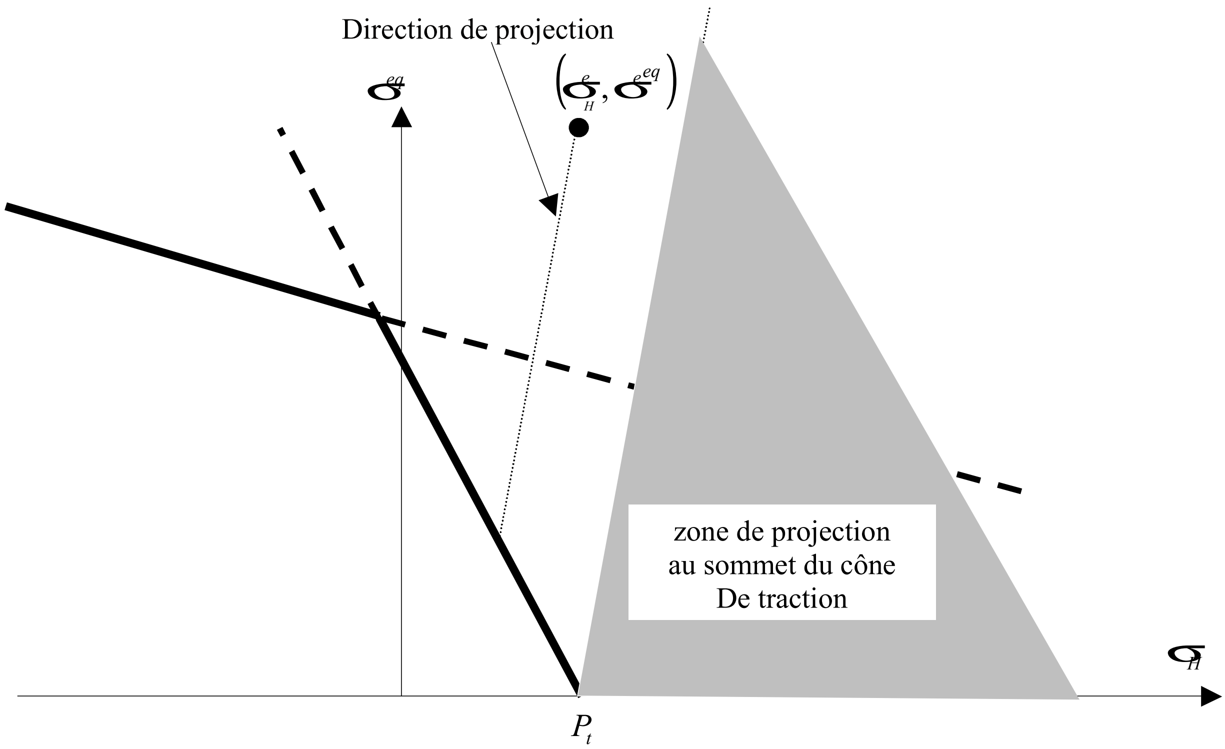


Figure 5 - 5.4.2.1-a

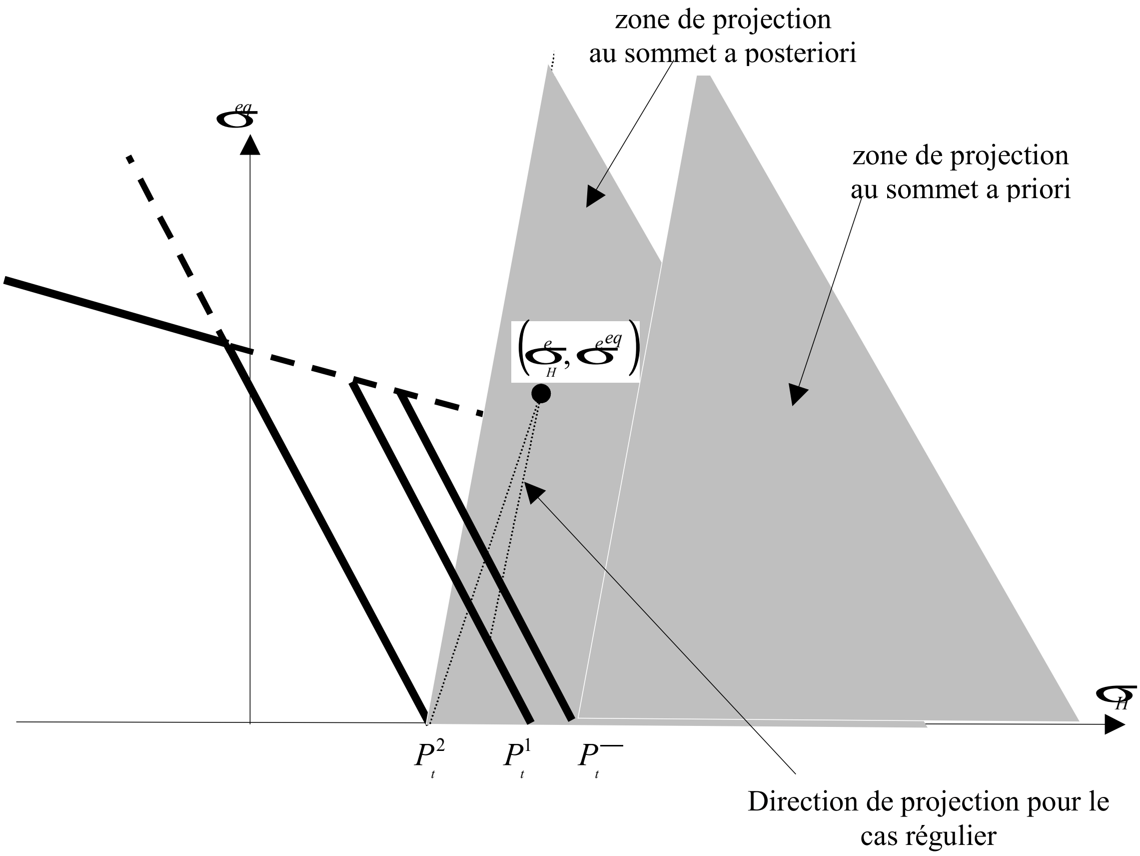
These conditions have a simple graphic interpretation. One can see easily that, in the case of a regular solution, one a:

$$\frac{\sigma^{e^{eq}}-\sigma^{eq}}{\sigma^{e}_{H}-\sigma_{H}}=\frac{\sqrt{2}\mu}{cK}$$

That shows that the solution in stress is obtained by projecting the point $\left(\sigma_H^e, \sigma^{e^{eq}}\right)$ parallel to a direction $\left(cK, \sqrt{2}\mu\right)$ in a diagram $\left(\sigma_H, \sigma^{eq}\right)$, as indicated on [Figure 5.4.2-a]. The zones of acceptability of projection at the top are cones of which the top and that of the cone of reversibility and delimited on the one hand by the axis $\sigma_H > OP_t$ and a half-line resulting from the same point and of direction $\left(cK, \sqrt{2}\mu\right)$.

5.4.3 Existence of a regular solution and a solution singular.

If projection at the top of the cone is acceptable a posteriori, it may be which exists also a regular solution as one can see it on [Figure 5.4.2-b].



Appear 5.4.2-b

the top of the cone of tension before hardening is noted P_t^- , that of the cone hammer-hardened with an increase in variable of hardening $\Delta\kappa_t^{p^1}$ is noted P_t^1 , that of the cone hammer-hardened with an increase in variable of hardening $\Delta\kappa_t^{p^2} > \Delta\kappa_t^{p^1}$ is noted P_t^2 . It is seen that there exists a regular solution with $\Delta\kappa_t^{p^1}$ and a solution with projection at the top of the cone for $\Delta\kappa_t^{p^2}$. Since the regular solution corresponds to a less hardening, in the process of evolution, it will be met before the solution with projection at the top: it is thus the regular solution which it is necessary to adopt.

For this reason the sequence of the regular searches of solution and with projection at the top is the following:

I projection at the acceptable top a priori : $\sigma^{e^{eq}} \frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma^{-}_{H}$

Computation of the solution with projection at the top: $\Delta \kappa_t^p$ by [éq 5.4-4]

So not

Search regular solution

So not convergence or not checking condition positivity

Computation of the solution with projection at the top: $\Delta \kappa_t^p$ by [éq 5.4-4]

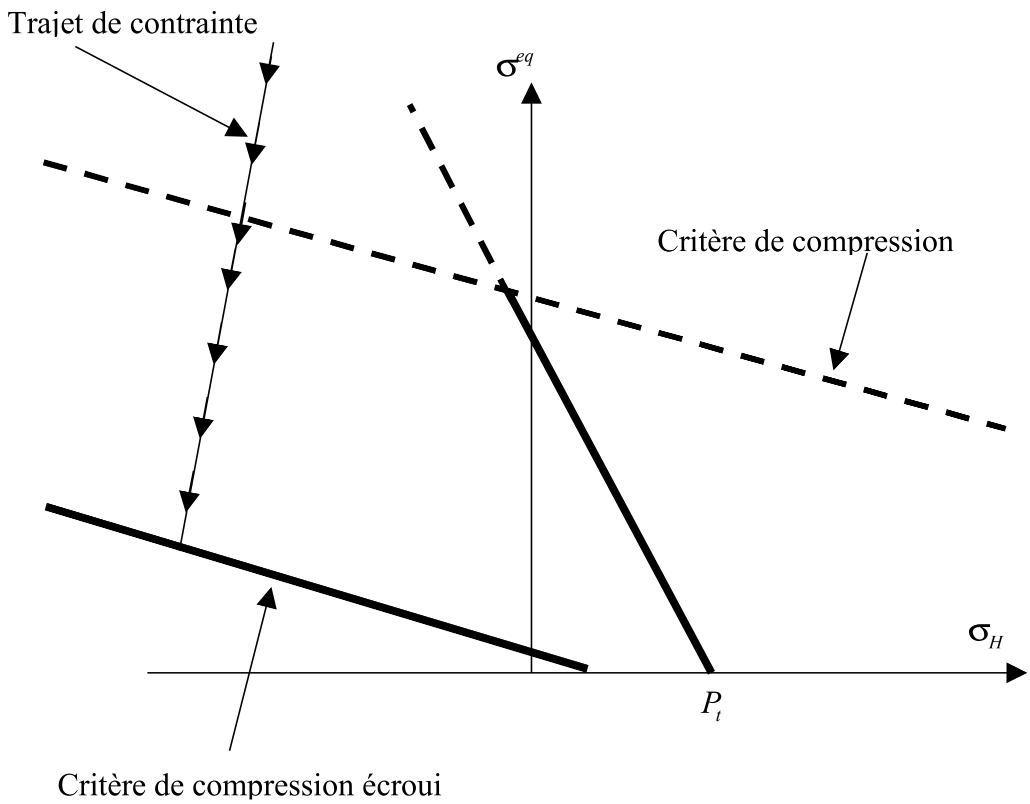
Checking of acceptability a posteriori: $\sigma^{e^{eq}} \frac{cK}{\sqrt{2}\mu} \leq \sigma^{e}_{H} - \sigma_{H}$

If not acceptable : $\sigma^{e^{eq}} \frac{cK}{\sqrt{2}\mu} > \sigma^{e}_{H} - \sigma_{H}$

Stop on diagnostic of nonchecking of condition of applicability

5.4.4 Inversion of the tops of the cones of tension and compression

A priori, the top of the cone of compression corresponds to a hydrostatic pressure of tension much larger than that of the top of the cone of tension. But, as one can see it on [Figure 5.4.4-a], one can find a load history which never activates the criterion of tension, which activates and strongly hammer-hardens the criterion of compression until it to return strictly included in the field of reversibility of the criterion of tension.

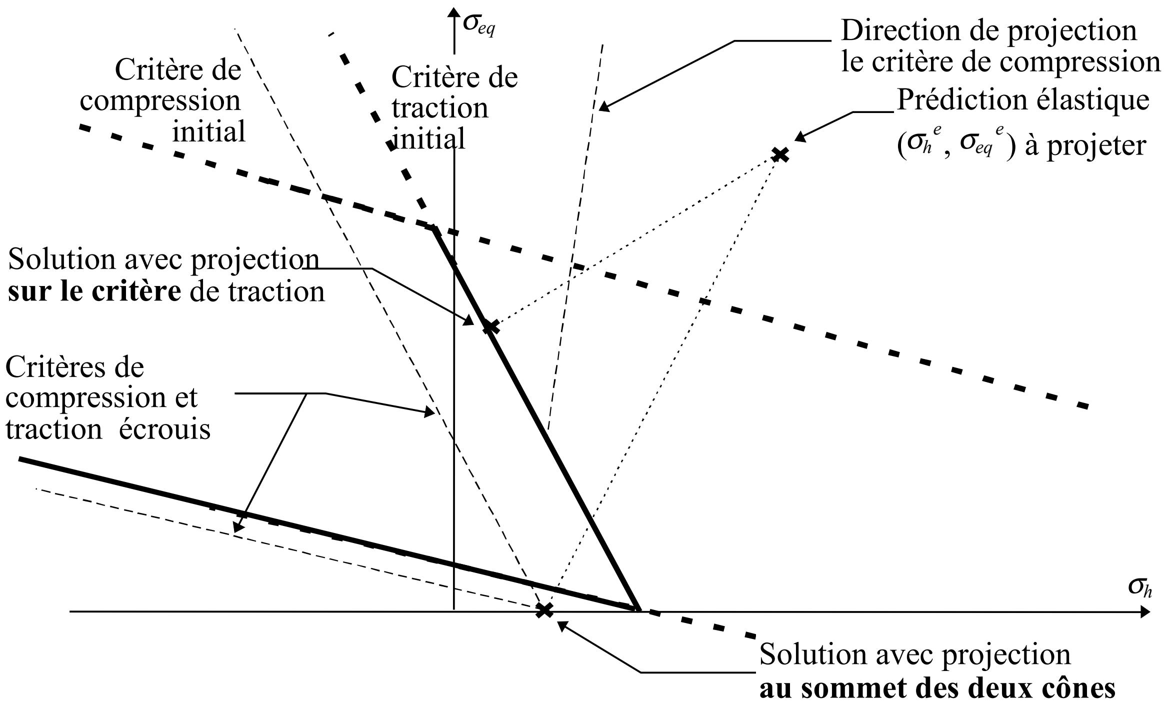


Appear 5.4.4-a

When the two criteria were thus reversed, should not intervene more a priori but the criterion of compression. It may be whereas the solution is a projection at the top of the cone of compression, which is treated exactly like projection with the top of the cone of tension.

5.4.5 Projection at the top of the two cones

If the two cones were inverted and if the elastic prediction violates the two criteria, it may be which are finally acceptable at the same time the solution of projection at the top of the cone of compression and the top of the cone of tension. In these situations, no criterion makes it possible to select a solution rather than the other and one will thus search a simultaneous projection with the top of the two cones, which will have to thus share the same top, as indicated on [Figure 5.4.5-a].



Appear 5.4.5-the

solution with projection in the two tops is obtained by solving the system:

$$\sigma_H^e - \frac{a}{b} K \Delta \kappa_c^p - \frac{c}{d} K \Delta \kappa_t^p = \frac{b}{a} f_c\left(\kappa_c^{p-} + \Delta \kappa_c^p\right) \quad \text{éq 5.4.5-1}$$

$$\sigma_H^e - \frac{a}{b} K \Delta \kappa_c^p - \frac{c}{d} K \Delta \kappa_t^p = \frac{d}{c} f_t\left(\kappa_t^{p-} + \Delta \kappa_t^p\right) \quad \text{éq 5.4.5-2}$$

the stress state is given by:

$$\mathbf{s} = 0$$
$$\sigma_H = \frac{d}{c} f_t\left(\kappa_t^{p-} + \Delta \kappa_t^p\right) = \frac{b}{a} f_c\left(\kappa_c^{p-} + \Delta \kappa_c^p\right) \quad \text{éq 5.4.5-3}$$

5.5 Determination of the tangent operator

During iterations of the algorithm of Newton-Raphson, it is necessary to calculate the tangent stiffness matrix. The construction of this one plays an important role in stability, the speed and the accuracy of the method of resolution. To preserve these properties, the tangent stiffness matrix must be built from an operator binding the increment of stress to the increment of strain in a precise way at the end of the

process of return on surfaces of load. The matrix of Hooke, as well as the thermal strains intervene like constants at the time of the determination of the coherent tangent operator, built at the end of the iteration in the increment concerned.
The computation of the operator of coherent tangent behavior takes into account plastic strains. For reasons of simplicity, we chose to calculate the operator of tangent behavior of velocity.

5.5.1 Tangent operator of velocity with only one active criterion

In the case of an only active criterion, for example, the criterion in compression, the computation of the operator of tangent behavior velocity is the following:
One thus uses the equations of velocity, in elastoplastic load:

$$\dot{\boldsymbol{\sigma}}-\boldsymbol{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\kappa}}_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\right)=\boldsymbol{0} \qquad \textbf{éq 5.5.1-1}$$

$$\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{comp}}{\partial\kappa_c^p}\dot{\boldsymbol{\kappa}}_c^p=0 \qquad \textbf{éq 5.5.1-2}$$

the tangent operator of velocity is defined by:

$$\dot{\boldsymbol{\sigma}}=\boldsymbol{D}\,\dot{\varepsilon} \qquad \textbf{éq 5.5.1-3}$$

While identifying [éq 5.5.1-3] with [éq 5.5.1-1] and [éq 5.5.1-2], one finds classically:

$$\boldsymbol{D}=\boldsymbol{H}-\frac{1}{\delta}\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\boldsymbol{H} \qquad \textbf{éq 5.5.1-4}$$

with:

$$\delta=\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial\kappa_c^p} \qquad \textbf{éq 5.5.1-5}$$

5.5.2 tangent Operator of velocity with two active criteria

If the two criteria are activated, the criterion in compression and the criterion in tension, the computation of the operator of tangent behavior velocity is the following:
One leaves:

$$\dot{\boldsymbol{\sigma}}-\boldsymbol{H}\left(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\kappa}}_c^p\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\dot{\boldsymbol{\kappa}}_t^p\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\right)=\boldsymbol{0} \qquad \textbf{éq 5.5.2-1}$$

$$\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{comp}}{\partial\kappa_c^p}\dot{\boldsymbol{\kappa}}_c^p=0 \qquad \textbf{éq 5.5.2-2}$$

$$\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\dot{\boldsymbol{\sigma}}+\frac{\partial f_{trac}}{\partial\kappa_t^p}\dot{\boldsymbol{\kappa}}_t^p=0 \qquad \textbf{éq 5.5.2-3}$$

One leads to:

$$\boldsymbol{D}=\boldsymbol{H}-\boldsymbol{H}\left[\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\left(\delta_{cc}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T+\delta_{ct}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\right)+\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\left(\delta_{tc}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T+\delta_{tt}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\right)\right] \qquad \text{éq 5.5.2-4}$$

with:

$$\delta_{cc}=\frac{\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial\kappa_t^p}\right)}{\left(\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial\kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial\kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial\boldsymbol{\sigma}}^T\boldsymbol{H}\frac{\partial f_{comp}}{\partial\boldsymbol{\sigma}}\right)}$$

éq 5.5.2-5

$$\delta_{ct}=\frac{-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

éq 5.5.2-6

$$\delta_{ct}=\frac{-\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

é **Q 5.5.2-7**

$$\delta_{cc}=\frac{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)}{\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{comp}}{\partial \kappa_c^p}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}-\frac{\partial f_{trac}}{\partial \kappa_t^p}\right)-\left(\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}\right)\left(\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}^{T}\boldsymbol{H}\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}\right)}$$

éq 5.5.2-8

the statement seems expensive to express in term of products of matrix and computation. But, when the operations are made in the order which is appropriate, it is enough to calculate terms little. Moreover, in fact the same terms intervene on several occasions. It is necessary to calculate derivatives of the criteria compared to the forced, and compared to the plastic multipliers, then the sums and the produced with the actual values, to finish by the constitution of the matrixes and theirs sums.
Lastly, the resulting matrix with the advantage of being symmetric, which is appropriate for the standard resolution with *the Code_Aster* .

5.5.3 Successive derivatives of the criteria in tension and compression

5.5.3.1 successive Drifts of the criteria compared to the forced

the derivatives of the components isotropic and deviatoric of the stresses compared to the stress tensor are expressed in the following way:

By defining the vector $\boldsymbol{\pi_0}=\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$

the derivatives of the criteria compared to the stress tensor are expressed in the following way:

$$\frac{\partial f_{comp}}{\partial \boldsymbol{\sigma}}=\frac{\boldsymbol{s}}{\sqrt{2}\, b\, \sigma^{eq}}+\frac{a}{3\mathrm{b}}\boldsymbol{\pi_0}$$

$$\frac{\partial f_{trac}}{\partial \boldsymbol{\sigma}}=\frac{\boldsymbol{s}}{\sqrt{2}\, d\, \sigma^{eq}}+\frac{c}{3\mathrm{d}}\boldsymbol{\pi_0}$$

5.5.3.2 Successive drifts of the criteria compared to the plastic multipliers

Derived from the criterion of compression in the case of a curved linear post-peak:

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=\frac{-4}{3}.\, f'_c.\left(\frac{1}{\kappa_e}-\frac{\kappa_c^p}{{\kappa_e}^2}\right)\quad si\quad \kappa_c^p\le\kappa_e$$

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=f'_c.\left(\frac{1}{(\kappa_t^u-\kappa_e)}\right)\quad si\quad \kappa_e\le\kappa_c^p\le\kappa_t^u$$

Derived from the criterion of compression in the case of a curved nonlinear post-peak:

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=\frac{-4}{3}.\, f'_c.\left(\frac{1}{\kappa_e}-\frac{\kappa_c^p}{{\kappa_e}^2}\right)\quad si\quad \kappa_c^p\le\kappa_e$$

$$\frac{\partial f_{comp}}{\partial \kappa_c^p}=2.\, f'_c.\left(\frac{(\kappa_c^p-\kappa_e)}{(\kappa_c^u-\kappa_e)^2}\right)\quad si\quad \kappa_e\le\kappa_c^p\le\kappa_c^u$$

Derived from the criterion of tension in the case of a curved linear post-peak:

$$\frac{\partial f_{trac}}{\partial \kappa_t^p}=\frac{f_t^{'}}{\kappa_t^u} \quad si \quad \kappa_t^p \le \kappa_t^u$$

Derived from the criterion of tension in the case of a curved post-peak exponential:

$$\frac{\partial f_{trac}}{\partial \kappa_t^p}=f_t^{'}\left(\frac{a}{\kappa_t^u}\right)e^{-\frac{a.\kappa_t^p}{\kappa_t^u}}$$

5.6 Local variables of the model

We assemble here the local variables stored in each Gauss point in the implementation of the model

Number of physical	local variable Meaning
1	κ_c^p : plastic strain cumulated in compression
2	κ_t^p : plastic strain cumulated in tension
3	θ : maximum temperature attack at the point of gauss
4	Indicator of plasticity

5.7 Top-level flowchart of resolution

the flow chart understands the various stages of the resolution, with the processing of projections at the tops of the cones of compression and tension in the following way:

at the beginning of algorithm,

one carries out a projection at the top of the cone of tension:

- when the elastic prediction checks the condition of projection **a priori in tension** ,
- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of tension and compression were inverted on the hydrostatic axis,

one carries out a simultaneous projection with the tops of the cones of tension and compression:

- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of tension and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of tension did not give a valid solution,

one carries out a projection at the top of the cone of compression:

- when the elastic prediction checks the condition of projection **a priori in compression** and that the tops of the cones of tension and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of tension did not give a valid solution, and that simultaneous projection with the tops of the two cones did not give a valid solution,

in medium of algorithm,

one carries out one, two or three standards resolutions with projection on the criterion of compression or the criterion of tension or the two criteria at the same time,

and **at the end of the algorithm** ,

when that the standards resolutions with activation of a criterion (tension or compression) or of the two criteria at the same time did not give a solution,

one carries out a projection at the top of the cone of tension:

- when the elastic prediction checks the condition of projection **a posteriori in tension** ,
- when the elastic prediction checks the condition of projection **a posteriori in compression** (and that the tops of the cones of tension and compression were inverted on the hydrostatic axis,

one carries out a simultaneous projection with the tops of the cones of tension and compression:

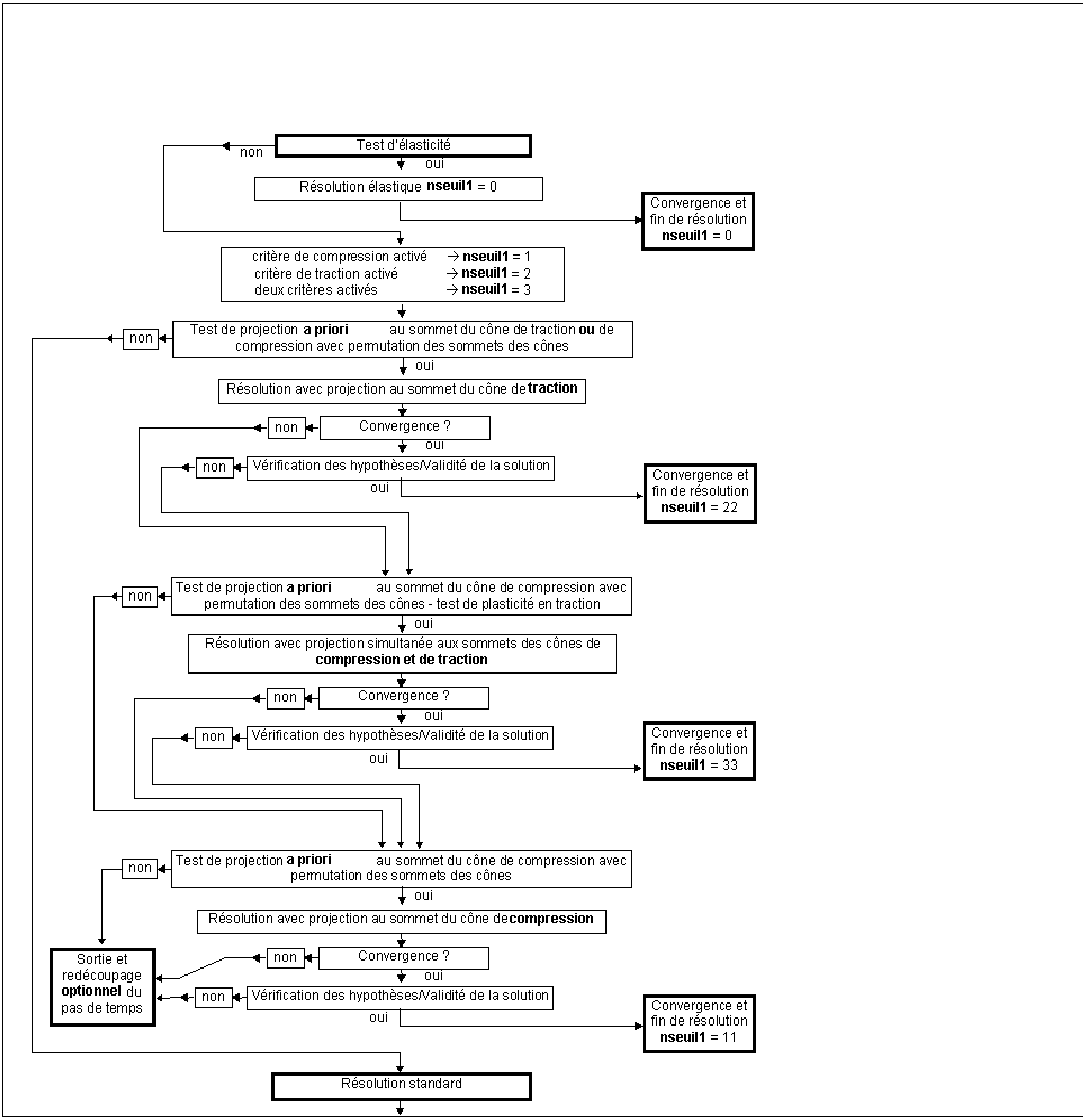
- when the elastic prediction checks the condition of projection **a posteriori in compression** and that the tops of the cones of tension and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of tension did not give a valid solution,

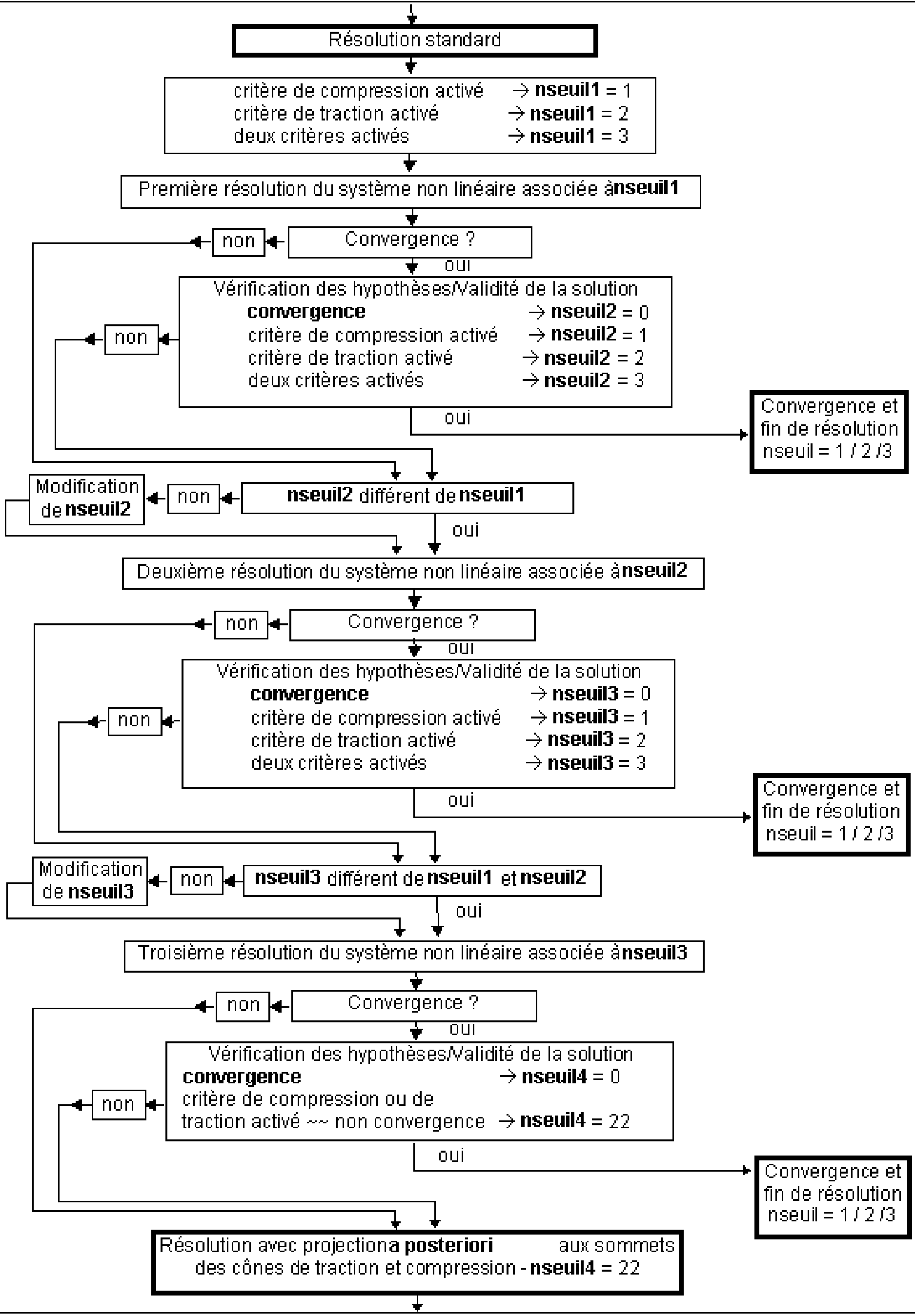
one carries out a projection at the top of the cone of compression:

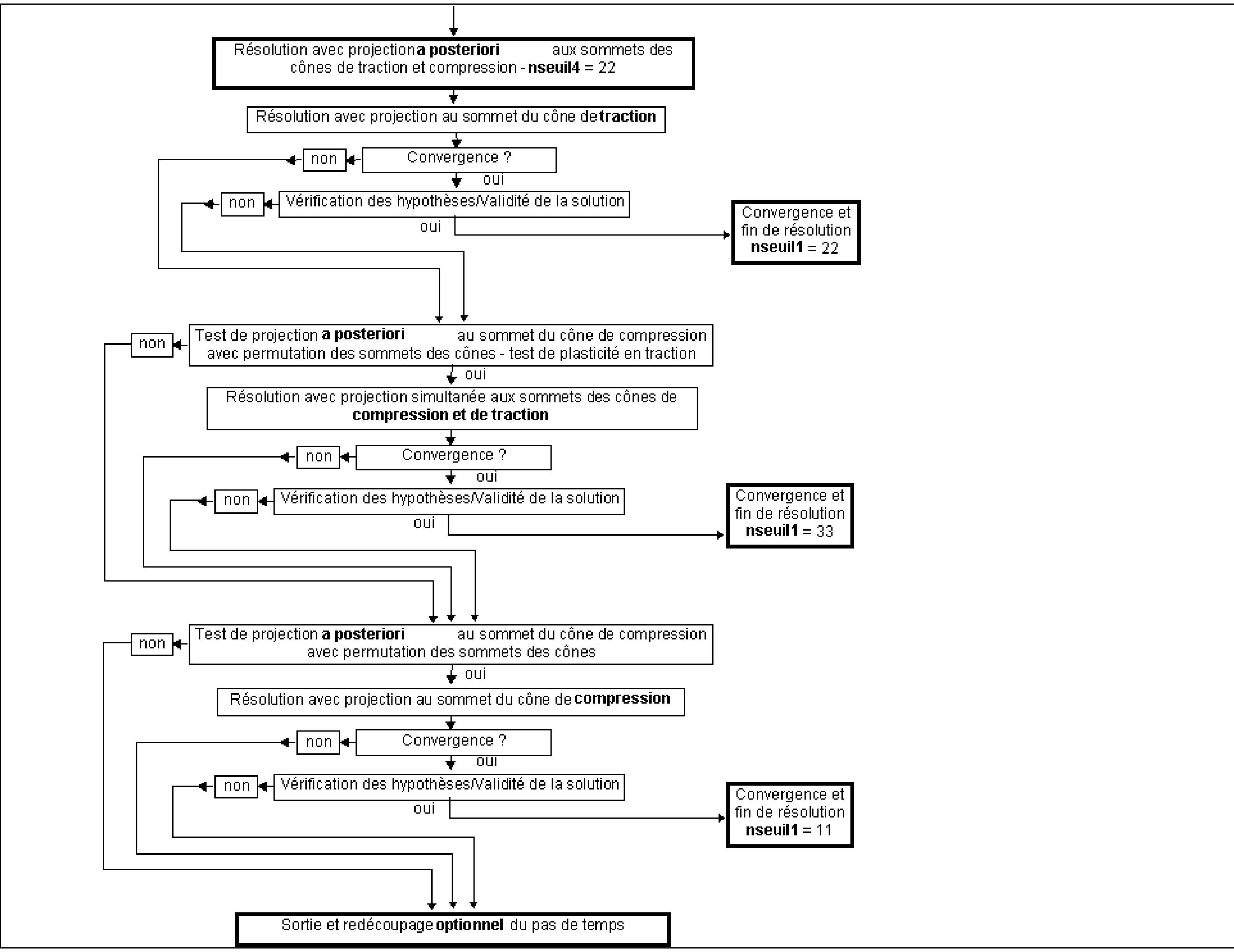
- when the elastic prediction checks the condition of projection **a posteriori in compression** and that the tops of the cones of tension and compression were inverted on the hydrostatic axis, and that projection at the top of the cone of tension did not give a valid solution, and that simultaneous projection with the tops of the two cones did not give a valid solution.

At the conclusion of each resolution having converged, one carries out the checks of conformity of the following solution:

- validity of the solution compared to the second criterion, when the resolution was made with only one criterion. In all the cases, it is enough to check that the two computed criterions with the final stress, are negative or null,
- conformity of the solution: one calculates in the course of resolution the final equivalent stress. It happens sometimes that the solution is beyond the top of the cone which is hammer-hardened, which leads to a “negative” equivalent stress. Numerically, that results in a strictly positive final criterion. To check the conformity of the solution, it is enough to check that the two computed criterions with the final stress, are negative or null,
- validity of projections at the tops of the cones. It should be checked that after resolution, when the hardening of the criterion is known, the slope of the right connecting the elastic prediction to projection is lower than the slope of the direction of projection. (condition of projection a posteriori at the top of the cones). In the contrary case, that means that there exists a solution with standard resolution.


Test d'élasticité
non
oui
Résolution élastique nseuil1 = 0
Convergence et fin de résolution nseuil1 = 0
critère de compression activé → nseuil1 = 1
critère de traction activé → nseuil1 = 2
deux critères activés → nseuil1 = 3
Test de projection a priori au sommet du cône de traction ou de compression avec permutation des sommets des cônes
Résolution avec projection au sommet du cône de traction
Convergence ?
Vérification des hypothèses/Validité de la solution
Convergence et fin de résolution nseuil1 = 22
Test de projection a priori au sommet du cône de compression avec permutation des sommets des cônes - test de plasticité en traction
Résolution avec projection simultanée aux sommets des cônes de compression et de traction
Convergence et fin de résolution nseuil1 = 33
Test de projection a priori au sommet du cône de compression avec permutation des sommets des cônes
Résolution avec projection au sommet du cône de compression
Sortie et redécoupage optionnel du pas de temps
Convergence et fin de résolution nseuil1 = 11
Résolution standard


Résolution standard
critère de compression activé → nseuil1 = 1
critère de traction activé → nseuil1 = 2
deux critères activés → nseuil1 = 3
Première résolution du système non linéaire associée à nseuil1
non
Convergence ?
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil2 = 0
critère de compression activé → nseuil2 = 1
critère de traction activé → nseuil2 = 2
deux critères activés → nseuil2 = 3
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Modification de nseuil2
non
nseuil2 différent de nseuil1
oui
Deuxième résolution du système non linéaire associée à nseuil2
non
Convergence ?
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil3 = 0
critère de compression activé → nseuil3 = 1
critère de traction activé → nseuil3 = 2
deux critères activés → nseuil3 = 3
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Modification de nseuil3
non
nseuil3 différent de nseuil1 et nseuil2
oui
Troisième résolution du système non linéaire associée à nseuil3
non
Convergence ?
oui
Vérification des hypothèses/Validité de la solution
convergence → nseuil4 = 0
critère de compression ou de traction activé ~~ non convergence → nseuil4 = 22
non
oui
Convergence et fin de résolution nseuil = 1 / 2 /3
Résolution avec projection a posteriori aux sommets des cônes de traction et compression - nseuil4 = 22



Note:

In the case of projections at the tops of the cones, one starts systematically with projection at the top of the cone of tension. If this solution is valid, this one is preserved. In the contrary case, if the criterion of tension is activated, one carries out a resolution with projection at the tops of the two cones. If the new solution is valid, that one is preserved. If not, one carries out a resolution with projection at the top of the cone of compression alone.

Note:

> *If the conditions of projection at the tops of the cones are activated, one of the three solutions must be valid, but for particularly important elastic jumps, it may be that the resolution does not succeed. The solution is then to carry out a recutting of time step.*

Note:

> *Projection at the tops of the two cones simultaneously supposes that the criterion of tension is activated. It may be very well that in the event of permutation of the tops, only the criterion of compression is found activated. It is necessary to then make a projection at the top of the cone of compression alone.*

Annexe 1snap-back with the initial values of the coefficients C and D

We show in this appendix the problem of snap-back met in the simulation of a traction test followed compression simple, if the choice of the coefficients c and d criterion of tension corresponds to a situation where the criterion of tension cuts the axes in a plane diagram of stress, i.e. the choice of the coefficients [éq 3.3-1] and [éq 3.3-2] leading to a field of reversibility represented on [Figure 3.3-b]:
The assumptions are the following ones:

- one takes into account only the criterion of tension,
- the marrow of hardening is of the type: $f_t\left(\kappa_t^p\right)=f_t'+h\,\kappa_t^p$,
- one notes simply: $\lambda=\kappa_t^p$ so that the curve of hardening is written: $f_t=f_t'+h\lambda$,
- the null Poisson's ratio,
- hardening is negative,
- the condition of applicability is filled: $-E<h<0$.

One considers an axial plain test controlled in strain according to X, as indicated on Figure 5 -

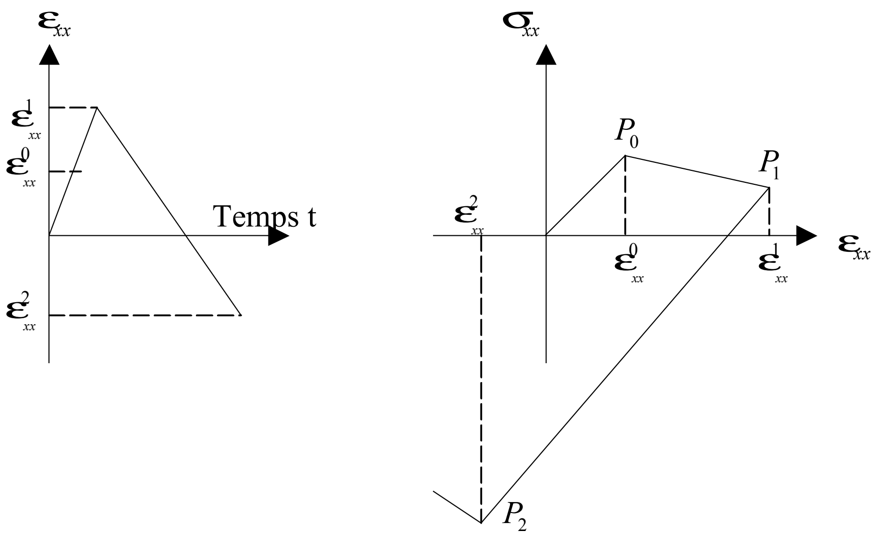


Figure 5 - has

In the other directions y and z , the imposed conditions are null constraints: $\sigma_{yy}=\sigma_{zz}=0$.
One starts by imposing a strain of tension ε_{xx}^1 such as there is a plasticization in tension, but without the limit of tension falling down to 0. It is the point P_1 in the diagram stress-strain. One notes ε_{xx}^0 the strain for which the rupture limit in tension appears for the first time. Beyond the point P_1 , one imposes a decrease of the strain, which involves an elastic unloading of the material, up to the value $\varepsilon_{xx}^2<0$ of the strain for which one has a plasticization again, but this time under compressive stress. The object of this appendix is primarily to study the behavior of the model not retained (that corresponding to the formulas [éq 3.3-1] and [éq 3.3-2]) beyond the point P_2 .

A1.1 Computation of the stresses and the strains during the loading

Taking into account the assumptions pointed out higher, the invariants of stress are worth:

$$I_1=\sigma_{xx} \quad \textbf{éq A1.1-1}$$

$$\sqrt{J_2}=\frac{|\sigma_{xx}|}{\sqrt{3}} \quad \textbf{éq A1.1-2}$$

$$\sigma^{eq}=|\sigma_{xx}| \quad \textbf{éq A1.1-3}$$

yielding is calculated by:

$$\dot{\varepsilon}^p_{xx}=\frac{c+\sqrt{2}}{3d}\dot{\lambda} \quad \textbf{éq A1.1-4}$$

A1.1.1 Paths 0P $_0$ and P $_0$ P1

Taking into account the relations [éq A1.1-1] with [éq A1.1-4], and the fact that along this way the stresses σ_{xx} are positive, one finds easily

$$\sigma_{xx_1}=E\left(\frac{h\,\varepsilon_{xx_1}+\dfrac{c+\sqrt{2}}{3\mathrm{d}}f'_t}{h+\dfrac{(c+\sqrt{2})^2E}{9\mathrm{d}^2}}\right) \quad \textbf{éq A1.1.1-1}$$

A1.1.2 Path P1P2

By definition, $P1P2$ is a path of elastic discharge, the point $P2$ being such as the criterion is again reached there

$$\sigma_{xx_2}=\frac{3\mathrm{d}}{c-\sqrt{2}}\sigma_{xx_1} \quad \textbf{éq A1.1.2-1}$$

$$\varepsilon_{xx_2}=\left(\frac{3\mathrm{d}}{c-\sqrt{2}}\right)\frac{\sigma_{xx_1}}{E}+\varepsilon_{xx_1} \quad \textbf{éq A1.1.2-2}$$

A1.1.3 With beyond point P2

One is interested now in the slope of the curve at the point $P2$ in the reference $(\sigma_{xx},\varepsilon_{xx})$. More precisely one is interested in the slope of this curve for a dissipative solution.
By writing that the material remains plastic beyond the point $P2$, i.e. that the stress state remains on criterion-which is hammer-hardened, one finds:

$$\dot{\sigma}_{xx}=\frac{3\mathrm{d}}{c-\sqrt{2}}h\dot{\lambda} \quad \textbf{éq A1.1.3-1}$$

In addition, by calculating the increase in yielding, and by deferring it in the computation of the increase in stress, one finds:

$$\dot{\varepsilon}_{xx}=\frac{\dot{\sigma}_{xx}}{E}+\frac{\dot{\lambda}}{3\mathrm{d}}\left(c-\sqrt{2}\right) \qquad \textbf{éq A1.1.3-2}$$

One can then eliminate $\dot{\lambda}$ between [éq A1.1.3-1] and [éq A1.1.3-2] and one obtains:

$$\frac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}=\frac{h\,E}{h+E\dfrac{\left(c-\sqrt{2}\right)^2}{9\mathrm{d}^2}}=E_{T_2} \qquad \textbf{éq A1.1.3-3}$$

This formula gives the slope of the response in the plane $(\sigma_{xx},\varepsilon_{xx})$
the numerator is always negative since h is negative.

The sign of $\left.\dfrac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ depends on the sign of the denominator, thus two cases are posed:

So $h<-E\dfrac{\left(c-\sqrt{2}\right)^2}{9\mathrm{d}^2}$ then $\left.\dfrac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ one is positive has a configuration of snap-back.

So $h>-E\dfrac{\left(c-\sqrt{2}\right)^2}{9\mathrm{d}^2}$ then $\left.\dfrac{\dot{\sigma}_{xx}}{\dot{\varepsilon}_{xx}}\right|_2=E_{T_2}$ is negative and there is no snap-back.

A new condition appears to avoid the snap back, condition which we compare with that already evoked, but which related to in fact possible the snap back at the point $P1$.

Slope at the point $P1$ in the reference $(\varepsilon_{xx},\sigma_{xx})$: $E_{T_1}=\dfrac{h\,E}{h+E}$,

Slope at the point $P2$ in the reference $(\varepsilon_{xx},\sigma_{xx})$: $E_{T_2}=\dfrac{h\,E}{h+E\dfrac{\left(c-\sqrt{2}\right)^2}{9\mathrm{d}^2}}$.

Gold $\dfrac{\left(c-\sqrt{2}\right)^2}{9\mathrm{d}^2}=3\,\alpha^2=3\left(\dfrac{f'_t}{f'_c}\right)^2$.

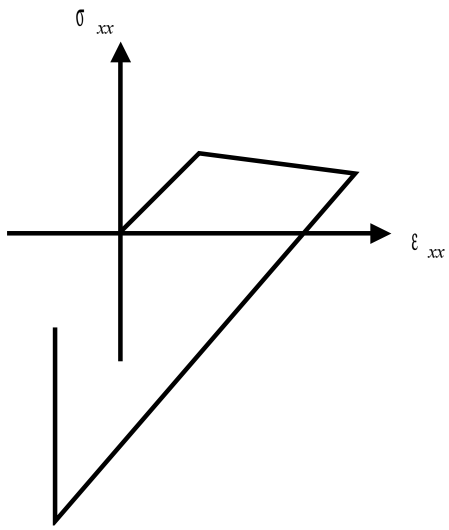
One can for example express E_{T_2} according to E_{T_1} while eliminating h :

$$E_{T_2}=\frac{E_{T_1}E}{3\,\alpha^2 E+E_{T_1}\left(1-3\,\alpha^2\right)} \ .$$

For example,

$E_{T_2}=\infty$ for $E_{T_1}=-\frac{3\alpha^2 E}{(1-3\alpha^2)}$.

As example, for $E=32000\,Mpa$, $f'_t=3\,Mpa$ and $f'_c=38{,}3\,Mpa$ one finds $E_{T_1}\approx-601$. Thus E_{T_1} is very weak compared to E . as illustrated on [A1.1.3-a Figure].



Appear A1.1.3-a

Thus, a condition implying that there is no snap-back at the point $P2$ would be too restrictive and would lead to practically choose a material not brittle in tension.
For this reason we preferred to modify the statement of the coefficients c and d as indicated in the paragraph [§ 3.3].

This said, and even if the adopted solution, consisting in modifying the coefficients c and d seems reasonable, the example treated in this appendix shows that a very simple problem can finally be a structure problem: there is in this example of the equilibrium conditions, they are the conditions: $\sigma_{yy}=\sigma_{zz}=0$.

6 Bibliography

•J.F. GEORGIN "Contribution to the modelization of the concrete under fast request of dynamics. The taking into account of the effect velocity by viscoplasticity" - Thesis (1/15/98).
•G. HEINFLING "Contribution to the numerical modelization of the behavior of the reinforced concrete concrete and structures under thermomechanical requests at high temperature" - Thesis (1/14/98)
•R.T. ROCKAFELLAR "Convex analysis" Princeton University Close 1972
•B. HALPHEN and NGUYEN QUOC HIS, "On the generalized standard materials" Newspaper of Mechanical Flight 14 n° 1,1975
•N. TARDIEU, I. VAUTIER, E. LORENTZ "quasi static nonlinear Algorithm" document de référence Aster [R5.03.01].

7 Functionalities and checking

This document relates to constitutive law `BETON_DOUBLE_DP` (key word `COMP_INCR` of `STAT_NON_LINE`) and its associated material `BETON_DOUBLE_DP` (command `DEFI_MATERIAU`).

This constitutive law is checked by the cases following tests:

SSNP116	Coupling creep/cracking - uniaxial Tension	[V6.03.116]
SSNV143	biaxial Tension with constitutive law `BETON_DOUBLE_DP`	[V6.04.143]
SSNV150	triaxial Tension with constitutive law `BETON_DOUBLE_DP`	[V6.04.150]
SSNV151	Tension/Compression with constitutive law `BETON_DOUBLE_DP`	[V6.04.151]

8 Description of the versions of the document

Version Aster	Author (S) Organization (S)	Description of the modifications
5	***C. CHAVANT***, *EDF-R&D/AMAB.CIREE* *CS-SI*	initial Text